Numerical Modelling of Hyperbolic Balance Laws with Application in Geophysics

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Nonlinear PDE's **Praha, 13-15/12/2009**

Overview

1. Shallow water eqs. with source terms:
   - bottom topography
   - Coriolis forces
   - friction effects
   - multi-layer SWE

2. **Well-balanced FVEG schemes** and theory of bicharacteristics and evolution operators

3. Theoretical analysis of the well-balanced properties

4. Numerical experiments

Figure 1: Floods in summer 2000 near Kaiserslautern, photo by G.Kries [Hilden]

Figure 2: Oceanographic flows

Shallow water equations with source terms

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}_1(\boldsymbol{U}) + \partial_y \boldsymbol{F}_2(\boldsymbol{U}) = \boldsymbol{T} \tag{1}$$

$$\boldsymbol{U} := \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \boldsymbol{F}_1 := \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{pmatrix} \boldsymbol{T} = \begin{pmatrix} 0 \\ -gh\partial_x b - fhv \\ -gh\partial_y b + fhu \end{pmatrix}$$

$c = \sqrt{gh}$ ... wave celerity, g ... gravitational const.,
f ... Coriolis parameter, $b(x, y)$ ... bottom elevation,

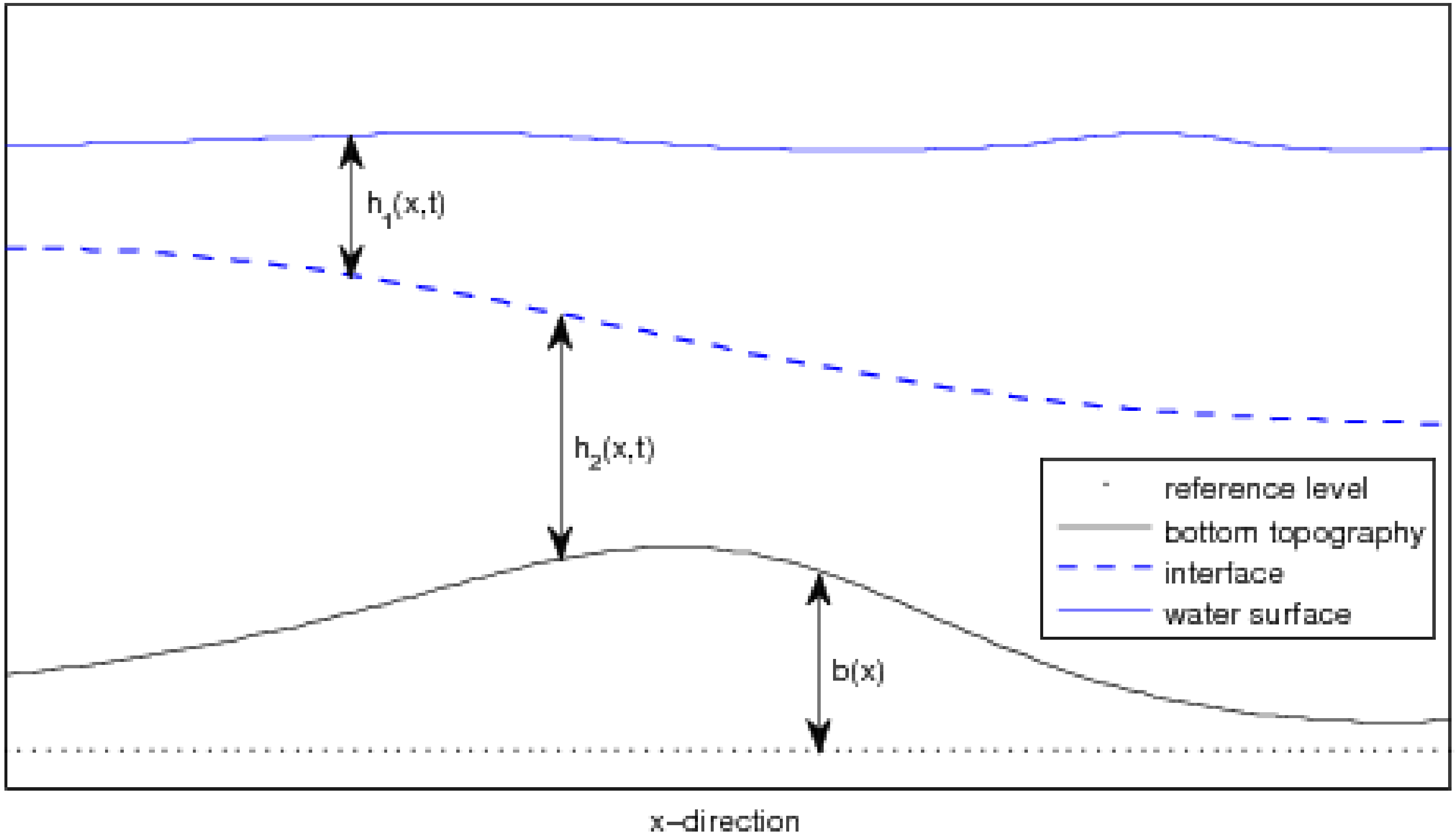
Two-layer shallow water equations with source terms

$$\begin{aligned}
\partial_t h_1 + \partial_x(h_1 u_1) + \partial_y(h_1 v_1) &= 0 \\
\partial_t(h_1 u_1) + \partial_x\left(h_1 u_1^2 + \tfrac{g}{2}h_1^2\right) + \partial_y(h_1 u_1 v_1) &= -g h_1 \partial_x(b + h_2) - f h_1 v_1 \\
\partial_t(h_1 v_1) + \partial_x(h_1 u_1 v_1) + \partial_y\left(h_1 v_1^2 + \tfrac{g}{2}h_1^2\right) &= -g h_1 \partial_y(b + h_2) + f h_1 u_1
\end{aligned}$$

