Nonlinear PDE's, Prague, December 13–15, 2009

MEMORIAL SEMINAR

dedicated to 80^{th} anniversary of birth of Professor Jindřich Nečas

ON TWO-SCALE CONVERGENCE

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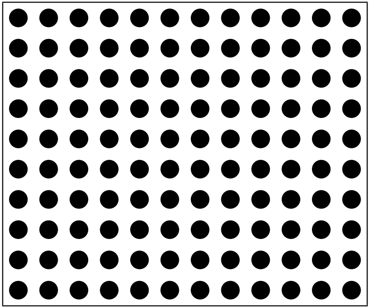
Motivation — Homogenization

- Physical setting



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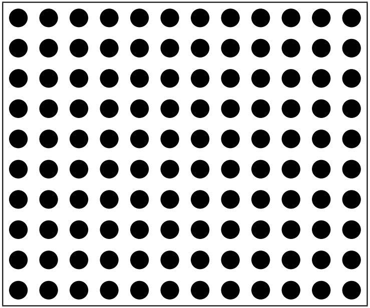

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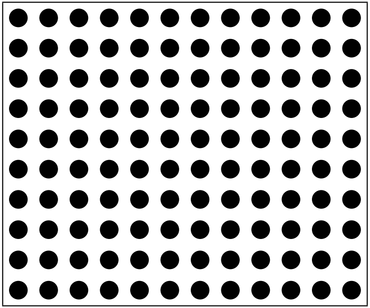

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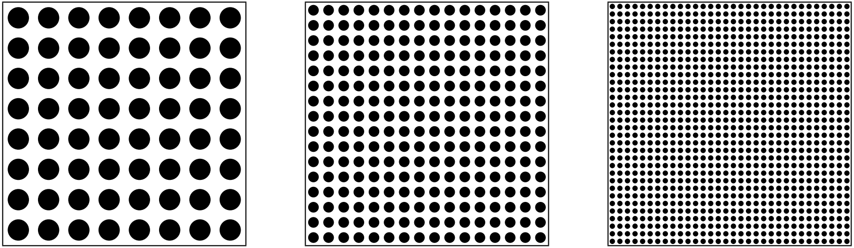
- For computation reason: fine structure needs fine discretization and large number of equations.

Homogenization-Mathematical Approach

- Sequence of problems with diminishing period (Babuška 1972)



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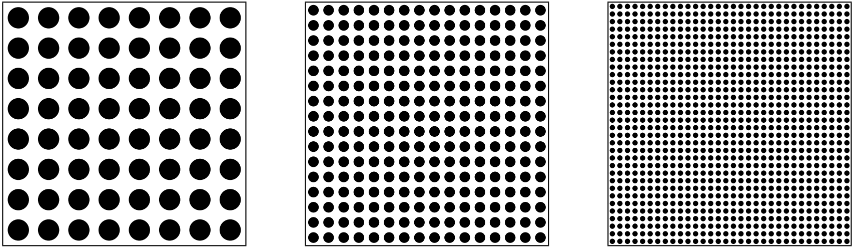

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$$-\mathrm{div}\,(a^{\varepsilon}(x)u^{\varepsilon}) = f \qquad a^{\varepsilon}(x) = a\left(\frac{x}{\varepsilon}\right) \qquad a(y) - Y\text{−periodic}$$

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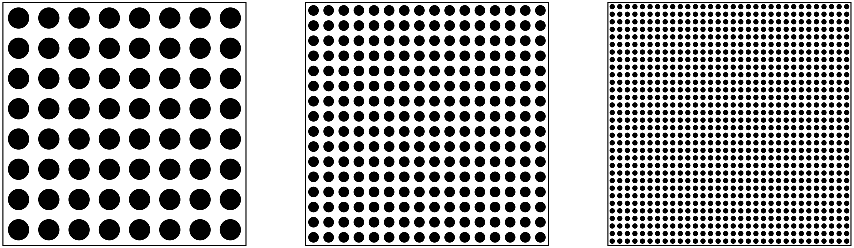

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- Questions:
  - Convergence of the solutions u^ε
  - The form of the limit problem
  - Formulae for the homogenized coefficients b,

Sketch of the proof

$$-\operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) = f \qquad \text{with} \qquad a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$$

Weak solutions u^ε — exist, bounded, $\eta^\varepsilon \equiv a^\varepsilon \cdot \nabla u^\varepsilon$ — bounded

