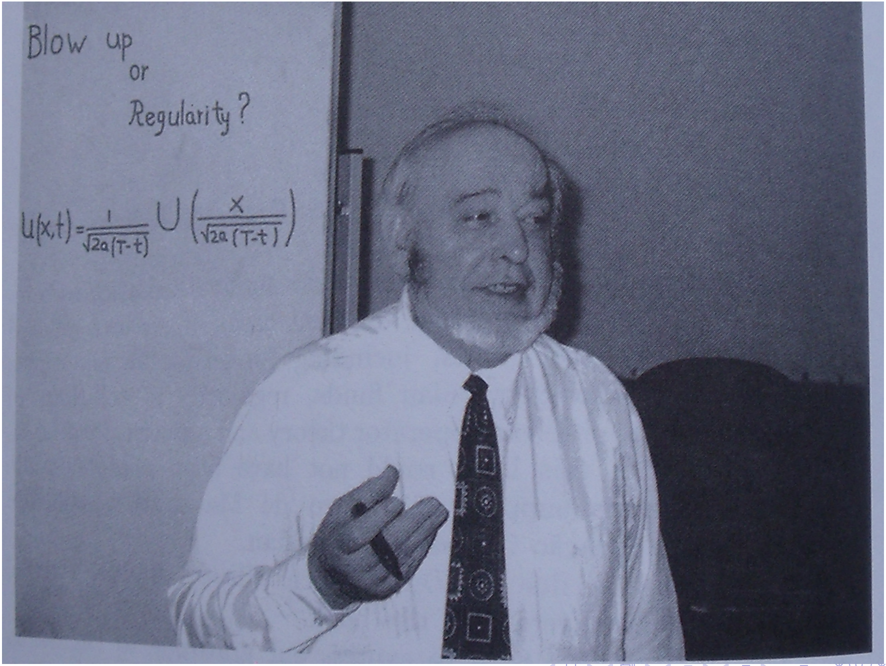
Blow up
or
Regularity?
$u(x,t) = \frac{1}{\sqrt{2a(T-t)}} U\left(\frac{x}{\sqrt{2a(T-t)}}\right)$

Steady-State Navier-Stokes Flow Past a Rotating Body: Leray Solutions are Physically Reasonable

Giovanni P. Galdi

Department of Mechanical Engineering & Materials Science
and
Department of Mathematics

University of Pittsburgh

Prague, December 14 2009

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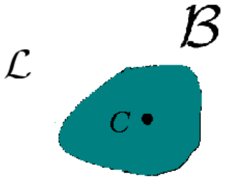

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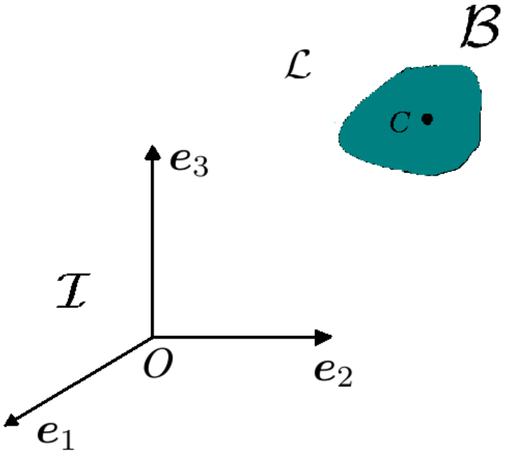

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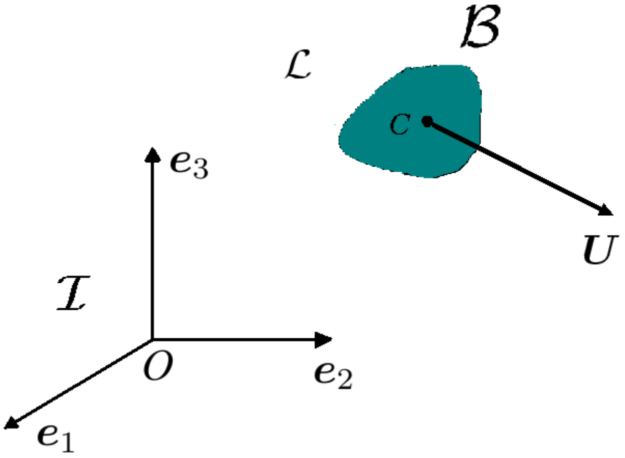

