Regularity results for nonlinear elliptic and parabolic differential equations

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General remarks about the system

Introduction

We consider

$$\dot{u}_\nu + \mathcal{L} u_\nu + \alpha_\nu u_\nu = H_\nu(x, u, \nabla u)$$

$$\nu = 1, \ldots, N\,, \qquad \text{in } \Omega \subset \mathbb{R}^N$$

where

$$\mathcal{L}_\nu v = -\sum_{i,k=1}^{n} D_i\big(a_{ik}^\nu(x) D_k v\big)$$

and $H = (H_{1,\ldots,} H_N)$ is a Caratheodory function

$$H : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times N} \to \mathbb{R}^N$$

with **quadratic growth** in ∇u

$$\big|H(x, u, \nabla u)\big| \le K|\nabla u|^2 + K_0$$

Well known: Quadratic behaviour of H creates singularities:

scalar case:

$$u = \ln \big| \ln |x| \big|$$

this implies

$$-\triangle u = |\nabla u|^2$$

u bounded & H^1, scalar $\rightarrow$ regularity (Ladyzhenskaya, Uralzeva)

Counterexamples for elliptic systems

Bounded solutions to elliptic systems, may be irregular even for dimensions $\boldsymbol{n \geq 2}$

scalar complex valued example

$$u = e^{i \ln \left| \ln |x| \right|}$$

solves

$$-\triangle u = (1+i)|\nabla u|^2 u$$

this can be interpreted as real valued system
Harmonic mappings ($n \geq 3$): $\dfrac{x}{|x|}$ solves

$$\int |\nabla u|^2 dx = min, \text{ subject to } |u| = 1$$

$$-\triangle u = u|\nabla u|^2$$

Consequence: additional structure for H is needed

A standard structure condition where the step

$$L^\infty \Rightarrow H^1 \cap C^\alpha \cap W^{2,p}$$

works is

For **2** equations

$$\begin{cases} |H_1(x,p)| & \leq K|p|^2 + K \\ |H_2(x,p)| & \leq K|p_2|^2 + K|p_1|\,|p_2| + K \end{cases}$$

For **3** equations

$$\begin{cases} |H_1(x,p)| & \leq K|p_1|^2 + K|p_1|\,|p_2| + K|p_1|\,|p_3| + K \\ |H_2(x,p)| & \leq K|p_1|^2 + K|p_2|^2 + K|p_2|\,|p_3| + K \\ |H_3(x,p)| & \leq K|p_1|^2 + K|p_2|^2 + K|p_3|^2 + K. \end{cases}$$

These conditions arise from stochastic differential games

Special structure of H

$$H_\nu(x,u,\nabla u) = H_{0\nu}(x,u,\nabla u_i) - \nabla u_i \cdot L(x,u,\nabla u) - \\ - u_i F_i(x,u,\nabla u) + f_i(x)\,,$$

with

$$\left.\begin{array}{l} H_{0i}(x,u,\nabla u) \\ F_i(x,u,\nabla u) \end{array}\right\} \quad \text{quadratic growth in } \nabla u$$

$$L(x,u,\nabla u) \qquad \text{linear growth in } \nabla u$$

The case $H_{0\nu} = 0,\ F_i \geq 0$ has been treated by **Wiegner**, say

$$-\triangle u + u|\nabla u|^2 = f$$

permits **full regularity**, i.e. $W^{2,p}$-solutions

open:

$$-\triangle u_\nu + u_\nu|\nabla u|^2 = f + b_\nu|\nabla u_\nu|^2$$

Applications to stochastic differential games

Stochastic Differential Games with Discount Control

Consider N **players**, who can modify the evolution of a dynamic system

$$\boldsymbol{d}y = g(y, v_1, \ldots, v_N)\,\boldsymbol{d}t + \sigma(y)\,\boldsymbol{d}w$$

$$y(0) = x$$

$g(t)$	drift term
$\sigma(t)$	diffusion term
$w(t)$	Wiener process
$v_1(t), \ldots, v_n(t)$	controls
$y(t)$	state at time t

Cost functional of player i

$$\mathcal{J}_i\big(x, v(\cdot)\big) = E_i\Big[\int_0^\tau l_i\big(y(t); v(t)\big) \exp\Big(-\int_0^t c_i\big(y(s), v(s)\big)\, dx\Big) + \\ + \phi_i\big(y(\tau)\big) \exp\left(-\int_0^\tau c_i(y(t)), v(t)\, dt\right)\Big]$$

τ exit time of $y(t)$
The factor

$$\exp\left(-\int_0^t c_i\big(y(s), v(s)\big)\, dt\right)$$

is the **discount factor** of the $\boldsymbol{i}$-th player which can be influenced by him/her.