$$\begin{aligned}
\partial_t h_2 + \partial_x(h_2 u_2) + \partial_y(h_2 v_2) &= 0 \\
\partial_t(h_2 u_2) + \partial_x\left(h_2 u_2^2 + \tfrac{g}{2}h_2^2\right) + \partial_y(h_2 u_2 v_2) &= -g h_2 \partial_x(b + r h_1) - f h_2 v_2 \\
\partial_t(h_2 v_2) + \partial_x(h_2 u_2 v_2) + \partial_y\left(h_2 v_2^2 + \tfrac{g}{2}h_2^2\right) &= -g h_2 \partial_y(b + r h_1) + f h_2 u_2
\end{aligned}$$

h_1, u_1, v_1 . . . first layer, h_2, u_2, v_2 . . . second layer

$r = \frac{\rho_1}{\rho_2} < 1$


h_1(x,t)
h_2(x,t)
b(x)
reference level
bottom topography
interface
water surface
x–direction

SWE (one- or two- layer) can be written in quasi-linear form

$$\boldsymbol{U}_t + \widetilde{\boldsymbol{A}}_1(\boldsymbol{U})\boldsymbol{U}_x + \widetilde{\boldsymbol{A}}_2(\boldsymbol{U})\boldsymbol{U}_y = 0,$$

$$\boldsymbol{U} = (h, hu, hv, b, x, y)^T \text{ or } \boldsymbol{U} = (h_1, h_1u_1, h_1v_1, h_2, h_2u_2, h_2v_2, b, x, y)^T$$

$$\widetilde{\boldsymbol{A}}_1 = \begin{pmatrix} \boldsymbol{A}_1(\boldsymbol{u}_1) & \boldsymbol{C}_1(\boldsymbol{u}_1) & -\boldsymbol{S}_1(\boldsymbol{u}_1) & -\boldsymbol{S}_3(\boldsymbol{u}_1) & 0 \\ \widetilde{\boldsymbol{C}}_1(\boldsymbol{u}_2) & \boldsymbol{A}_1(\boldsymbol{u}_2) & -\boldsymbol{S}_1(\boldsymbol{u}_2) & -\boldsymbol{S}_3(\boldsymbol{u}_2) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{S}_1(\boldsymbol{u}_i) = \begin{pmatrix} 0 \\ -gh_i \\ 0 \end{pmatrix}, \boldsymbol{S}_2(\boldsymbol{u}_i) = \begin{pmatrix} 0 \\ 0 \\ -gh_i \end{pmatrix}$$

$$\boldsymbol{S}_3(\boldsymbol{u}_i) = \begin{pmatrix} 0 \\ fh_iv_i \\ 0 \end{pmatrix}, \boldsymbol{S}_4(\boldsymbol{u}_i) = \begin{pmatrix} 0 \\ 0 \\ -fh_iu_i \end{pmatrix}, \quad i = 1, 2$$

$$\boldsymbol{C}_1(\boldsymbol{u}_1) = \begin{pmatrix} 0 & 0 & 0 \\ c_1^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\boldsymbol{C}}_2(\boldsymbol{u}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ rc_2^2 & 0 & 0 \end{pmatrix}$$

The one-layer system is weakly hyperbolic

-eigenstructure is readily available

- eigenvalues of matrix pencil $\underline{\underline{A}}_1 \cos\theta + \underline{\underline{A}}_2 \sin\theta, \quad \theta \in [0, 2\pi]$

$$\begin{aligned}
\lambda_1 &= u\cos\theta + v\sin\theta - c \\
\lambda_2 &= u\cos\theta + v\sin\theta \\
\lambda_3 &= u\cos\theta + v\sin\theta + c \\
\lambda_{4,5,6} &= 0
\end{aligned}$$

This multi-layer system is:

- non-strictly hyperbolic !
- non-conservative !
- eigenstructure is not readily available !
- lost of hyperbolicity !
- has physically interesting equilibria that should be preserved !

- **equilibria**

1. rest state (lake at rest):

$$h_1 + h_2 + b = \text{const.},\; rh_1 + h_2 + b = \text{const.},\; u_1 = v_1 = 0 = u_2 = v_2$$

2. geostrophic equilibrium (jet in the rotational frame):

$$g\partial_x(h_1 + h_2 + b) = -fv_1 \,\&\, u_1 = 0$$

$$g\partial_x(rh_1 + h_2 + b) = -fv_2 \,\&\, u_2 = 0$$

• non-conservative terms

- weak solutions: non-conservative terms are interpreted as Boler measures (Dal Maso et al.):

