On Kelvin-Voigt model and its generalizations

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In memory of Professor Jindřich Nečas

Contents

Part #1

System of PDEs and its properties

Problem description

For given

- $\Omega \subset \mathbb{R}^d$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$; $\quad T > 0$
- $\mathbf{u}_0$, $\mathbf{v}_0$ (initial data); $\quad \mathbf{u}_D$, $\mathbf{g}$ (boundary data)
- $\varrho_0 : \Omega \mapsto \mathbb{R}$ density
- $r, q \in (1, \infty)$

to find $\mathbf{u} : (0, T) \times \Omega \to \mathbb{R}^d$

$$
\begin{aligned}
\varrho_0 \mathbf{u}_{,tt} - \operatorname{div} \mathbf{T} &= \mathbf{0} && \text{in } \Omega \times (0, T) \\
\mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } \Omega \\
\mathbf{u}_{,t}(0, \cdot) &= \mathbf{v}_0 && \text{in } \Omega \\
\mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma_D \times (0, T) \\
\mathbf{T}\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_N \times (0, T)
\end{aligned}
$$

$$
\mathbf{T} := \mu_*(1 + |\mathbf{D}(\mathbf{u})|^2)^{(q-2)/2}\mathbf{D}(\mathbf{u}) + \nu_*(1 + |\mathbf{D}(\mathbf{u}_{,t})|^2)^{(r-2)/2}\mathbf{D}(\mathbf{u}_{,t})
$$

A Jindřich Nečas program

Mathematical analysis of nonlinear (quasilinear) elliptic, parabolic and hyperbolic problems

Concept of (generalized) solution, its **existence**, uniqueness and **regularity**

Question: **Provided that data are smooth, is the generalized solution of the problem a classical solution?**

$$\Delta_p \mathbf{z} := \operatorname{div}\left((1+|\mathbf{D}(\mathbf{z})|^2)^{(p-2)/2}\mathbf{D}(\mathbf{z})\right) \qquad 2\mathbf{D}(\mathbf{z}) := \nabla\mathbf{z} + (\nabla\mathbf{z})^T$$

$$\begin{aligned} -\Delta_r \mathbf{v} &= \mathbf{0} \\ \mathbf{v}_{,t} - \Delta_r \mathbf{v} &= \mathbf{0} \\ \mathbf{u}_{,tt} - \Delta_q \mathbf{u} &= \mathbf{0} \\ \mathbf{u}_{,tt} - \Delta_q \mathbf{u} - \Delta_r \mathbf{u}_{,t} &= \mathbf{0} \end{aligned}$$

Several results

$$-\Delta_r \mathbf{v} = \mathbf{0} \qquad (1)$$

$$\mathbf{v}_{,t} - \Delta_r \mathbf{v} = \mathbf{0} \qquad (2)$$

$$\mathbf{u}_{,tt} - \Delta_q \mathbf{u} = \mathbf{0} \qquad (3)$$

$$\mathbf{u}_{,tt} - \Delta_q \mathbf{u} - \Delta_r \mathbf{u}_{,t} = \mathbf{0} \qquad (4)$$

Several results

$$-\Delta_r \mathbf{v} = \mathbf{0} \quad (1)$$

$$\mathbf{v}_{,t} - \Delta_r \mathbf{v} = \mathbf{0} \quad (2)$$

$$\mathbf{u}_{,tt} - \Delta_q \mathbf{u} = \mathbf{0} \quad (3)$$

$$\mathbf{u}_{,tt} - \Delta_q \mathbf{u} - \Delta_r \mathbf{u}_{,t} = \mathbf{0} \quad (4)$$

- existence, uniqueness of weak solution for (1) or (2) - monotone operator theory
- positive answer to full regularity for $d = 2$ - $C^{1,\alpha}$-regularity of weak solution - stationary problems (1967-71, with J. Stará), evolutionary problem (1991, with V. Šverák), extended to generalized Stokes systems (since 1996 by P. Kaplický, J. Stará)
- measure-valued solution for (3) for the scalar case (together with M. Růžička and M. Rokyta)
- existence, uniqueness, $C^{1,\alpha}$-regularity for (4) - with A. Friedman (1988), also by T. Roubíček

PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 1, 1988

SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH NONLINEAR VISCOSITY

AVNER FRIEDMAN AND JINDRICH NECAS

An equation of the form

$$\ddot{u} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial W(p)}{\partial p_i} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial V(q)}{\partial q_i} = f$$

where $p = \nabla u$, $q = \nabla \dot{u}$, $\dot{u} = \partial u/\partial t$, $\ddot{u} = \partial^2 u/\partial t^2$ represents, for suitable functions $W(p)$, $V(q)$, a nonlinear hyperbolic equation with nonlinear viscosity and it appears in models of nonlinear elasticity. In this paper existence and regularity of solutions for the Cauchy problem will be established. In particular, if $n = 2$, or if $n \geq 3$ and the eigenvalues of $(\partial^2 V/\partial q_j \partial q_j)$ belong to a "small" interval, then the solution is classical. These results will actually be established for a system of equations of the above type.

Assumptions/1 - homogeneous Dirichlet bc's

For given

- $\Omega \subset \mathbb{R}^d$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$; $T > 0$
- $\mathbf{u}_0$, $\mathbf{v}_0$ (initial data); $\mathbf{g}$ (boundary data)
- $\varrho_0 : \Omega \mapsto \mathbb{R}$ density
- $r, q \in (1, \infty)$

to find $\mathbf{u} : (0, T) \times \Omega \to \mathbb{R}^d$

$$\boxed{\begin{aligned} \varrho_0 \mathbf{u}_{,tt} - \operatorname{div} \mathbf{T} &= \mathbf{0} && \text{in } \Omega \times (0, T) \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } \Omega \\ \mathbf{u}_{,t}(0, \cdot) &= \mathbf{v}_0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D \times (0, T) \\ \mathbf{T}\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_N \times (0, T) \end{aligned}}$$

$$\boxed{\mathbf{T} := \mathbf{E}(\mathbf{D}(\mathbf{u})) + \mathbf{S}(\mathbf{D}(\mathbf{u}_{,t}))}$$

$$(\mathbf{S}(\mathbf{D}_1) - \mathbf{S}(\mathbf{D}_2), \mathbf{D}_1 - \mathbf{D}_2) \sim |\mathbf{D}_1 - \mathbf{D}_2|^2 \int_0^1 (1 + |\mathbf{D}_1 - s(\mathbf{D}_1 - \mathbf{D}_2)|)^{r-2} ds$$

$$(\mathbf{E}(\mathbf{D}_1) - \mathbf{E}(\mathbf{D}_2), \mathbf{D}_1 - \mathbf{D}_2) \sim |\mathbf{D}_1 - \mathbf{D}_2|^2 \int_0^1 (1 + |\mathbf{D}_1 - s(\mathbf{D}_1 - \mathbf{D}_2)|)^{q-2} ds$$

If $\mathbf{S}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{0}) = \mathbf{0}$ then

$$\mathbf{S}(\mathbf{D}) \cdot \mathbf{D} \geq C_1(|\mathbf{D}|^r - 1) \qquad \mathbf{E}(\mathbf{D}) \cdot \mathbf{D} \geq C_1(|\mathbf{D}|^q - 1)$$

$$|\mathbf{S}(\mathbf{D})| \leq C_2(1 + |\mathbf{D}|)^{r-1} \qquad |\mathbf{E}(\mathbf{D})| \leq C_2(1 + |\mathbf{D}|)^{q-1}$$

Definition of solution

- $\mathbf{g} \in L^{r'}(0, T; (W^{1-\frac{1}{r},r}(\Gamma_N)^d)^*)$
- $\mathbf{u}_0 \in W^{1,q}_{\Gamma_D}(\Omega)^d$, $\mathbf{v}_0 \in L^2(\Omega)^d$
- $0 < m \le \varrho_0 \le M < \infty$ a.e. in Ω