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Passing to the limit in the weak formulation of the problem:

$$\int_\Omega \nabla v \cdot a^{\varepsilon'} \cdot \nabla u^{\varepsilon'} \mathrm{d}x \equiv \int_\Omega \nabla v \cdot \eta^{\varepsilon'} \mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

$$\downarrow$$

$$\int_\Omega \nabla v \cdot \eta^* \mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

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Problem – product of two weakly converging sequences

$$\left.\begin{matrix} f_n \longrightarrow f^* \\ g_n \longrightarrow g^* \end{matrix}\right\} \Longrightarrow f_n\, g_n \rightarrow f^*\, g^*$$

An example in $L^2(0, 2\pi)$:

The sequences

$$\begin{aligned} u_k(x) = \sin(kx) \quad &\rightharpoonup \quad 0 \\ v_k(x) = \sin(kx) \quad &\rightharpoonup \quad 0 \end{aligned}$$

but

$$u_k(x) \cdot v_k(x) = \sin^2(kx) \quad = \quad \frac{1}{2} \neq 0 \cdot 0$$

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- General case: Σ convergence — Nguetseng (2004)

Two-scale convergence – classical approach

G. Nguetseng   Université de Yaoundé, Cameroon

$\Omega \subset \mathbb{R}^N$, Y – unit cube in $\mathbb{R}^N$, $\{u^\varepsilon\}_\varepsilon$ sequence in $L^p(\Omega)$,

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– 2 variables: $x \in \Omega$ — global $y \in Y$ — local behavior.

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for all $\varphi(x, y)$ Y-periodic.

If moreover $\|u^\varepsilon\|_{p;\Omega} \to \|u^0\|_{p;\Omega\times Y}$
then the convergence is called 2-scale strong.

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Let $f(x), g(x)$ in $L^p(\Omega)$

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Local behavior of u^ε and $\varphi(x, x/\varepsilon)$ — are in "resonance"

If not in resonance – e.g. $u^\varepsilon(x,y) = f(x) \cdot \psi\left(\frac{x}{\sqrt{2}\varepsilon}\right) + g(x)$,
then u^ε 2-scale converge (only weakly) $u^0(x,y) = g(x)$
— local behavior in the limit u^0 is lost.

Fundamental properties

Comparison of convergences in $L^p(\Omega)$:

strong $\Rightarrow$ strong two-scale $\Rightarrow$ weak two-scale $\Rightarrow$ weak.

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Theorem – Convergence result *Let the sequences*

$$u^\varepsilon \rightharpoonup u^0 \quad \text{2-scale (weakly) in } L^p(\Omega) \text{ and}$$

$$v^\varepsilon \rightarrow u^0 \quad \text{2-scale strongly in } L^q(\Omega).$$

Then

$$u^\varepsilon\, v^\varepsilon \rightharpoonup u_0\, v_0 \quad \text{2-scale (weakly) in } L^r(\Omega).$$

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Then

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Particularly for any $\varphi \in L^s(\Omega)$ $\quad \frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$

$$\int_\Omega u^\varepsilon(x)\, v^\varepsilon(x)\varphi(x)\mathrm{d}x \longrightarrow \iint_{\Omega\times Y} u^0(x,y)\, v^0(x,y)\, \varphi(x)\mathrm{d}x\mathrm{d}y\,.$$

Application to the homogenization problem:

$$\begin{array}{rcl} \int_\Omega \nabla v \cdot a^{\varepsilon'} \cdot \nabla u^{\varepsilon'} \mathrm{d}x & = & \int_\Omega f v \, \mathrm{d}x \\ \downarrow \qquad \downarrow & & \\ \int_\Omega \nabla v \cdot \eta^* \mathrm{d}x & = & \int_\Omega f v \, \mathrm{d}x \end{array}$$

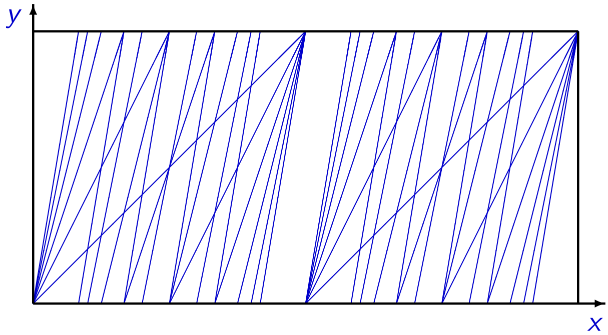
$a^\varepsilon(x)$ converge 2-scale strong to $a(y)$

$\nabla u^{\varepsilon'}(x) = \xi^{\varepsilon'}(x)$ converge 2-scale weak to some $\xi^0(x, y)$.

thus the limit is $\int_Y a(y)\xi^0(x, y)\, \mathrm{d}y$.