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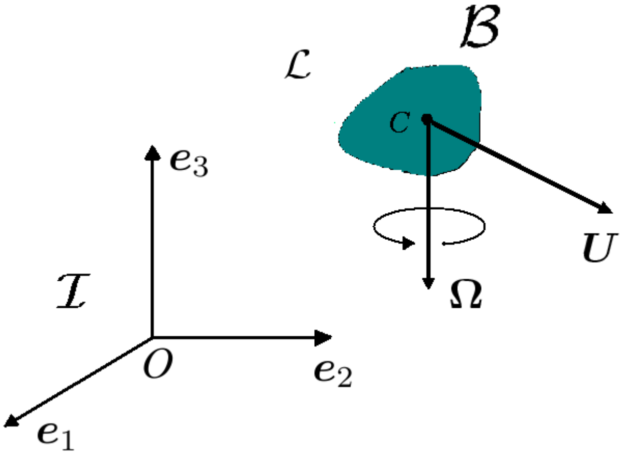

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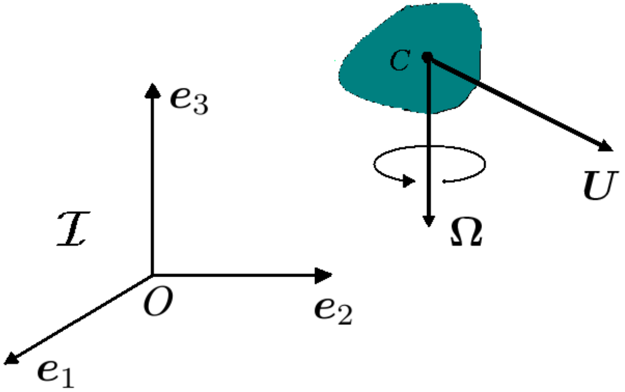

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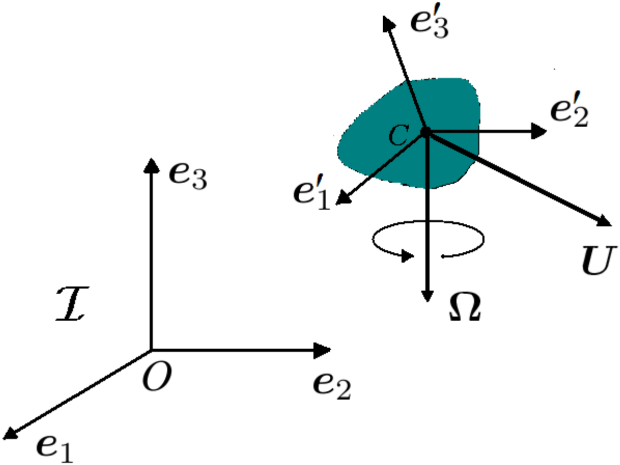

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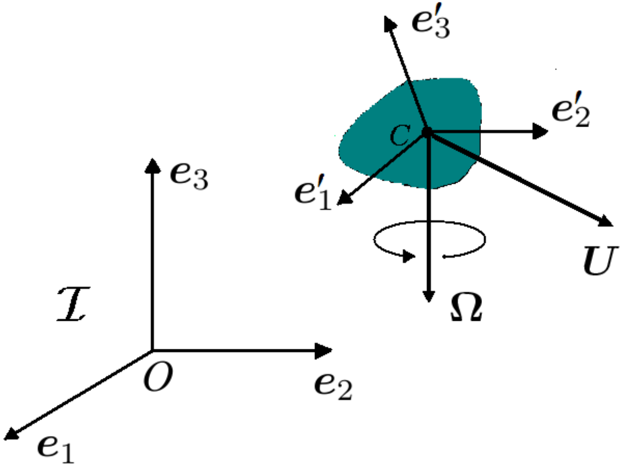