Then $\mathbf{u}$ is a weak solution to the problem if

$$\mathbf{u} \in L^\infty(0, T; W^{1,q}_{\Gamma_D}(\Omega)^d) \cap W^{1,\infty}(0, T; L^2(\Omega)^d) \cap W^{1,r}(0, T; W^{1,r}(\Omega)^d)$$

$$\varrho_0 \mathbf{u}_{,tt} \in L^{\min(r',q')}(0, T; (W^{1,\max(r,q)}_{\Gamma_D}(\Omega)^d)^*)$$

$$\int_0^T \langle \varrho_0 \mathbf{u}_{,tt}, \varphi \rangle + \big(\mathbf{S}(\mathbf{D}(\mathbf{u}_{,t})) + \mathbf{E}(\mathbf{D}(\mathbf{u})), \mathbf{D}(\varphi)_\Omega\big)\, dt = \int_0^T \langle \mathbf{g}, \varphi \rangle_{\Gamma_N}\, dt$$

$$\text{for all } \varphi \in L^{\max(r,q)}(0, T; W^{1,\max(r,q)}_{\Gamma_D}(\Omega)^d)$$

$$\lim_{t \to 0_+} \left(\|\mathbf{u}_{,t}(t) - \mathbf{v}_0\|_2^2 + \|\mathbf{u}(t) - \mathbf{u}_0\|_{1,2}^2\right) = 0.$$

Theorem 1 (M. Bulíček, J. Málek, K.R. Rajagopal)

Theorem (Existence and Uniqueness)

Let **S** and **E** satisfy the monotone, growth and coercivity conditions with

$$\boxed{1 < q \leq 2 \leq r < \infty}$$

Then, for any set of data Ω, T, ϱ_0, $\mathbf{u}_0$, $\mathbf{v}_0$ and $\mathbf{g}$ there is unique weak solution to the problem.

Novel aspects

- $r > 2$ (nonlinearity in $\nabla \mathbf{u}_{,t}$)
- variable density ϱ_0
- no potential structure
- Friedman, Nečas - nonlinear **S** of the potential structure with $r = 2$ and $d = 2$ or eigenvalues of $\frac{\partial \mathbf{S}}{\partial \mathbf{D}}$ lie in small interval

Theorem 2 (M. Bulíček, J. Málek, K.R. Rajagopal)

Theorem (Regularity)

(1) If in addition $\varrho_0 \in \mathcal{C}^{0,1}(\Omega)$ and $\mathbf{v}_0 \in W^{1,2}_{loc}(\Omega)^d$ then

$$
\begin{aligned}
(1 + |\mathbf{D}(\mathbf{v})|)^{\frac{r-2}{2}} \mathbf{D}(\nabla \mathbf{v}) &\in L^2(0, T; L^2_{loc}(\Omega)^{d\times d}) \\
\mathbf{v} &\in L^\infty(0, T; W^{1,2}_{loc}(\Omega)^d) \\
\mathbf{v}_{,t} &\in L^{r'}(0, T; L^{r'}_{loc}(\Omega)^d)
\end{aligned}
$$

(2) If $\varrho_0 \in \mathcal{C}^{0,1}(\overline{\Omega})$, $\mathbf{v}_0 \in W^{1,2}(\Omega)^d$, $\mathbf{g} \in W^{1,r}(0, T; W^{1+\frac{1}{r},r}(\Gamma_N))$ and $\mathbf{S}_{ij}(\mathbf{D}) = \partial_{\mathbf{D}_{ij}} U_{\mathbf{S}}(\mathbf{D})$ then

$$
\mathbf{v}_{,t} \in L^2(0, T; L^2(\Omega)^d) \quad \mathbf{v} \in L^\infty(0, T; W^{1,r}(\Omega)^d)
$$

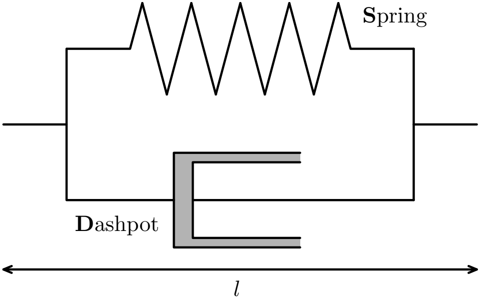
Elliptic regularity then implies (for smooth Ω) that

$$
(1 + |\mathbf{D}(\mathbf{v})|)^{\frac{r-2}{2}} \mathbf{D}(\nabla \mathbf{v}) \in L^2(0, T; L^2(\Omega)^{d\times d})
$$