Problem: Choice of the space for test functions

In definition $\varphi \in L^p(\Omega \times Y)$ yields undefined $\varphi(x, x/\varepsilon)$



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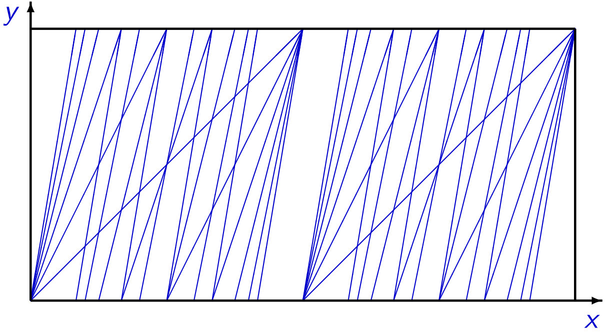

An "adjoint" definition of two-scale convergence removes the problem.

Adjoint definition using periodic unfolding

Arbogast-Douglas-Hornung 1990, Casado 2000,
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test function $\varphi(x, y)$ converted into $\varphi(x, x/\varepsilon)$
and convergence tested in $L^p(\Omega)$

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Alternative two-scale convergence – using periodic unfolding:
$u^\varepsilon(x)$ converted into $\widehat{u^\varepsilon}(x, y)$
and the convergence

$$\widehat{u^\varepsilon}(x, y) \rightarrow u^0(x, y)$$

is tested in the classical $L^p(\Omega \times Y)$

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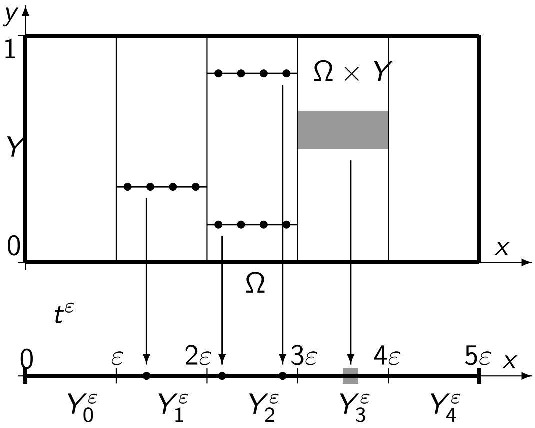
where the two-scale transform is define by means of the mapping:

$$\tau^\varepsilon : (x, y) \mapsto \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y$$

and

$$\widehat{u^\varepsilon}(x, y) = u^\varepsilon(\tau^\varepsilon(x, y) \equiv u^\varepsilon(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y)$$