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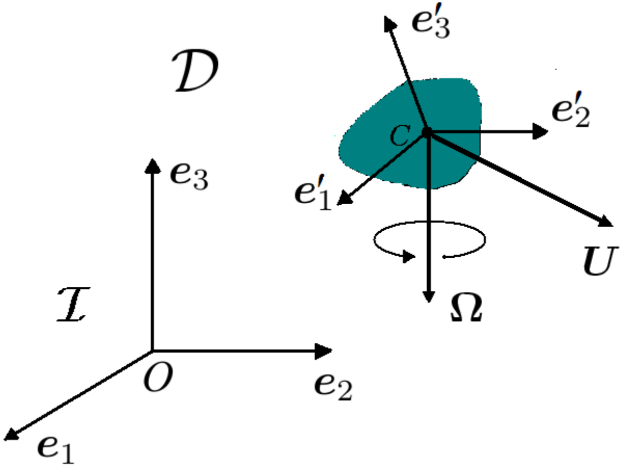

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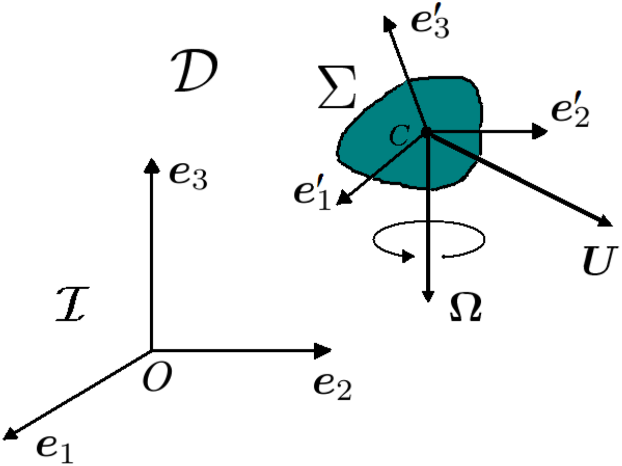

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Unbounded coefficient $|\boldsymbol{\omega} \times \boldsymbol{x}| \to \infty$ **as** $|x| \to \infty$

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All properties listed in (A), (B) and (C) are related to the **Asymptotic Spatial Structure** of the velocity field $\boldsymbol{v}$.

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Asymptotic properties:

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Both questions have been thoroughly investigated for > 50 years. Here is some basic facts:

LERAY (1933): Existence of smooth solutions $(\boldsymbol{v}, p)$ for **all** data, with $\mathrm{grad}\,\boldsymbol{v} \in L^2(\mathcal{D})$.
Asymptotic properties:

$$\int_\Omega |\mathrm{grad}\,\boldsymbol{v}|^2 < \infty \Longrightarrow \int_\Omega |\boldsymbol{v}|^6 < \infty \text{ (by Sobolev inequality)}\,.$$

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FINN (1965): Existence of PR solutions for data of **restricted** size

BABENKO (1972), GPG (1992): **Every Leray solution is PR**

Rotation and Translation: Existence of Solutions

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Unbounded Coefficient! $|\boldsymbol{e}_1 \times \boldsymbol{x}| \to \infty$ as $|x| \to \infty$

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Objective of this talk is to prove (or to give a flavor of the proof) that both Questions 1 and 2 are **positively answered.**

In other words, **for data of arbitrary size, there is always a corresponding, smooth PR solution.**

Rotation and Translation: Leray Solutions

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The "rotational term" satisfies the fundamental property:

$$\int_{\mathcal{D}} (\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \operatorname{grad} \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) \cdot \boldsymbol{u} = 0\,, \quad \text{for all } \boldsymbol{u} \in C_0^\infty(\mathcal{D}),\ \operatorname{div} \boldsymbol{u} = 0\,.$$

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Thanks to this property, the above problem (1) admits the formal *a priori* **global** estimate

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By coupling this inequality with,*e.g.*, Galerkin's method, one can show the following result.