Nash Point

Find $\hat{v}_1(\cdot), \dots, \hat{v}_N(\cdot)$ such that

$$\mathcal{J}_i\big(x, \hat{v}_1(\cdot), \dots, \hat{v}_i(\cdot), \dots, \hat{v}_N(\cdot)\big) \le \mathcal{J}_i\big(x, \hat{v}_1, \dots, \boldsymbol{v}(\cdot), \dots \hat{v}_N(\cdot)\big)$$

$l_i(x, v_1, \dots, v_N)$, $c_i(x, v_1, \dots, v_N)$, $\phi_i(x)$
are given functions

The Hamiltonian

$$\mathscr{L}_i(x, \lambda_i, p_i, v) = l_i(x, v) + p_i g(x, v) - \lambda_i c_i(x, v)$$

Fixing x, λ_i, p_i, we look for a **Nash point** $\hat{V}_1(x, \lambda, p), \ldots, \hat{V}_N(x, \lambda, p)$ for the functionals L_i.

$$\begin{aligned} \lambda = & \quad (\lambda_i, \ldots, \lambda_N) \\ p = & \quad (p_1, \ldots, p_N) \end{aligned}$$

We define

$$H_i(x, \lambda, p) = \mathscr{L}_i\big(x, \lambda_i, p_i, \hat{V}(x, \lambda, p)\big)$$

The Hamiltion-Bellmann equations

$$Au_i = H_i(x, u, Du)\,, \qquad\qquad u_i = 0 \text{ on } \partial\Omega$$

$$A = -\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i, \partial x_j}$$

Once we have found a regular solution , say $u_i \in W^{2,p}(\Omega), p > n$, we may set

$$\hat{v}_i(x) = \hat{V}_i\big(x, u(x), Du(x)\big)$$

and obtain an optimal feedback for the player i, in the sense that

$$\hat{v}_i(t) = \hat{v}_i\big(y(t)\big)$$

A Standard Example for Lagrangians Modelling Discount Control

A reasonably simple class of Lagrangians $\mathscr{L}_i$ is

$$\mathscr{L}_i(x, \lambda_i, p_i, v) = \frac{1}{2} v_i \cdot B_i v_i + p_i \sum_{\nu=1}^{N} A_\nu v_\nu - \\ - \frac{1}{2} \lambda_i \Big(\sum_{\nu=1}^{N} v_\nu \cdot C_\nu^i v_\nu \Big) + f_i(x) \, .$$

A **Nash point** v^* of $\mathscr{L}_i$ satisfies

$$B_i v_i^* + A_i^T p_i - \lambda_i C_i^i v_i^* = 0 \qquad \text{and}$$

$$v_i^* = -(B_i - \lambda_i C_i^i)^{-1} A_i^T p_i$$

$$
\begin{aligned}
H_i(x,u,\nabla u) = {} & \frac{1}{2}E_i\nabla u_i \cdot B_i E_i \nabla u_i - \\
& - \nabla u_i \cdot \sum_{\nu=1}^{N} A_\nu E_\nu \nabla u_\nu - \frac{1}{2}u_i \sum_{\nu=1}^{N} E_\nu \nabla u_\nu \cdot C_\nu^i E_\nu \nabla u_\nu + f_i(x) =: \\
=: {} & H_{0i}(x,u,\nabla u_i) - \nabla u_i \cdot L(x,u,\nabla u) - u_i F_i(x,u,\nabla u)
\end{aligned}
$$

where

$$
E_i := -(B_i - u_i C_i)^{-1} A_i^T \, .
$$

Cyclic Games

Interesting examples for Lagrangians: **Cyclic games**

$$\mathscr{L}\nu(x,v) := \frac{1}{2}|v_\nu|^2 + \sum_{\substack{i=1\\ i\neq\nu}}^{N} \theta v_\nu v_i + p_\nu \cdot \sum_{i=1}^{N} A_i v_i$$

Theory works for $|\boldsymbol{\theta}| < \mathbf{1}$

Stackelberg games:

The $\boldsymbol{j}$**-player** knows the strategy of the players $1, \ldots, j-1$

Analysis of elliptic systems arising from problems with discount controls

Discussion of the Hamiltonian with Respect to PDE-Theory

The Hamiltonian is of the form

$$H_i(x,u,\nabla u) = H_{0i}(x,u,\nabla u_i) - \nabla u_i L(x,u,\nabla u) - \\ - u_i F_i(x,u,\nabla u) + f_i \,.$$

a priori condition

$$\lambda_0 Id \leq B_i - u_i C_i^i \leq \Lambda_0 Id$$

H_{0i}, L, F_i *are Lipschitz continuous on compact subsets of* $\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{nm}$

$|H_{0i}(x,u,\eta_i)| \leq K|\eta_i|^2 + K_0$

$0 \leq F_i(x,u,\eta) \leq K|\eta|^2 + K_0$

$|L(x,u,\eta)| \leq K|\eta| + K_0$

Theorem

Let H satisfy the structure condition

$$|H_{0i}(x,u,\eta_i)| \le K|\eta_i|^2 + K_0$$
$$0 \le F_i(x,u,\eta) \le K|\eta|^2 + K_0$$
$$|L(x,u,\eta)| \le K|\eta| + K_0$$

Let $f \in L^\infty$ and $\boldsymbol{n = 2}$
then there exists a regular solution.