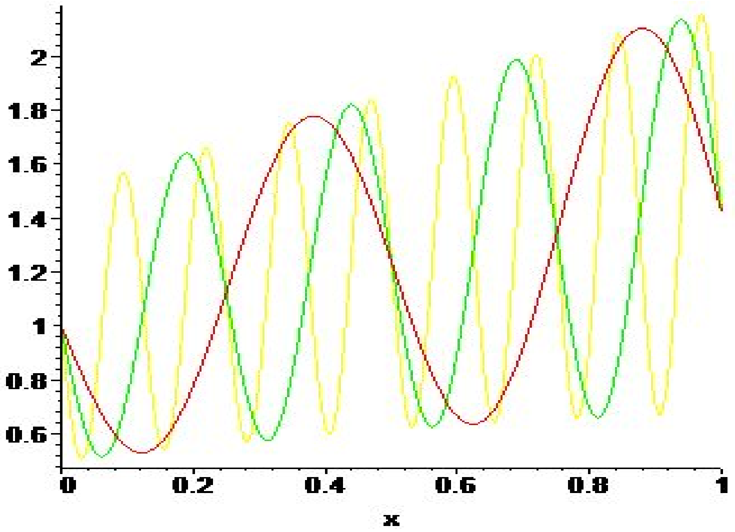
Two-scale mapping – periodic unfolding



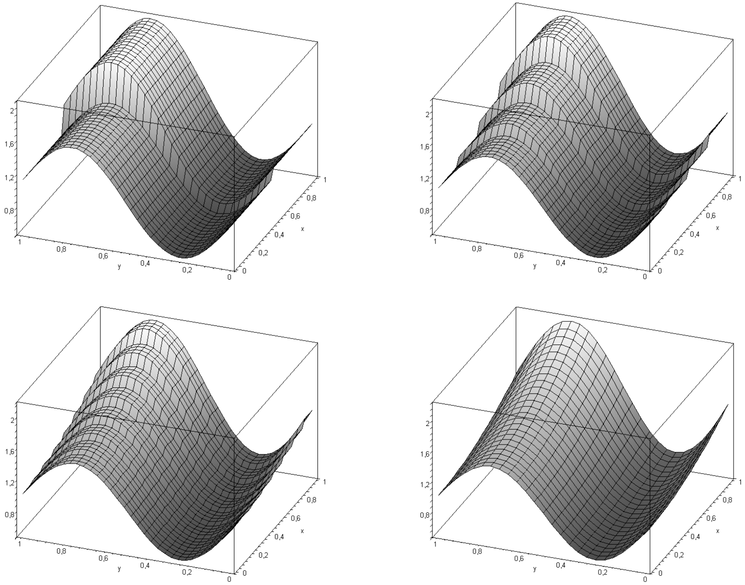
two-scale mapping: $t^\varepsilon : \Omega \times Y \to \Omega$

$$u^\varepsilon(x) \mapsto \widehat{u^\varepsilon}(x,y) = u^\varepsilon(t^\varepsilon(x,y))$$

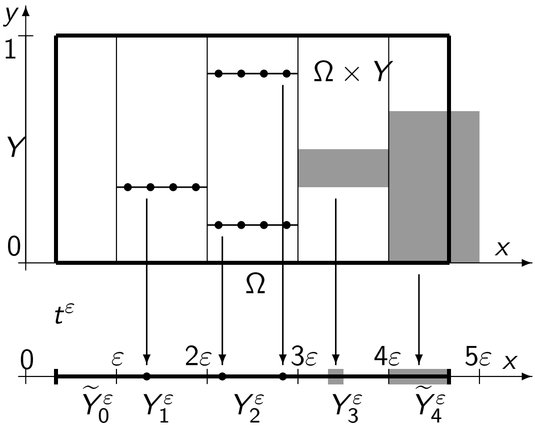
Example



Transformed u^{ε} for $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and u^0



Problem: boundary cells



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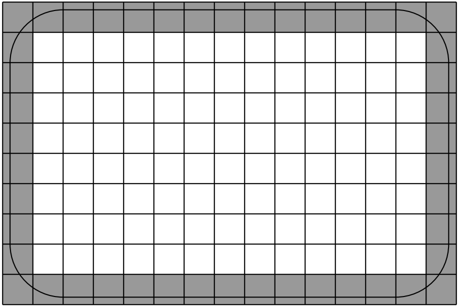

Boundary incomplete cells – undefined – zero extension

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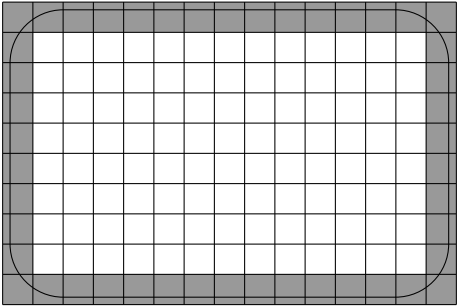

Boundary incomplete cells – undefined – zero extension
Problem: measure conserving property does not hold

$$\|\widehat{u^\varepsilon}\|_{L^p(\Omega\times Y)} = \|u^\varepsilon\|_{L^p(\Omega)}$$

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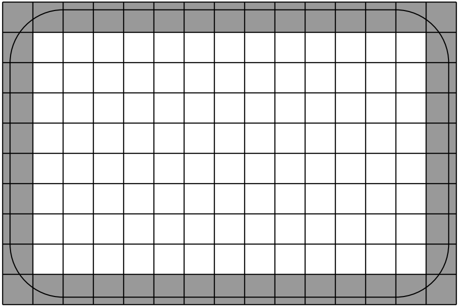