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Theorem 1 (Weinberger, 1982; Serre, 1987; Borchers, 1992)

Let $\mathcal{D}$ be an exterior domain in $\mathbb{R}^3$. For **any** $\mathrm{Re} > 0$ *and* $\mathrm{Ta} \ge 0$, *there exists at least one* $(\boldsymbol{v}, p) \in C^\infty(\mathcal{D}) \times C^\infty(\mathcal{D})$ (Leray solution) *to problem* (1) .

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- Use a perturbation argument around the solution to the **linear problem** in the domain $\mathcal{D}$.
- By a "localization procedure", reduce this latter to the study of the asymptotic properties and corresponding estimates of solutions $(\boldsymbol{u}, \mathsf{p})$ to the linear problem in the whole space:

$$\left.\begin{aligned} &\Delta \boldsymbol{u} + \mathrm{Re}\frac{\partial \boldsymbol{u}}{\partial x_1} + \mathrm{Ta}\,(\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \operatorname{grad} \boldsymbol{u} - \boldsymbol{e}_1 \times \boldsymbol{u}) = \operatorname{grad} \mathsf{p} + \boldsymbol{f} \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \text{in } \mathbb{R}^3$$

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- Obtain the asymptotic properties of $\boldsymbol{u}$ and the corresponding estimates by means of its representation through the fundamental tensor solution $\mathfrak{G}$:

$$\boldsymbol{u}(x) = \int_{\mathbb{R}^3} \mathfrak{G}(x, y) \cdot \boldsymbol{f}(y) dy \,.$$

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- Unlike the case $\mathrm{Ta} = 0$ (no rotation), the fundamental tensor $\mathfrak{G}$ does **not** satisfy the uniform estimate (that would be the starting point to establish asymptotic properties):

$$|\mathfrak{G}(x,y)| \le \frac{C}{|x-y|}\,, \quad \text{for all } x,y \in \mathbb{R}^3$$

for some C independent of x, y (Farwig, Hishida & Müller, 2004).

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Therefore, one would like to argue in a different way.

Rotation and Translation: PR Solutions

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Let

$$\boldsymbol{Q}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathrm{Ta}\, t) & -\sin(\mathrm{Ta}\, t) \\ 0 & \sin(\mathrm{Ta}\, t) & \cos(\mathrm{Ta}\, t) \end{bmatrix}, \quad t \geq 0 \ \text{(rotation matrix around } \boldsymbol{e}_1\text{)}$$

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Define

$$\boldsymbol{w}(\boldsymbol{y}, t) := \boldsymbol{Q}(t) \cdot \boldsymbol{u}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})\,, \quad \pi(\boldsymbol{y}, t) := \mathsf{p}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})\,,$$

$$\boldsymbol{F}(\boldsymbol{y}, t) := \boldsymbol{Q}(t) \cdot \boldsymbol{f}(\boldsymbol{Q}^\top(t) \cdot \boldsymbol{y})\,.$$

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In these new variables, the original problem

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transforms into the following Oseen **initial-value problem**

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \mathrm{Re}\frac{\partial \boldsymbol{w}}{\partial y_1} - \mathrm{grad}\,\pi - \boldsymbol{F}\\ &\mathrm{div}\,\boldsymbol{w} = 0\end{aligned}\right\}\ \text{ in } \mathbb{R}^3 \times (0, \infty)$$

$$\boldsymbol{w}(\boldsymbol{y}, 0) = \boldsymbol{u}(y)$$

Rotation and Translation: PR Solutions

If $\boldsymbol{u}$ and $\boldsymbol{f}$ have a mild degree of regularity as $|x| \to \infty$,
e.g. $\boldsymbol{u} \in L^{q_1}(\mathbb{R}^3)$, $\boldsymbol{f} \in L^{q_2}(\mathbb{R}^3)$, for some $q_i \in [1, \infty]$, $i = 1, 2$,
then $\boldsymbol{w}(y, t)$ can be represented as follows:

$$\boldsymbol{w}(y,t) = \int_{\mathbb{R}^3} \boldsymbol{\Gamma}(x-y,t)\cdot\boldsymbol{u}(y)dy + \int_{\mathbb{R}^3}\int_0^t \boldsymbol{\Gamma}(x-y,t-\tau)\cdot\boldsymbol{F}(y,\tau)\,d\tau\,dy\,, \quad t > 0\,.$$