$U \in L^\infty$ is a weak solution to

$$U_t + A(U)U_x = 0$$

iff the measure $[A(U)U_x]_\Psi$ satisfies

$$U_t + [A(U)U_x]_\Psi = 0.$$

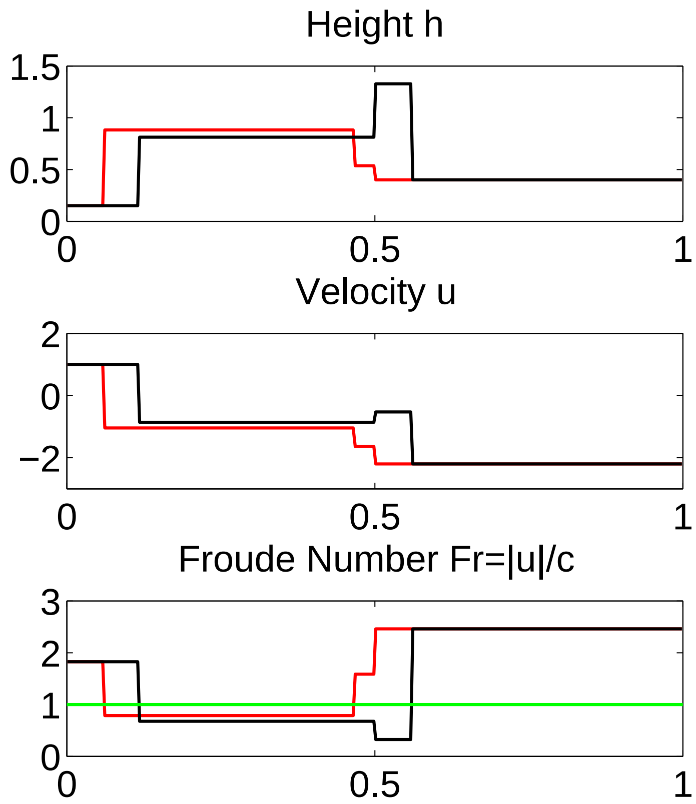
Here Ψ is a suitable Lipschitz-cont. path in a phase space, $[A(U)U_x]_\Psi$ Borel measure

$$[A(U)U_x]_\Psi(M) := \int_M A(U)U_x + \sum_{x_0} \int_0^1 A(U(\Psi(\sigma))\Psi'(\sigma)\mathrm{d}\sigma$$

Question: How does the solution depends on Ψ ?

- one-layer 1-D SWE with initial conditions [N. Andrianov]:

$$(h,u,b) := \begin{cases} (0.222, -1, 2) & x \leq 0.5 \\ (0.7246, -1.6359, 0.1) & x > 0.5 \end{cases}$$

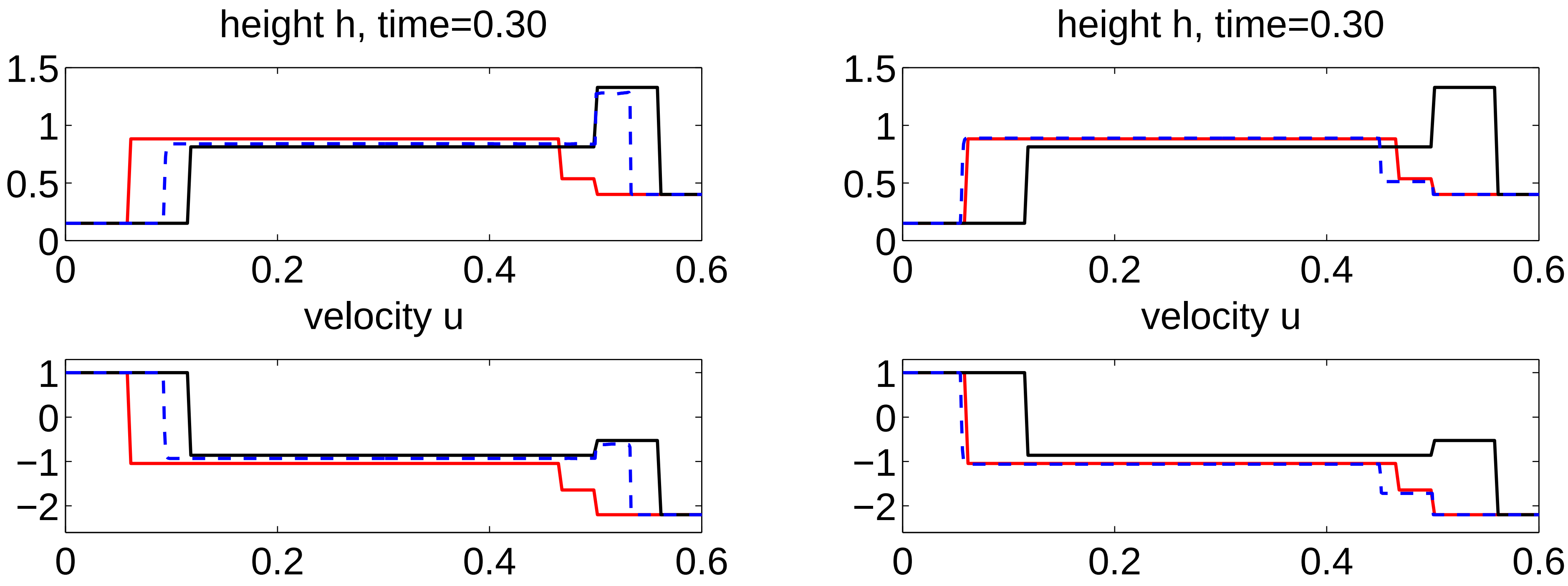


Path 1

$$\Psi_1 := \begin{cases} \begin{pmatrix} h_l \\ b_l + 2s(b_r - b_l) \end{pmatrix} & s \in [0, 0.5] \\ \begin{pmatrix} h_l + 2(s-0.5)(h_r - h_l) \\ b_r \end{pmatrix} & s \in [0.5, 1] \end{cases}$$

Path 2

$$\Psi_2 := \begin{cases} \begin{pmatrix} h_l + 2s(h_r - h_l) \\ b_l \end{pmatrix} & s \in [0, 0.5] \\ \begin{pmatrix} h_r \\ b_l + 2(s-0.5)(b_r - b_l) \end{pmatrix} & s \in [0.5, 1] \end{cases}$$



Path-consistent FV Roe method left: Ψ_1 path; right Ψ_2, 1000 cells, CFL=0.5

Open problems:

- The solution depends on a path Ψ !
- Which path to choose ?
- Additional condition to specify solution?