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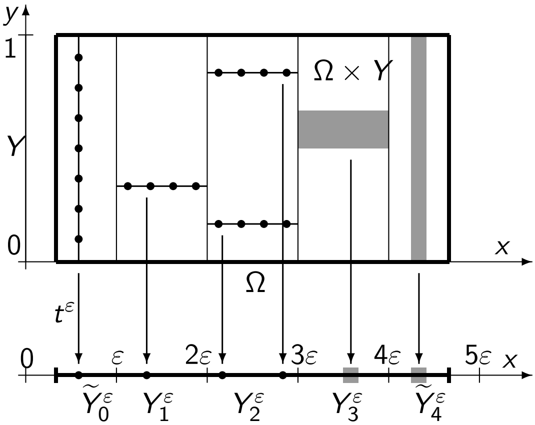
solved by a new concept in Cioranescu-Damlamian-Griso 2008.

A new simple solution



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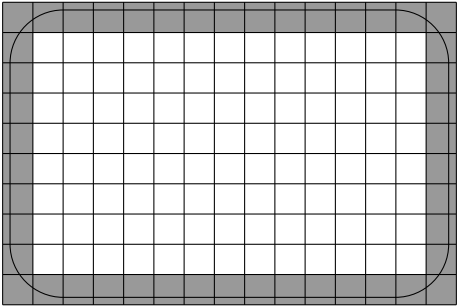
Survey of the modified two-scale transform



Inner cells: two-scale transform

$$\widehat{u^\varepsilon}(x, y) = u^\varepsilon(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y)$$

Incomplete boundary cells: no transform

$$\widehat{u^\varepsilon}(x, y) = u^\varepsilon(x)$$

measure conserving property holds.

Survey of the modified two-scale transform

- No problem with the space for test function
- The whole space $L^2(\Omega \times Y)$ for u^0 and the same for test functions
- Natural definition of the weak and strong two-scale convergence
- With measure conserving property the proofs from L^p-theory
  - Compactness
  - Passing to the limit is possible in case of:
    weak 2-scale $\times$ strong 2-scale

New general approach by Nguetseng (2004)

It covers periodic, quasi-periodic and non-periodic structures

Let Π – bounded continuous functions f on $\mathbb{R}^N$ having the mean value — the $L^\infty(\mathbb{R}^N)$-weak*limit $M(f) = \lim_{\varepsilon\to 0} f\left(\frac{x}{\varepsilon}\right)$

$(\Pi, \|\cdot\|_{\max})$ – Banach algebra = Banach space + multiplication:

$$f, g \in \Pi \Rightarrow f \cdot g \in \Pi$$

here pointwise multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$.

Structural representation —

— a countable multiplicative subgroup Γ in Π .

***H*-structure** — a class Σ of structural representations Γ generating the same linear subspace in Π.

***H*-algebra** — Banach algebra $A \equiv A_\Sigma$ in Π spanned by Σ.

The key notion – Spectrum of an algebra

Given algebra A – its spectrum $\Delta(A)$
— a subset of the dual A^*
— the set of all nonzero continuous multiplicative linear functionals on A:

$$F(f \cdot g) = F(f) \cdot F(g).$$

$\Delta(A)$ in weak topology: compact space.

Gelfand representation: $A \longleftrightarrow \mathcal{C}(\Delta(A))$

$f \in A \mapsto \mathcal{G}(f) \in \mathcal{C}(\Delta(A))$ defined by $\mathcal{G}(f)(s) = s(f) \;\; \forall\, s \in \Delta(A)$

Example: If A – Y-periodic functions, then $\Delta(A) \approx Y$

Structures on the spectrum

Radon measure β on $\Delta(A)$ — induced by the mean value $M(f)$

$$M(f) = \int_{\Delta(A)} \mathcal{G}(f)(s)\mathrm{d}\beta(s) \qquad \forall f \in A.$$

Lebesgue spaces on $\Delta(A)$ $\quad \mathcal{X}^p_\Sigma$

— closure of the Banach algebra $A = A_\Sigma$ in the norm

$$\sup_{0<\varepsilon\leq 1} \left(\int_{|x|<1} \left| u\left(\frac{x}{\varepsilon}\right) \right|^p \right)^{1/p}$$

Gelfand mapping can be extended to $\mathcal{G} : \mathcal{X}^p_\Sigma \to L^p(\Delta(A))$.