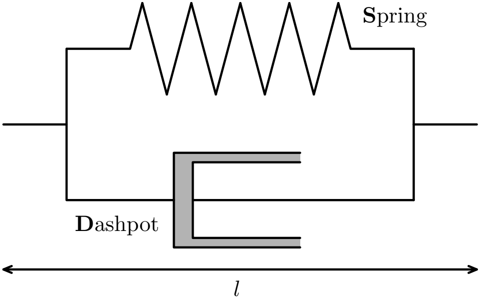
Part #2

Kelvin-Voigt solid and its generalizations

Kelvin-Voigt solid - mechanical analog

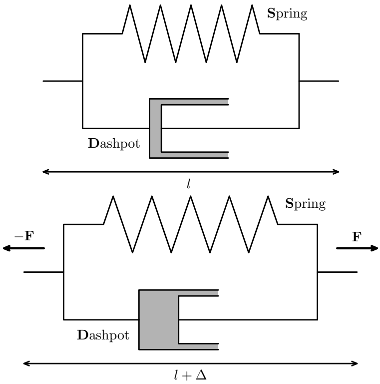


Kelvin-Voigt solid - mechanical analog

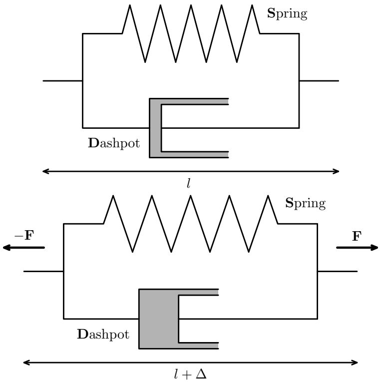


$$\sigma_S = E\varepsilon_S \quad \text{and} \quad \sigma_D = \nu\dot{\varepsilon}_D$$

Kelvin-Voigt solid - mechanical analog



Kelvin-Voigt solid - mechanical analog



$$F = F_S + F_D \qquad \text{and} \qquad \Delta = \Delta_S = \Delta_D$$
$$\sigma = \sigma_S + \sigma_D \qquad \text{and} \qquad \varepsilon = \varepsilon_S = \varepsilon_D \implies = \dot{\varepsilon} = \dot{\varepsilon}_S = \dot{\varepsilon}_D$$
$$\sigma_S = E\varepsilon_S \qquad \text{and} \qquad \sigma_D = \nu\dot{\varepsilon}_D$$

$$\boxed{\sigma = E\varepsilon_S + \nu\dot{\varepsilon}_D}$$

Kelvin-Voigt solid and its linearization

Response corresponds to the mixture of solid and fluid components with neo-Hookean solid and Newtonian fluid

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \mathbf{T}_e = \mu_* \mathbf{B} \quad \mathbf{T}_v = \nu_* \mathbf{D}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ (right Cauchy-Green stretch tensor) and $\mathbf{D} = \mathbf{D}(\mathbf{v}) := \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)$. Then

$$\mathbf{T} = \mu \mathbf{B} + \eta \mathbf{D}$$

Linearized elastic solid $\varepsilon := \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$

$$\mathbf{T} = \mu \varepsilon + \eta \mathbf{D} = \mu \varepsilon + \eta \dot{\varepsilon}$$

Question: **Is the model that is non-linear in the linearized strain justifable?**
Yes, but one has to take a different point of view.

Implicit constitutive theory (K.R. Rajagopal, 2003)

General implicit relations of the form

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \boxed{\mathbf{f}(\mathbf{T}_e, \mathbf{B}) = \mathbf{0}} \qquad \boxed{\mathbf{g}(\mathbf{T}_v, \mathbf{D}) = \mathbf{0}}$$

include two explicit constitutive theories:

Implicit constitutive theory (K.R. Rajagopal, 2003)

General implicit relations of the form

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \boxed{\mathbf{f}(\mathbf{T}_e, \mathbf{B}) = \mathbf{0}} \qquad \boxed{\mathbf{g}(\mathbf{T}_v, \mathbf{D}) = \mathbf{0}}$$

include two explicit constitutive theories:

$$\mathbf{T}_e = \mathbf{f}_1(\mathbf{B}) \qquad \text{and } \mathbf{T}_v = \mathbf{g}_1(\mathbf{D})$$

Implicit constitutive theory (K.R. Rajagopal, 2003)