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- The solution depends on a path Ψ !
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Finite Volume Evolution Galerkin Schemes

- predictor - corrector method
- corrector: path-consistent FV update [Castro, Parés, et al.]
- predictor: (approximate) evolution along (bi-) characteristics
- path Ψ is dictated by the evolution operator
- $\rightarrow$ **well-balanced path-consistent FVEG method**

Path-Consistent Finite Volume Evolution Galerkin Method

- corrector step

$$\boldsymbol{U}_{k\ell}^{n+1} = \boldsymbol{U}_{k\ell}^{n} \qquad - \frac{\Delta t}{\Delta x \Delta y} \sum_{\ell' \in L} \alpha_{\ell'} \left(\boldsymbol{D}_{k+1/2,\ell'}^{i,-} + \boldsymbol{D}_{k-1/2,\ell'}^{i,+} \right)$$
$$- \frac{\Delta t}{\Delta x \Delta y} \sum_{k' \in K} \beta_{k'} \left(\boldsymbol{D}_{k',\ell+1/2}^{i,-} + \boldsymbol{D}_{k',\ell-1/2}^{i,+} \right)$$

$$\boldsymbol{D}_{k+1/2,\ell'}^{i,-} := \int_0^1 \widetilde{\boldsymbol{A}}_1^{i}(\Psi(s; \boldsymbol{W}_{k,\ell'}^{*}, \boldsymbol{W}_{k+1/2,\ell'}^{*})) \frac{\partial \Psi}{\partial s}(s; \boldsymbol{W}_{k,\ell'}^{*}, \boldsymbol{W}_{k+1/2,\ell'}^{*})\, ds$$

- predictor step

$$\boldsymbol{W}^{*} = E_{\Delta t/2} \boldsymbol{W}^{n} \qquad \boldsymbol{W} \ldots \text{ primitive variables}$$

• Theory of bicharateristics / Evolution operator

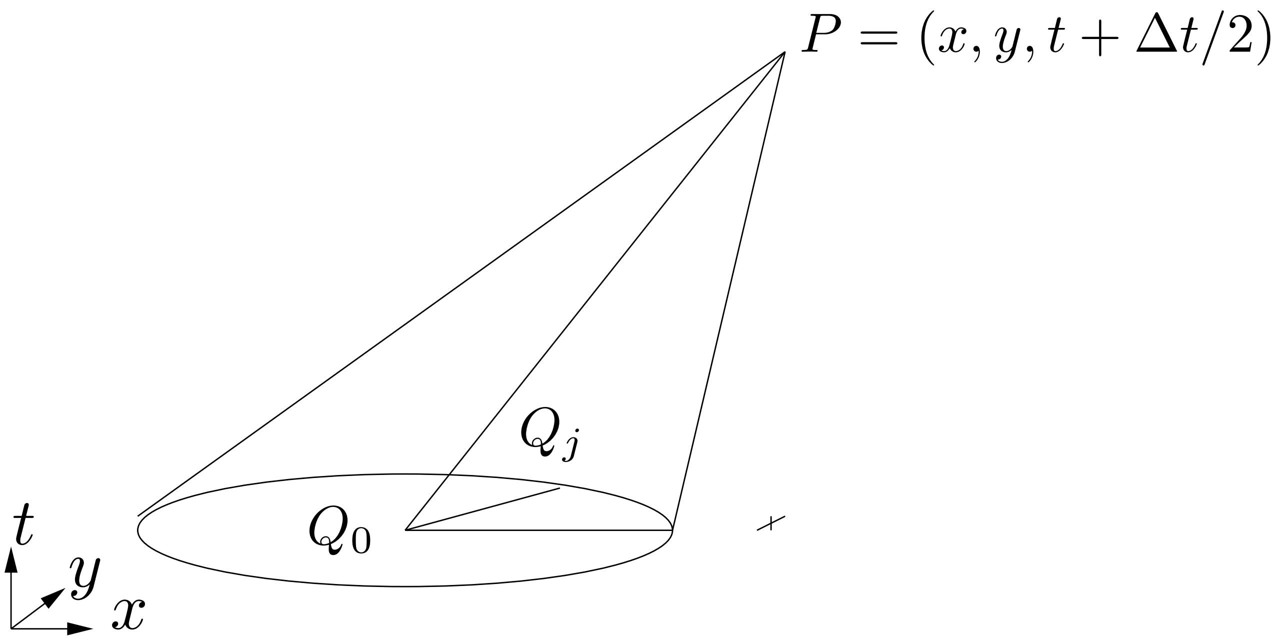


Figure 3: Bicharacteristics along the Mach cone

M. Lukacova, K.W. Morton, G. Warnecke, SIAM J.Sci.Comp. 2004

M. Lukacova, K.W. Morton: Overview paper, 2009

Strategy to derive exact / approximate evolution operator:

- Take an integration point $P = (x, y, t_n + \Delta t/2)$ on cell interfaces, linearize at (x, y). Apply the characteristic decomposition for each direction $\boldsymbol{n} = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$.

- Integrate along bicharateristics from t_n up to $t_n + \Delta t/2$.