Sobolev-type space $H^1(\Delta(A))$ — Gelfand mapping yields
derivatives in A $\quad \longleftrightarrow \quad$ derivatives in $\mathcal{C}(\Delta(A))$
Completion of smooth functions A^∞ — $W^{1,p}(\Delta(A))$.

Σ-convergence

Generalization of 2-scale convergence:

DEFINITION $\{u_\varepsilon\}_\varepsilon$ in $L^2(\Omega)$ weakly Σ-converge to an $u_0 \in L^2(\Omega, \Delta(A))$ if

$$\int_\Omega u_\varepsilon(x) v^\varepsilon(x)\,\mathrm{d}x \to \iint_{\Omega\times\Delta(A)} u_0(x,s)\widehat{v}(x,s)\mathrm{d}x\,\mathrm{d}\beta(s)$$

for each $v \in L^2(\Omega; A)$, where $v^\varepsilon(x) = v(x, x/\varepsilon)$ and $\widehat{v} = \mathcal{G} \circ v$.

Compactness: each sequence u_ε bounded in $L^2(\Omega)$ contains a subsequence $u_{\varepsilon'}$ weakly Σ-converging to an $u_0 \in L^2(\Omega, \Delta(A))$.

A stronger version is called strong Σ-convergence.

H-structure Σ — is proper if it satisfies some density, regularity and reflexivity conditions.

The H-structures of periodic and almost periodic functions are proper.

Some of my souvenirs to professor Jindřich Nečas

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In 1977–80 I was a Ph.D. student of prof. Nečas.

In the beginning he choose literature to my study: looking into his bookshelves he pulled out 14 books for me to study – to write down their titles – there were: 3 in English, 4 in French, 3 in Russian, 2 in Italian and 2 in Czech – and all were very thick.

To the rigorous exam he order me to learn the last chapter of a book on P.D.E.: Minimal surface equation. To study the last chapter I had to study almost all preceding chapters, I spent many days by trying to learn it but with quite weak result – I could not say that I learned it.

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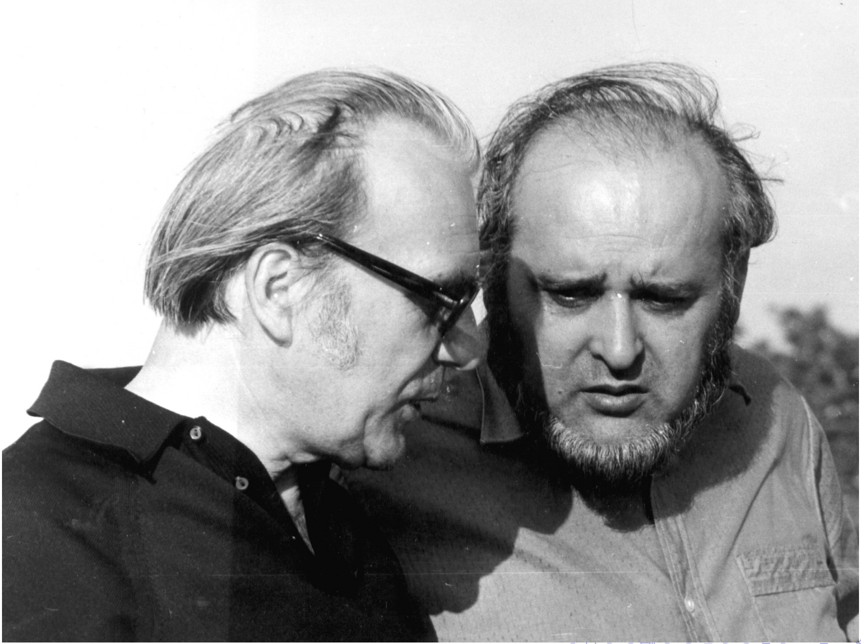
In 1978 prof. Nečas accepted from prof. Václav Horák a proposal to study a new method described in papers written by Ivo Babuška.

Since he was busy, he gave me the papers to refer it in his seminar and write a report on it and promised me a part of the money he would received for it.

In this way I met homogenization and started to be interested in it. He was lending me all papers on the homogenization he was receiving. Besides the seminar I wrote my dissertation and several further papers on the topic.

Professor Nečas in my pictures – with prof. Jan Polášek



Professor Nečas in middle of students in Olomouc 1999