General implicit relations of the form

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \boxed{\mathbf{f}(\mathbf{T}_e, \mathbf{B}) = \mathbf{0}} \qquad \boxed{\mathbf{g}(\mathbf{T}_v, \mathbf{D}) = \mathbf{0}}$$

include two explicit constitutive theories:

$$\mathbf{T}_e = \mathbf{f}_1(\mathbf{B}) \qquad \text{and } \mathbf{T}_v = \mathbf{g}_1(\mathbf{D})$$

and

$$\mathbf{B} = \mathbf{f}_2(\mathbf{T}_e) \qquad \text{and } \mathbf{D} = \mathbf{g}_2(\mathbf{T}_v)$$

Implicit constitutive theory (K.R. Rajagopal, 2003)

General implicit relations of the form

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \boxed{\mathbf{f}(\mathbf{T}_e, \mathbf{B}) = \mathbf{0}} \qquad \boxed{\mathbf{g}(\mathbf{T}_v, \mathbf{D}) = \mathbf{0}}$$

include two explicit constitutive theories:

$$\mathbf{T}_e = \mathbf{f}_1(\mathbf{B}) \qquad \text{and } \mathbf{T}_v = \mathbf{g}_1(\mathbf{D})$$

and

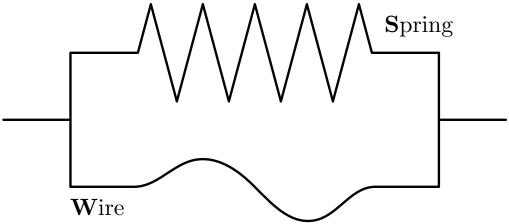
$$\mathbf{B} = \mathbf{f}_2(\mathbf{T}_e) \qquad \text{and } \mathbf{D} = \mathbf{g}_2(\mathbf{T}_v)$$

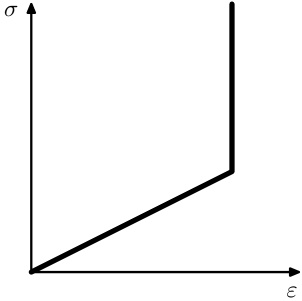
Linearization of the second set leads to

$$\varepsilon = \mathbf{f}_2(\mathbf{T}_e) \qquad \text{and } \mathbf{D} = \mathbf{g}_2(\mathbf{T}_v)$$

If this relations are invertible we obtain the model under considerations that can shear thin or shear thicken, and creep (as a viscoelastic material).

Implicit constitutive theory - Example





Special deformation applied to full nonlinear model

Consider the special shearing motion given by

$$x = X + u(Y, t)\,, \quad y = Y\,, \quad z = Z$$

and apply it to the generalized Kelvin-Voigt solid

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_v \qquad \mathbf{T}_e = \mathbf{f}(\mathbf{B}) \quad \mathbf{T}_v = \mathbf{g}(\mathbf{D})$$

Then one obtains the scalar equation of the above form without restricting to small gradients of the displacement.

Part #3

Proof of the existence result

Definition of solution

- $\mathbf{S}$, $\mathbf{E}$ are uniformly monotone (growth, coercivness assumptions) $\boxed{1 < q \leq 2 \leq r < +\infty}$
- $\mathbf{g} \in L^{r'}(0, T; (W^{1-\frac{1}{r}, r}(\Gamma_N)^d)^*)$
- $\mathbf{u}_0 \in W^{1,q}_{\Gamma_D}(\Omega)^d$, $\mathbf{v}_0 \in L^2(\Omega)^d$
- $0 < m \leq \varrho_0 \leq M < \infty$ a.e. in Ω

Then $\mathbf{u}$ is a weak solution to the problem if

$$\mathbf{u} \in L^\infty(0, T; W^{1,q}_{\Gamma_D}(\Omega)^d) \cap W^{1,\infty}(0, T; L^2(\Omega)^d) \cap W^{1,r}(0, T; W^{1,r}(\Omega)^d)$$

$$\varrho_0 \mathbf{u}_{,tt} \in L^{r'}(0, T; (W^{1,r}_{\Gamma_D}(\Omega)^d)^*)$$

$$\int_0^T \langle \varrho_0 \mathbf{u}_{,tt}, \varphi \rangle + \big(\mathbf{S}(\mathbf{D}(\mathbf{u}_{,t})) + \mathbf{E}(\mathbf{D}(\mathbf{u})), \mathbf{D}(\varphi)\big)_\Omega \, dt = \int_0^T \langle \mathbf{g}, \varphi \rangle_{\Gamma_N} \, dt$$

$$\text{for all } \varphi \in L^r(0, T; W^{1,r}_{\Gamma_D}(\Omega)^d)$$

$$\lim_{t \to 0_+} \left(\|\mathbf{u}_{,t}(t) - \mathbf{v}_0\|_2^2 + \|\mathbf{u}(t) - \mathbf{u}_0\|_{1,2}^2 \right) = 0.$$