- Transform back to the physical variables. Integrate the resulting system along $\theta \in [0, 2\pi)$.

- Approximate time integrals, keep equilibrium conditions

• 1-layer SWE

well-balanced approx. evol. operator for *p.w. constant* data $\boldsymbol{W}^n$ ... E_Δ^{const}

$$
\begin{aligned}
h(P) &= -b(P) + \frac{1}{2\pi}\int_0^{2\pi} (h(Q) + b(Q)) \\
&\qquad - \frac{\tilde{c}}{g} u(Q)\,\mathrm{sgn}(\cos\theta) - \frac{\tilde{c}}{g} v(Q)\,\mathrm{sgn}(\sin\theta)\,\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
u(P) &= -\frac{g}{\tilde{c}}\frac{1}{2\pi}\int_0^{2\pi} (h(Q) + b(Q) - V(Q))\,\mathrm{sgn}(\cos\theta) \\
&\qquad + u(Q)\left(\frac{1}{2} + \cos^2\theta\right) + v(Q)\ \sin\theta\ \cos\theta\mathrm{d}\theta
\end{aligned}
$$

$V(x) := \int_{x_0}^{x} fv$

M. Lukacova, S. Noelle, M. Kraft, J. Comp. Phys., 2007

$\longrightarrow$ well-balanced approximate evolution operator E_{Δ}^{const}

$$\boldsymbol{W}^{*}(\boldsymbol{P}) = \boldsymbol{W}(\cdot, t_{n+1/2}) = E_{\Delta}^{const}\boldsymbol{W}(\cdot, t_n)$$

- higher order methods: $\longrightarrow$ approximate evolution operator for continuous p.w. bilinear data $R_h^C \boldsymbol{U}^n$ . . . E_{Δ}^{bilin} ... analogous

... corrector step: FV - update keeps balance between fluxes and sources at the discrete level

• AIM: apply for 2-layer SWE

BUT: eigenstructure is not readily available ! $\Longrightarrow$ SPLITTING

operator T^1

$$\partial_t \boldsymbol{u}_1 + \boldsymbol{A}_1(\boldsymbol{u}_1)\partial_x \boldsymbol{u}_1 + \boldsymbol{A}_2(\boldsymbol{u}_1)\partial_y \boldsymbol{u}_1 = -\boldsymbol{C}_1(\boldsymbol{u}_1)\partial_x \boldsymbol{u}_2 - \boldsymbol{C}_2(\boldsymbol{u}_1)\partial_y \boldsymbol{u}_2$$
$$+\boldsymbol{S}_1(\boldsymbol{u}_1)\partial_x b + \boldsymbol{S}_2(\boldsymbol{u}_1)\partial_y b + \boldsymbol{S}_3(\boldsymbol{u}_1) + \boldsymbol{S}_4(\boldsymbol{u}_1)$$

operator T^2

$$\partial_t \boldsymbol{u}_2 + \boldsymbol{A}_1(\boldsymbol{u}_2)\partial_x \boldsymbol{u}_2 + \boldsymbol{A}_2(\boldsymbol{u}_2)\partial_y \boldsymbol{u}_2 = -\widetilde{\boldsymbol{C}}_1(\boldsymbol{u}_2)\partial_x \boldsymbol{u}_1 - \widetilde{\boldsymbol{C}}_2(\boldsymbol{u}_2)\partial_y \boldsymbol{u}_1$$
$$+\boldsymbol{S}_1(\boldsymbol{u}_2)\partial_x b + \boldsymbol{S}_2(\boldsymbol{u}_2)\partial_y b + \boldsymbol{S}_3(\boldsymbol{u}_2) + \boldsymbol{S}_4(\boldsymbol{u}_2)$$

$$\boldsymbol{W}^{n+1} = T^1_{\Delta t/2} T^2_{\Delta t} T^1_{\Delta t/2} \boldsymbol{W}^n$$

Theorem (M.L., M. Dudzinski (2009))

Let $\boldsymbol{W}^n$ satisfy equilibrium conditions: lake at rest & geostrophic balance.

$\Longrightarrow$ Then $\boldsymbol{W}^{n+1}$ obtained by the path-consistent FVEG scheme satisfies the same well-balanced conditions:

lake at rest . . . preserved exactly
geostrophic equilibrium . . . up to at least third order accuracy

Numerical experiments

1. Comparison of accuracy / shallow flow (no source)

$\|\boldsymbol{U}^n_{N/2} - \boldsymbol{U}^n_N\|_{L^1}/N(\mathbf{CFL = 0.2})$	FVEG	FV	DG
40	0.004640	0.009473	0.008227
80	0.001066	0.002313	0.002039
160	0.000261	0.000595	0.000506
320	0.000065	0.000152	0.000126
EOC	**1.999**	**1.969**	**2.005**

$\|\boldsymbol{U}^n_{N/2} - \boldsymbol{U}^n_N\|_{L^1}/N(\mathbf{CFL = 0.6})$	FVEG	FV
40	0.002290	0.010843
80	0.000405	0.002851
160	0.000088	0.000746
320	0.000021	0.000192
EOC	**2.044**	**1.958**