Iterated Exponential Test Functions

We work with the iterated exponential functions

$$\psi_i = \tau(e^{\beta u_i} - e^{-\beta u_i}) \exp\left(c \sum_{\nu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu}) \right)$$

In this simple case, double iterated epxonentials are sufficient

Effect of the test function on the right hand side of the Hamiltonian

$$-\Delta u_i + \alpha_i u_i = H_{0i}(x,u,\nabla u_i) - \nabla u_i L(x,u,\nabla u) - \\ - u_i F_i(x,u,\nabla u) + f_i(x)$$

$\Rightarrow$

$$\sum_{\nu=1}^{N} \int \big[|\nabla u_\nu|^2 (e^{\beta u_\nu} + e^{-\beta u_\nu}) + (\alpha_\nu u_\nu - f_\nu)(e^{\beta u_\nu} - e^{-\beta u_\nu}) \big] \times$$

$$\times \exp\big(c \sum_{\mu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu})\big) \tau \, dx \leq$$

$$\leq - \sum_{\nu=1}^{N} \int \nabla u_\nu (e^{\beta u_\nu} - e^{-\beta u_\nu}) \exp\big(c \sum_{\nu=1}^{N} (e^{\beta u_\nu} + e^{-\beta u_\nu})\big) \nabla \tau \, dx \, .$$

L^∞-Estimates

Lemma:
Let $w \in C(\overline{\Omega}) \cap H^1_{\Gamma_D}(\Omega)$ be a weak solution and assume our structure conditions. Then

$$\alpha_0^{-1} \operatorname{ess\,inf}[f]_- \leq w \leq \alpha_0^{-1} \operatorname{ess\,sup}[f]_+$$

Notation:

$$[f]_- = \min\{f, 0\}$$
$$[f]_+ = \max\{f, 0\}$$

A Logarithmic Morrey Estimate for ∇u

Lemma:
Let $u \in L^\infty \cap H^1$ be a solution of the system and assume the growth and sign conditions. Let $\alpha_i \geq 0$, $i = 1, \ldots, N$, and $f \in L^q$, with some $q > 1$. Then, for every $\alpha \in (0,1)$ there is a constant K_α such that

$$\int\limits_{B_{\frac{1}{2}}(x_0)\cap\Omega} |\nabla u|^2 \Big| \ln |x - x_0| \Big|^\alpha \, dx \leq K_\alpha \,, \qquad\qquad x_0 \in \Omega$$

C^α-Estimates

Combine the uniform estimate

$$\int_{B_R} |\nabla u|^2 dx \leq K |\ln R|^{-\delta}$$

with a **global hole filling** argument
$\Rightarrow$ estimate:

$$S_{R_0} := \sup_{\substack{B_R \\ R \leq R_0}} R^{-2\alpha} \int_{B_R} |\nabla u|^2 dx$$

$$S_{R_0} \leq S_{2R_0} + K$$

$\Rightarrow$ $u \in C^\alpha$ due to Morrey's lemma for $n = 2$

General case

The standard structure conditions

$$-\sum_{i,k=1}^{n} D_i\big(a_{ik}^{\nu}(x)D_k u)\big) = H_\nu(x,u,\nabla u) \qquad \nu = 1,\dots,N$$

$$|H_1(x,u,\nabla u)| \le K|\nabla u_1|^2 + K|\nabla u||\nabla u_1| + K$$

$$|H_2(x,u,\nabla u)| \le K|\nabla u_1|^2 + K|\nabla u_2|^2 + K|\nabla u||\nabla u_2| + K$$

$$|H_3(x,u,\nabla u)| \le K\big(|\nabla u_1|^2 + |\nabla u_2| + |\nabla u_3|\big) + K|\nabla u||\nabla u_3| + K$$

$$\vdots$$

$$|H_{n-1}(x,u,\nabla u)| \le K\big(|\nabla u_1| + \ldots + |\nabla u_{n-1}|\big) + K|\nabla u||\nabla u_{n-1}| + K$$

C^α-regularity for u in arbitrary dimension is obtained via testfunctions using iterated exponentials