Reformulation of the problem

$$\varrho_0 \mathbf{v}_{,t} - \operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{v})) - \operatorname{div} \mathbf{E}(\mathbf{D}(\mathbf{u})) = \mathbf{0} \quad \text{in } \Omega \times (0, T)$$

$$\mathbf{u}_{,t} = \mathbf{v} \quad \text{in } \Omega \times (0, T)$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0 \quad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_D \times (0, T)$$

$$(\mathbf{S}(\mathbf{D}(\mathbf{v})) + \mathbf{E}(\mathbf{D}(\mathbf{u})))\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N \times (0, T)$$

Step 1 - Galerkin Approximations

$$\mathbf{u}^n(t,x) := \sum_{i=1}^{n} c_i(t)\omega_i(x) \qquad \Longrightarrow \qquad \mathbf{v}^n := \mathbf{u}^n_{,t}$$

$$(\varrho_0 \mathbf{v}^n_{,t}, \omega_i)_\Omega + (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\omega_i))_\Omega + (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\omega_i))_\Omega = \langle \mathbf{g}, \omega_i \rangle_{\Gamma_N}$$
$$\text{for all } i = 1, 2, \ldots, n$$

$$\mathbf{u}^n(0,x) = \mathbf{u}^n_0(x)$$
$$\mathbf{v}^n(0,x) := \mathbf{u}^n_{,t}(0,x) = \mathbf{v}^n_0(x)$$

- The Carathéodory theory and uniform estimates give long-time existence for $\mathbf{u}^n$
- Uniform estimates

Step 2 - Uniform estimates

$$\boxed{(\varrho_0 \mathbf{v}^n_{,t}, \mathbf{v}^n)_\Omega + (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega + (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega = \langle \mathbf{g}, \mathbf{v}^n \rangle_{\Gamma_N}}$$

Step 2 - Uniform estimates

$$\boxed{(\varrho_0 \mathbf{v}^n_{,t}, \mathbf{v}^n)_\Omega + (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega + (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega = \langle \mathbf{g}, \mathbf{v}^n \rangle_{\Gamma_N}}$$

$$\begin{aligned} \frac{d}{dt}\left\{\|\sqrt{\varrho_0}\mathbf{v}^n\|_2^2 + \|\mathbf{D}(\mathbf{u}^n))\|_q^q\right\} + 2(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega \\ \leq C + |(\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega| \end{aligned}$$

Step 2 - Uniform estimates

$$\boxed{(\varrho_0 \mathbf{v}^n_{,t}, \mathbf{v}^n)_\Omega + (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega + (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega = \langle \mathbf{g}, \mathbf{v}^n \rangle_{\Gamma_N}}$$

$$\begin{aligned}\frac{d}{dt}\left\{\|\sqrt{\varrho_0}\mathbf{v}^n\|_2^2 + \|\mathbf{D}(\mathbf{u}^n))\|_q^q\right\} + 2(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega \\ \leq C + |(\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega|\end{aligned}$$

$$\sup_{t\in(0,T)} \left(\|\mathbf{v}^n\|_2^2 + \|\mathbf{u}^n\|_{1,2}^2\right) + \int_0^T \|\mathbf{v}^n\|_{1,r}^r \, dt \leq C$$