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where $\boldsymbol{\Gamma} = \{\boldsymbol{\Gamma}_{ij}(x, t; \mathrm{Re})\}$ is the fundamental tensor solution to the **time-dependent** Oseen problem:

$$\frac{\partial \Gamma_{ij}}{\partial \tau} = \mathrm{Re}\,\frac{\partial \Gamma_{ij}}{\partial x_1} + \Delta\Gamma_{ij} - \frac{\partial \gamma_i}{\partial x_j} + \delta_{ij}\delta(x)\delta(t)$$

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$$\int_0^\infty |\mathbf{\Gamma}(\xi,t)|dt \le \frac{2}{|\xi|(1+2\mathrm{Re}\, s(\xi))}$$

$$\int_0^\infty |\mathrm{grad}\, \mathbf{\Gamma}(\xi,t)|dt \le C_\beta \begin{cases} \mathrm{Re}^{\frac{1}{2}} |\xi|^{-3/2}(1+2\mathrm{Re}\, s(\xi))^{-3/2}, & \text{if } |\xi| \ge \beta/\mathrm{Re}, \\ |\xi|^{-2}, & \text{if } |\xi| \in (0, \beta/\mathrm{Re}) \,, \end{cases} \tag{1}$$

where $s(x) = |\boldsymbol{\xi}| + \boldsymbol{\xi} \cdot \boldsymbol{e}_1$. Moreover:

$$\left| \int_{\mathbb{R}^3} \mathbf{\Gamma}(y-z,t) \cdot \boldsymbol{u}(z) dz \right| \le C\, t^{-3/(2q_1)} \|\boldsymbol{u}\|_{q_1} \,.$$

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Remark

The estimates (2) **coincide** *with those well known for the* **time-independent** *Oseen fundamental tensor* $\mathbf{\Gamma}_0 = \{\Gamma_{0ij}(x;\mathrm{Re})\}$*:*

$$\Delta\Gamma_{0ij} + \mathrm{Re}\,\frac{\partial\Gamma_{0ij}}{\partial x_1} = \frac{\partial\gamma_{0i}}{\partial x_j} + \delta_{ij}\delta(x)\,, \qquad \frac{\partial\Gamma_{0ij}}{\partial y_i} = 0\,.$$

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Rotation and Translation: PR Solutions

Likewise,

$$\begin{aligned}|\operatorname{grad} \boldsymbol{u}(x)| \leq \quad & C_1 \int_{|x-z| \leq \frac{\beta}{\mathrm{Re}}} \frac{G(z)}{|x-z|^2} d\, y \\ & + C_2 \sqrt{\mathrm{Re}} \int_{|x-z| \geq \frac{\beta}{\mathrm{Re}}} \frac{G(z)}{|x-z|^{3/2} (1 + 2\mathrm{Re}\, s(x-z))^{3/2})} d\, y\end{aligned}$$

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has one and only one solution such that

$$\begin{aligned} &\sup_{x\in\mathbb{R}^3} \left[|\boldsymbol{u}(x)|(1+|x|)(1+\mathrm{Re}\, s(x))\right] < \infty \\ &\sup_{x\in\mathbb{R}^3} \left[|\mathrm{grad}\, \boldsymbol{u}(x)|(1+|x|)^{3/2}(1+\mathrm{Re}\, s(x))^{3/2}\right] < \infty\,, \end{aligned}$$

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and satisfying corresponding estimates. In particular, the solution is PR.