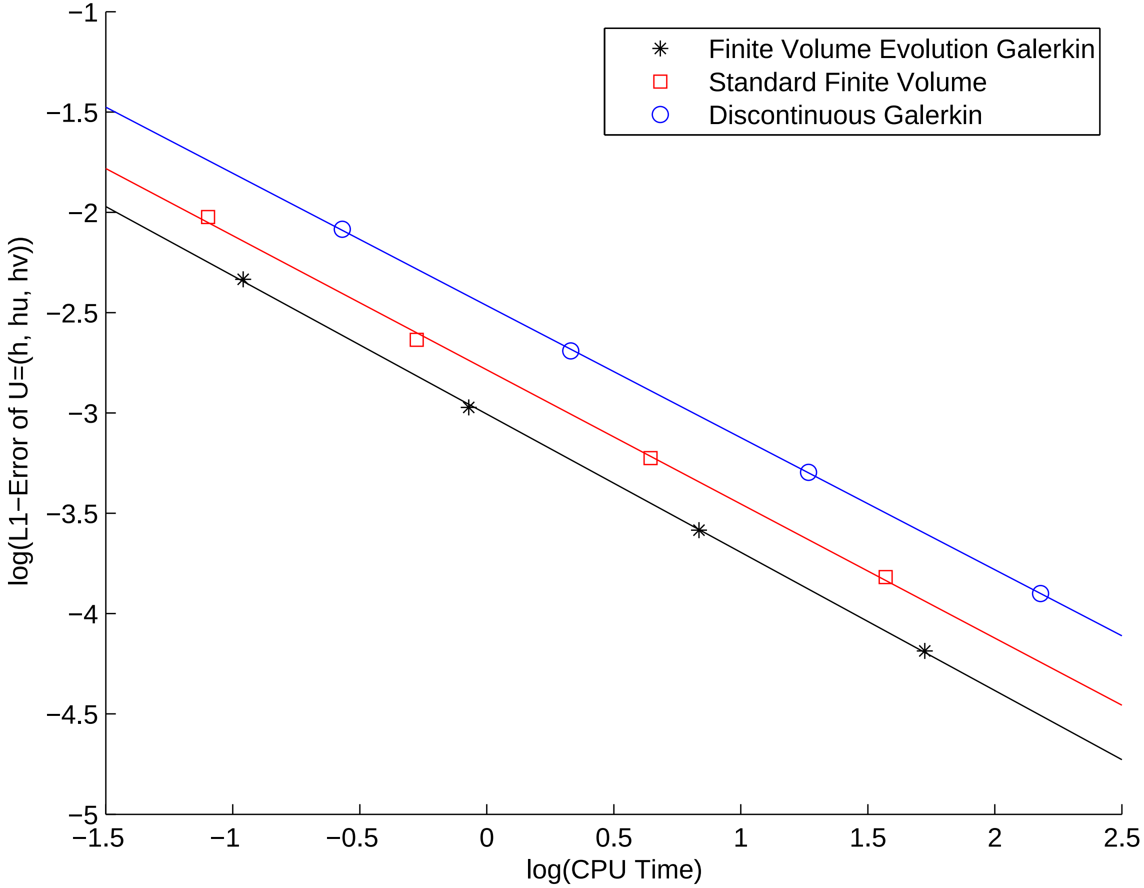


Figure 4: Efficiency study: relative L^1 error over CPU-time
FVEG scheme (stars), the FVM (boxes) and the DG method (circles).

2. Balance law / stationary equilibrium

$$b(x,y) = 0.8\exp\left(-5(x-0.5)^2 - 50(y-0.5)^2\right)$$

initial data:

$$\begin{aligned} h(x,y,0) + b(x,y) &= 1+\varepsilon; \quad \varepsilon = 0 \text{ or } 0.01 \text{ if } x \in [0.05; 0.15] \\ u(x,y,0) &= 0 = v(x,y,0) \end{aligned}$$

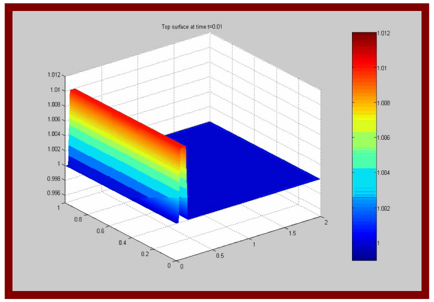


Table 1: L^1-error for $\varepsilon = 0$ using 20×20 mesh cells.

Method	$t = 0.2$	$t = 1$	$t = 10$
first order FVEG	2.35×10^{-17}	5.09×10^{-17}	5.02×10^{-17}
second order FVEG	4.97×10^{-17}	6.74×10^{-17}	1.53×10^{-16}

3. Rossby adjustments / steady jet in the rotational frame

initial data:

$$h(x, y, 0) = H,\ u(x, y, 0) = 0,\ v(x, y, 0) = VN(x)$$

$$H = 1,\ V = 2,\ N(x) \ldots \text{smooth normalized profile}$$

$$Fr = \frac{|\boldsymbol{u}|}{\sqrt{gH}} = 2 \ldots \text{Froude number}$$

$$Ro = \frac{V}{fL} = 1 \ldots \text{Rossby number}$$

$$Bu = \frac{gH}{f^2L^2} = 0.25 \ldots \text{Bugers number}$$

$$S_{fx} = 0 = S_{fy}, b = 0$$

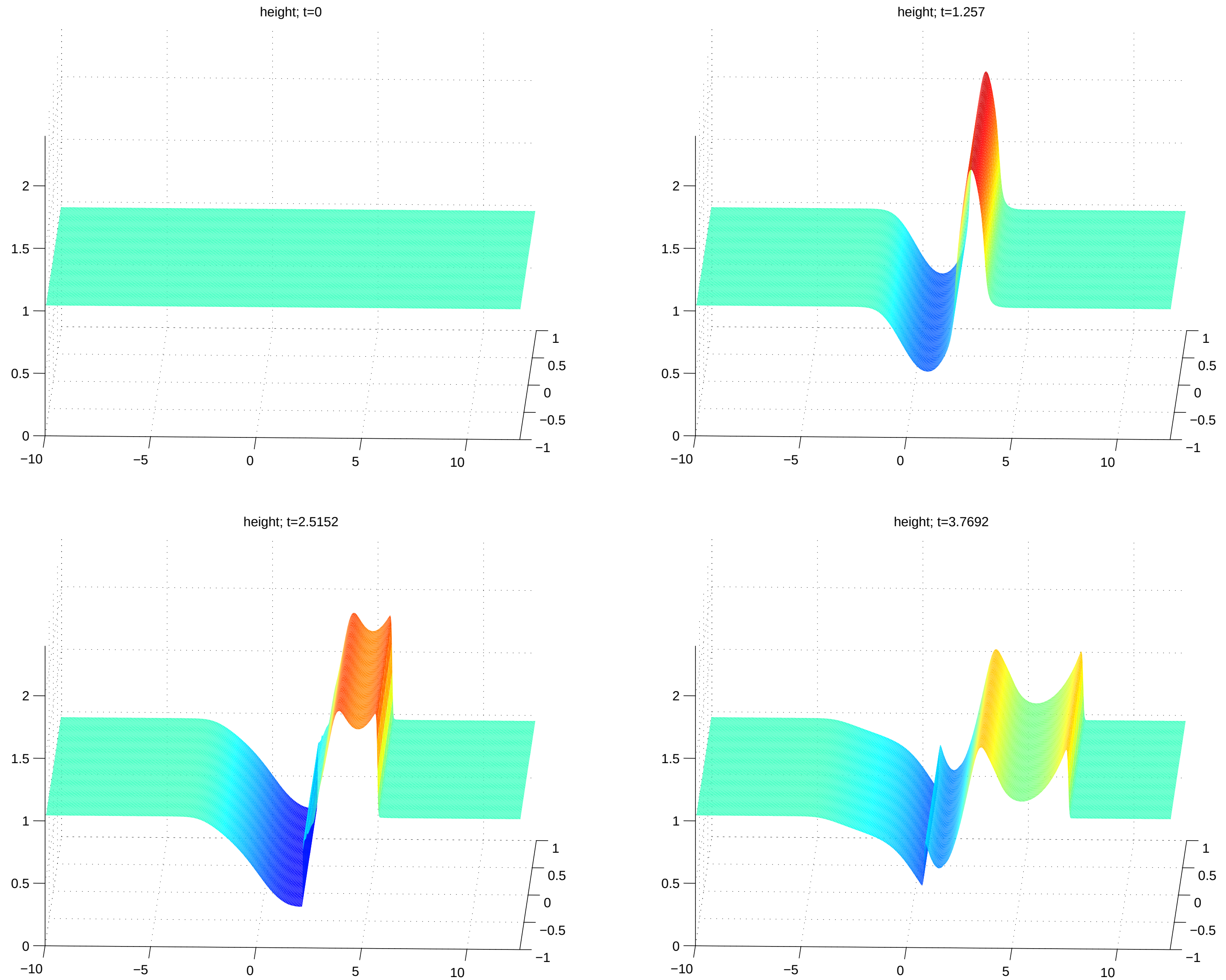


Figure 5: Water surface h at $T = 0, 1.2, 2.5, 3.7$ for jet

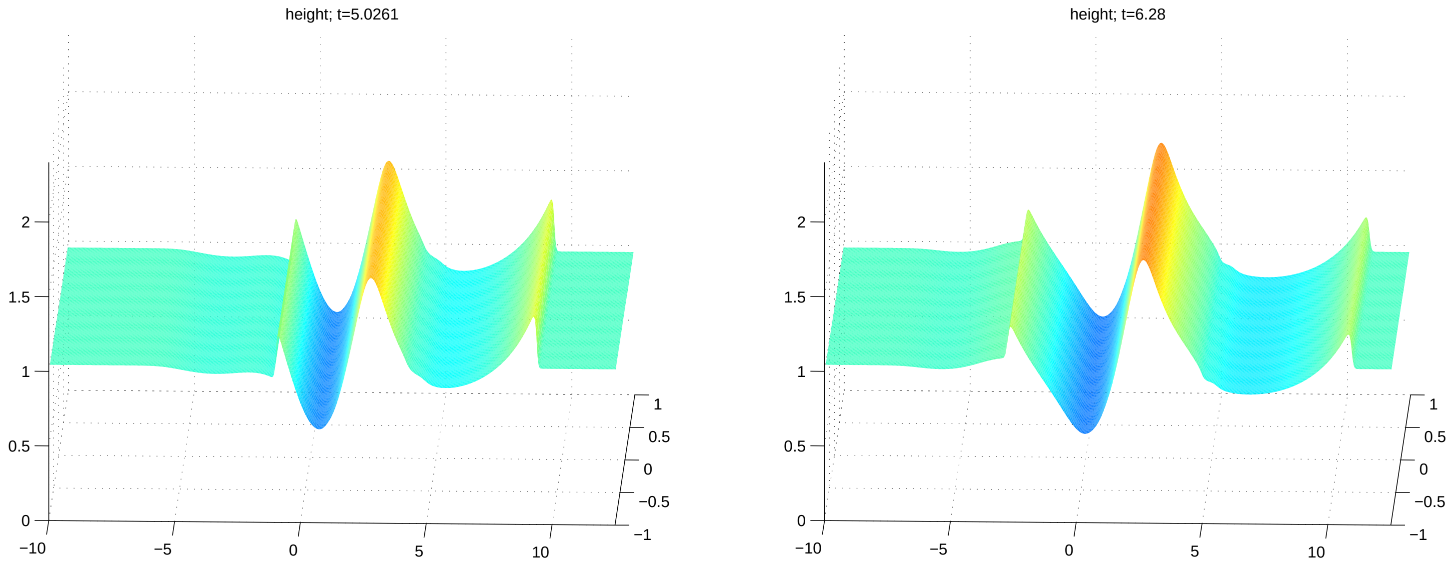
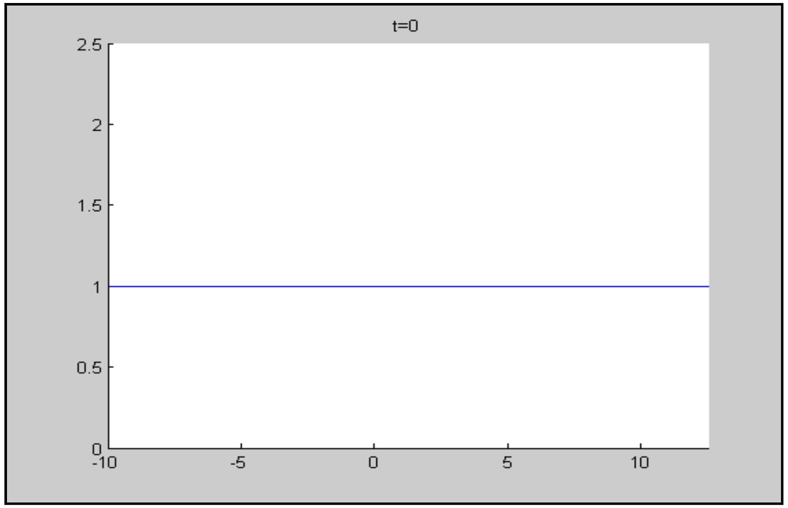
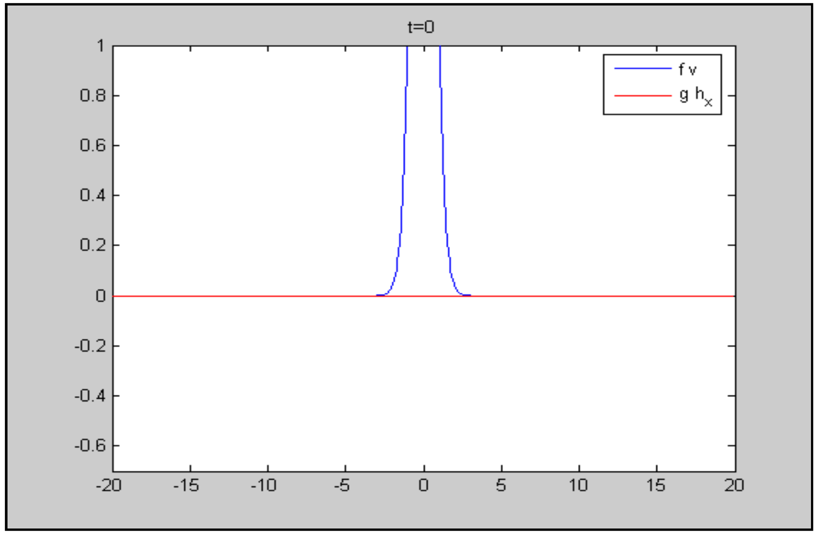