Step 2 - Uniform estimates

$$\boxed{(\varrho_0 \mathbf{v}^n_{,t}, \mathbf{v}^n)_\Omega + (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega + (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega = \langle \mathbf{g}, \mathbf{v}^n \rangle_{\Gamma_N}}$$

$$\frac{d}{dt}\left\{\|\sqrt{\varrho_0}\mathbf{v}^n\|_2^2 + \|\mathbf{D}(\mathbf{u}^n))\|_q^q\right\} + 2(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega \\ \leq C + |(\mathbf{E}(\mathbf{D}(\mathbf{u}^n)), \mathbf{D}(\mathbf{v}^n))_\Omega|$$

$$\sup_{t\in(0,T)} \left(\|\mathbf{v}^n\|_2^2 + \|\mathbf{u}^n\|_{1,2}^2\right) + \int_0^T \|\mathbf{v}^n\|_{1,r}^r \, dt \leq C$$

$$\sup_{t\in(0,T)} \|\mathbf{E}(\mathbf{D}(\mathbf{u}^n))\|_{q'}^{q'} + \int_0^T \|\mathbf{S}(\mathbf{D}(\mathbf{v}^n))\|_{r'}^{r'} \, ds \leq C$$

$$\int_0^T \|\varrho_0 \mathbf{v}^n_{,t}\|^{r'}_{(W^{1,r}_{\Gamma_D}(\Omega))^*} \, dt \leq C$$

Step 3 - Limit as $n \longrightarrow \infty$

$$\int_0^T \langle \varrho_0 \mathbf{v}_{,t}, \boldsymbol{\omega} \rangle + (\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{\omega}))_\Omega + (\overline{\mathbf{E}}, \mathbf{D}(\boldsymbol{\omega}))_\Omega \, dt = \int_0^T \langle \mathbf{g}, \boldsymbol{\omega} \rangle \, dt,$$

$$\text{for all } \boldsymbol{\omega} \in L^r(0,T; W^{1,r}_\Gamma(\Omega)^d)$$

$$\mathbf{v} = \mathbf{u}_{,t}$$

Key problem is the identification of $\overline{\mathbf{E}}$ and $\overline{\mathbf{S}}$. It suffices to to show that (for a subsequences $\{\mathbf{u}^n\}$ and $\{\mathbf{v}^n\}$)

$$\mathbf{D}(\mathbf{v}^n) \to \mathbf{D}(\mathbf{v}) \quad \text{and } \mathbf{D}(\mathbf{u}^n) \to \mathbf{D}(\mathbf{u}) \qquad \text{a.e. in } (0,T) \times \Omega \tag{5}$$

- Operators are strictly monotone
- $\mathbf{v}$ is admissible test function for the limit equation
- Limit eq. is sufficient to establish time-continuity and the attainment of the initial conditions
- $\mathbf{u}^n(t) - \mathbf{u}(t) = \int_0^t \mathbf{v}^n(\tau) - \mathbf{v}(\tau) \, d\tau + \mathbf{u}^n_0 - \mathbf{u}_0$

Step 4 - Almost everywhere convergence of $\mathbf{D}(\mathbf{v}^n)/1$

$$\limsup_{n\to\infty}\int_0^{t^*}(\mathbf{S}(\mathbf{D}(\mathbf{v}^n)),\mathbf{D}(\mathbf{v}^n))_\Omega+(\mathbf{E}(\mathbf{D}(\mathbf{u}^n)),\mathbf{D}(\mathbf{v}^n))_\Omega\,dt$$
$$\leq -\frac{1}{2}\|\sqrt{\varrho_0}\mathbf{v}(t^*)\|_2^2+\frac{1}{2}\|\sqrt{\varrho_0}\mathbf{v}_0\|_2^2$$

Next, taking $\boldsymbol{\omega} := \mathbf{v}$ in the limit Eq., integrating the result over time interval $(0,t^*)$ and comparing the result with the above Ineq. (use $\mathbf{v}(0)=\mathbf{v}_0$)

$$\limsup_{n\to\infty}\int_0^{t^*}(\mathbf{S}(\mathbf{D}(\mathbf{v}^n))+\mathbf{E}(\mathbf{D}(\mathbf{u}^n)),\mathbf{D}(\mathbf{v}^n))_\Omega;dt\leq\int_0^{t^*}(\overline{\mathbf{S}}+\overline{\mathbf{E}},\mathbf{D}(\mathbf{v}))_\Omega\,dt$$

This implies that for all $t^*\in[0,T]$

$$\boxed{\limsup_{n\to\infty}\int_0^{T^*}(\mathbf{S}(\mathbf{D}(\mathbf{v}^n))-\mathbf{S}(\mathbf{D}(\mathbf{v}))+\mathbf{E}(\mathbf{D}(\mathbf{u}^n))-\mathbf{E}(\mathbf{D}(\mathbf{u})),\mathbf{D}(\mathbf{v}^n-\mathbf{v}))_\Omega\leq 0}$$