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has a unique (smooth) PR solution, with the following asymptotic properties:

$$\sup_{x \in \mathbb{R}^3} \left[|\boldsymbol{v}(x)|(1 + |x|)(1 + \mathrm{Re}\,s(x))\right] < \infty$$

$$\sup_{x \in \mathbb{R}^3} \left[|\mathrm{grad}\,\boldsymbol{v}(x)|(1 + |x|)^{3/2}(1 + \mathrm{Re}\,s(x))^{3/2}\right] < \infty\,.$$

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Then, if for some $R > 0$,

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for all sufficiently large $|x|$ *we have*

$$|\boldsymbol{v}(x)| \leq \mathcal{V}_1(x) + \mathcal{V}_2(x)$$

where

$$\mathcal{V}_1(x) = O([(1+|x|)(1+\mathrm{Re}\, s(x))]^{-1})\,, \quad \mathcal{V}_2(x) = O(|x|^{-3/2+\delta}), \ \text{arbitrary } \delta > 0\,.$$

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and the regularity of $\boldsymbol{v}$ *it follows*

$$\boldsymbol{v} \in L^q(\mathcal{D}), \quad \text{for all } q > 2.$$

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It can be shown that

$$\boldsymbol{v} \in L^2(\mathcal{D}) \implies \boldsymbol{v}(x) \equiv \mathbf{0}.$$

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For a fixed and sufficiently large $\rho > 0$, take a smooth "cut-off" function $\psi_\rho = \psi_\rho(x)$ that is 0 if $|x| < R$ and is 1 if $|x| > 2\rho$, and set

$$\boldsymbol{u} := \psi_\rho \boldsymbol{v} - \boldsymbol{z}\,, \quad \operatorname{div} \boldsymbol{z} = \boldsymbol{v} \cdot \operatorname{grad} \psi_\rho\,, \quad \mathsf{p} := \psi_\rho\, p$$

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Then, the original problem for $(\boldsymbol{v}, p)$ goes into the following one:

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where $\boldsymbol{f}_c \in C_0^\infty(\mathbb{R}^3)$.

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Step 4: Estimates of the Functions $\boldsymbol{w}_i$, $i = 1, 2, 3$.

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We easily show that

$$|\boldsymbol{w}_1(y,t)| = \left| \int_{\mathbb{R}^3} \boldsymbol{\Gamma}(y-z,t) \cdot \boldsymbol{u}(z)\, dz \right| \leq C\, t^{-1/4} \|\boldsymbol{v}\|_6$$

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Moreover, using the estimate

$$\int_0^\infty |\boldsymbol{\Gamma}(\xi,t)| dt \leq \frac{2}{|\xi|(1+2\,s(\xi))}\,,$$

and the fact that $\boldsymbol{F}_c \in L^\infty(0,\infty; C_0^\infty(\mathbb{R}^3))$, we also (easily) show

$$|\boldsymbol{w}_2(y,t)| = \left| \int_{\mathbb{R}^3} \int_0^t \boldsymbol{\Gamma}(y-z,\tau) \cdot \boldsymbol{F}_c(z,t-\tau)\, d\tau\, dz \right| \leq \frac{\|\boldsymbol{f}_c\|_r}{(1+|y|)(1+s(y))}\,.$$

for some $r > 3$ and all $|y| \geq 2R$.

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Finally, using the estimate

$$\int_0^\infty |\mathrm{grad}\,\boldsymbol{\Gamma}(\xi, t)|dt \le C_\beta \begin{cases} |\xi|^{-3/2}(1 + 2s(\xi))^{-3/2}, & \text{if } |\xi| \ge \beta, \\ |\xi|^{-2}, & \text{if } |\xi| \in (0, \beta)\,, \end{cases}$$

Leray Solutions are PR Solutions: Sketch of Proof

Step 4: Estimates of the Functions $\boldsymbol{w}_i$, $i = 1, 2, 3$ (cont'd)