Figure 6: Water surface h at $T = 5, 6.2$ for jet



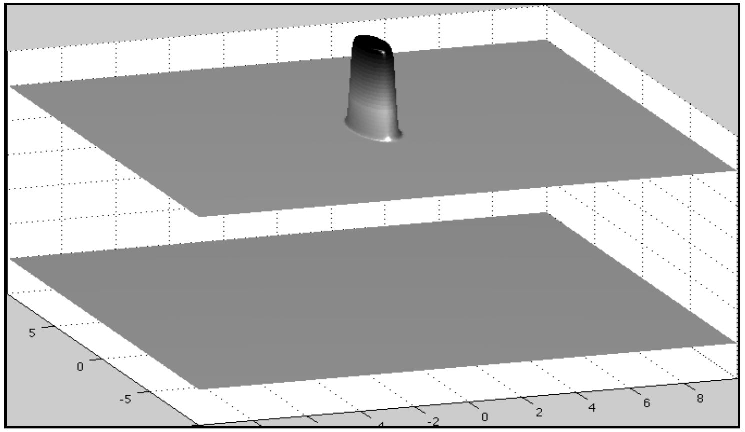


5. Geostrophic adjustment - 2 layer model

$$h_1(x,y,0) = 1 + \frac{A_0}{2}\left(1 - \tanh\left(\frac{\sqrt{(\sqrt{\lambda}x)^2 + (y/\sqrt{\lambda})^2} - R_i}{R_E}\right)\right)$$

$$h_2(x,y,0) = 1 \quad u_1(x,y,0) = v_1(x,y,0) = u_2(x,y,0) = v_2(x,y,0) = 0$$

$A_0 = 0.5$, $\lambda = 2.5$, $R_E = 0.1$, $R_i = 1$, $r = 0.98$, $g = 1$, $f = 1$



http://www.tu-harburg.de/mat/hp/lukacova

from 1.3. 2010 at Johannes Gutenberg University of Mainz