Step 5 - Almost everywhere convergence of $\mathbf{D}(\mathbf{v}^n)/2$

$$\int_0^{T_1} (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))_\Omega \, dt \le g(n) \qquad (\limsup_{n\to\infty} g(n) \to 0)$$
$$+ \int_0^{T_1} (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)) - \mathbf{E}(\mathbf{D}(\mathbf{u})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))_\Omega \, dt$$

Since $q \le 2 \le r$ and $\mathbf{u}^n(t) - \mathbf{u}(t) = \int_0^t \mathbf{v}^n(\tau) - \mathbf{v}(\tau)\, d\tau + \mathbf{u}_0^n - \mathbf{u}_0$

$$C_* \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2^2 \, dt \le C \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2 \|\mathbf{D}(\mathbf{u}^n - \mathbf{u})\|_2 \, dt + g(n)$$
$$\le C \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2 \Big\| \int_0^t \|\mathbf{D}(\mathbf{u}^n - \mathbf{u})\, ds \Big\|_2 \, dt + g(n)$$
$$\le (CT_1 + \epsilon) \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2^2 \, dt + g(n)$$

If T_1 is such that $\boxed{CT_1 < C_*}$ then a.e. convergence follows.

Step 5 - Almost everywhere convergence of $\mathbf{D}(\mathbf{v}^n)/2$

$$\int_0^{T_1} (\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))_\Omega \, dt \le g(n) \qquad (\limsup_{n\to\infty} g(n) \to 0)$$

$$+ \int_0^{T_1} (\mathbf{E}(\mathbf{D}(\mathbf{u}^n)) - \mathbf{E}(\mathbf{D}(\mathbf{u})), \mathbf{D}(\mathbf{v}^n - \mathbf{v}))_\Omega \, dt$$

Since $q \le 2 \le r$ and $\mathbf{u}^n(t) - \mathbf{u}(t) = \int_0^t \mathbf{v}^n(\tau) - \mathbf{v}(\tau)\, d\tau + \mathbf{u}_0^n - \mathbf{u}_0$

$$C_* \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2^2 \, dt \le C \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2 \|\mathbf{D}(\mathbf{u}^n - \mathbf{u})\|_2 \, dt + g(n)$$

$$\le C \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2 \Big\| \int_0^t \|\mathbf{D}(\mathbf{u}^n - \mathbf{u}) \, ds \Big\|_2 \, dt + g(n)$$

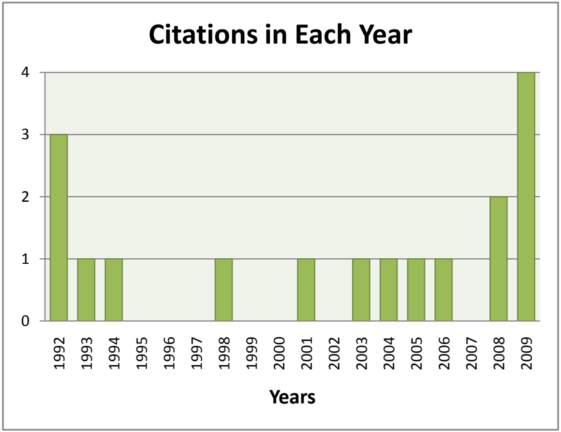
$$\le (CT_1 + \epsilon) \int_0^{T_1} \|\mathbf{D}(\mathbf{v}^n - \mathbf{v})\|_2^2 \, dt + g(n)$$

If T_1 is such that $\boxed{CT_1 < C_*}$ then a.e. convergence follows.
Inductively on $[0, T]$.

Concluding notes

- Long-time and Large-data analysis for generized Kelvin-Voigt models for solids (interesting for all possible values of r, q) limited to $\boxed{1 < q \le 2 \le r < +\infty}$
- Understanding of the model from the implicit constitute theory (or rather dual explicit constitutive theory when the kinematical quantities are functions of stresses)
- Recall the paper by A. Friedman and J. Nečas (WOS - 17 citations, MathSciNet - 13, Google Scholar - 37)

Citation of the A. Friedman and J. Nečas paper



Part #4

Final remarks