Finally, using the estimate

$$\int_0^\infty |\mathrm{grad}\, \boldsymbol{\Gamma}(\xi, t)| dt \leq C_\beta \begin{cases} |\xi|^{-3/2}(1 + 2s(\xi))^{-3/2}, & \text{if } |\xi| \geq \beta, \\ |\xi|^{-2}, & \text{if } |\xi| \in (0, \beta)\,, \end{cases}$$

we show

$$\begin{aligned} |\boldsymbol{w}_3(y, t)| &= \left| \int_{\mathbb{R}^3} \int_0^t \mathrm{grad}\, \boldsymbol{\Gamma}(y - z, \tau) : [\boldsymbol{V} \otimes \boldsymbol{V}](z, t - \tau) d\tau\, dz \right| \\ &\leq C_\theta \left(\int_{|z| \geq R} |\mathrm{grad}\, \boldsymbol{v}|^2 \right)^{1-\theta}, \end{aligned}$$

for all $\theta > 0$, all $|y| \geq 2R$, and arbitrary $R > \rho/2$.

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Collecting the previous inequalities and transforming back to the original variable $\boldsymbol{v}$ and x, one concludes

$$|\boldsymbol{v}(x)| \le C_\theta \left[t^{-1/4} \|\boldsymbol{v}\|_6 + \frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \left(\int_{|y|\ge R} |\operatorname{grad} \boldsymbol{v}|^2 \right)^{1-\theta} \right] .$$

for all $\theta > 0$ and all $|x| \ge 2R$.

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for all $\theta > 0$ and all $|x| \geq 2R$.

So, in the limit $t \to \infty$,

$$|\boldsymbol{v}(x)| \leq C_\theta \left[\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \left(\int_{|y|\geq R} |\operatorname{grad} \boldsymbol{v}|^2 \right)^{1-\theta} \right].$$

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Key Lemma

For all $\varepsilon > 0$*, there is* $C = C(\boldsymbol{v}, \varepsilon) > 0$ *such that*

$$\int_{|z|\ge R} |\mathrm{grad}\, \boldsymbol{v}|^2 \le C\, R^{-1+\varepsilon}$$

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A crucial step in the proof of this lemma is to show the following:

$$\boldsymbol{v} \in L^{s_1}(\mathcal{D}^R)\,, \quad \mathrm{grad}\, \boldsymbol{v} \in L^{s_2}(\mathcal{D}^R)\,, \quad D^2\boldsymbol{v} \in L^{s_3}(\mathcal{D}^R)\,,$$

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where $\mathcal{D}^R := \mathcal{D} \cap \{|y| \geq R\}$.

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$$\text{all } s_1 > 2,\ s_2 > 4/3,\ s_3 > 1\,,$$

where $\mathcal{D}^R := \mathcal{D} \cap \{|y| \geq R\}$. Notice that, at the outset, we **only** know

$$\boldsymbol{v} \in L^6(\mathcal{D}^R)\,, \quad \text{grad}\, \boldsymbol{v} \in L^2(\mathcal{D}^R)\,.$$

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Step 6: A First Pointwise Estimate for $\boldsymbol{v}(x)$.

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Replace the estimate

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into the inequality

$$|\boldsymbol{v}(x)| \leq C_\theta \left[\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \left(\int_{|y|\geq R} |\mathrm{grad}\, \boldsymbol{v}|^2 \right)^{1-\theta} \right].$$

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and choose $R = |x|$. We thus find

$$|\boldsymbol{v}(x)| \leq C_\eta \left(\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + \frac{1}{|x|^{1-\eta}} \right), \quad \text{for all } \eta > 0\,.$$

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Step 7: Boot-strap Argument: End of Proof.

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Since

$$|\boldsymbol{v}(x)| \le C\,\frac{\|\boldsymbol{f}_c\|_r}{(1+|x|)(1+s(x))} + |\boldsymbol{w}_3(y,t)|\,, \quad |y| = |x|$$

this ends the proof of the theorem .

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$$|\boldsymbol{v}(x)| \leq \mathcal{V}_1(x) + \mathcal{V}_2(x)$$

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arbitrary $\delta > 0$.

- This estimate is sharp, in the sense that

$$\boldsymbol{v} \in L^2(\mathcal{D}) \implies \boldsymbol{v}(x) \equiv \mathbf{0}\,.$$

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  arbitrary $\delta > 0$, but no proof is available (yet).