On weak solutions to model problems for turbulent flows

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Nonlinear PDE's
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1. Two coupled turbulent flows
   (joint work with J. WOLF)

2. Unbounded eddy viscosities
   (joint work with P.-E. DRUET)

1. Two coupled turbulent flows

1.1 The model

$\Omega_i \subset \mathbb{R}^d$ bounded domains ($i = 1, 2; d = 2$ or $d = 3$);
$\Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 =: \Gamma \subset \partial\Omega_i$
$\partial\Omega_i$ smooth, Γ relatively open.

Differential equations in Ω_i

$$\begin{aligned}
&(1) & \operatorname{div} \boldsymbol{u}_i &= \boldsymbol{0} \\
&(2) & -\operatorname{div}(\nu_i(k_i) D(\boldsymbol{u}_i)) + \nabla p_i &= \boldsymbol{f}_i, \\
&(3) & -\Delta k_i &= \mu_i(k_i)|D(\boldsymbol{u}_i)|^2,
\end{aligned}$$

$$\left[\begin{array}{l} D(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top), \quad \boldsymbol{v} = (v_1, \dots, v_d), \\ |D(\boldsymbol{v})|^2 = D(\boldsymbol{v}) : D(\boldsymbol{v}) \\ 0 < \nu_* \le \nu_i \le \nu^*, \quad 0 \le \mu_i(\xi) \le \mu^* \quad \forall \xi \in \mathbb{R} \end{array}\right.$$

Boundary conditions

(4) $\quad \boldsymbol{u}_i = \boldsymbol{0}, \quad k_i = 0 \quad \text{on} \quad \partial\Omega_i \setminus \Gamma,$

(5) $\quad \boldsymbol{u}_i \cdot \boldsymbol{n}_i = 0 \quad \text{on} \quad \Gamma,$

(6) $$\begin{cases} \nu_i(k_i)(D(\boldsymbol{u}_i)\boldsymbol{n}_i)_\tau + |\boldsymbol{u}_i - \boldsymbol{u}_j|(\boldsymbol{u}_i - \boldsymbol{u}_j)_\tau = \boldsymbol{0} \\ \text{on} \quad \Gamma, \quad \forall \tau \perp \boldsymbol{n}_i \quad (i \neq j), \end{cases}$$

$$(7) \quad k_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \quad \text{on} \quad \Gamma$$

$$\left[\begin{array}{l} 0 \leq G_i(\xi) \leq C_0\xi, \\ \\ |G_i(\xi) - G_i(\eta)| \leq C_1|\xi - \eta| \quad \forall \xi, \eta \in [0, +\infty). \end{array} \right.$$

- Lions; Temam; Wang (1993)
- Bernardi; Chacon; Lewandowski; Murat (2002)

1.2 Existence of weak solutions $\boxed{d = 3}$

$$\boldsymbol{V}_i := \Big\{ \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega_i)\,;\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega_i, \\ \boldsymbol{v} \cdot \boldsymbol{n}_i = 0 \text{ on } \Gamma, \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_i \setminus \Gamma \Big\}$$

Theorem 1 (N.; Wolf) Let $\boldsymbol{f}_i \in \boldsymbol{L}^{\frac{6}{5}}(\Omega_i)$. Then there exist

$$\text{(8)} \qquad \boldsymbol{u}_i \in \boldsymbol{V}_i, \quad k_i \in \bigcap_{1 \le q < \frac{3}{2}} W^{1,q}(\Omega_i)$$

such that

$$\text{(9)} \qquad k_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \text{ on } \Gamma, \quad k_i = 0 \text{ on } \partial\Omega_i \setminus \Gamma,$$

$$(10)\qquad \begin{cases} \int\limits_{\Omega_1} \nu_1(k_1) D(\boldsymbol{u}_1) : D(\boldsymbol{v}_1) + \int\limits_{\Omega_2} \nu_2(k_2) D(\boldsymbol{u}_2) : D(\boldsymbol{v}_2) + \\ + \int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2| (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) dS \\ = \int\limits_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 + \int\limits_{\Omega_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}_2 \quad \forall (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2, \end{cases}$$

$$(11)\qquad \begin{cases} \text{for any} \quad r \in (3, +\infty), \\ \int\limits_{\Omega_i} \nabla k_i \cdot \nabla \varphi = \int\limits_{\Omega_i} \mu_i(k_i) |D(\boldsymbol{u}_i)|^2 \varphi \quad \forall \varphi \in W_0^{1,r}(\Omega). \end{cases}$$

Remarks $\boxed{1}$ $|D(\boldsymbol{u}_i)|^2 \in L^1(\Omega_i) \subset W^{-1,q}(\Omega_i)$ $\left(1 \le q < \frac{3}{2}\right)$

$\boxed{2}$ $\boldsymbol{u}_i \in W^{1,2}(\Omega_i) \quad \Rightarrow \quad \gamma_i(\boldsymbol{u}_i) \in W^{\frac{1}{2},2}(\partial\Omega_i)$,

$\Rightarrow \quad \gamma_i(\boldsymbol{u}_i)|_\Gamma \in W^{\frac{1}{2},2}(\Gamma)$,

$\Rightarrow \quad \boldsymbol{g} := \left(\gamma_1(\boldsymbol{u}_1)|_\Gamma - \gamma_2(\boldsymbol{u}_2)|_\Gamma\right) \in W^{\frac{1}{2},2}(\Gamma)$, and

decay of $\boldsymbol{g}$ of order $|\xi|^3$ near to $\partial\Gamma$.

Extension of $\boldsymbol{g}$ by zero onto $\partial\Omega_i \setminus \Gamma$

$$\widetilde{\boldsymbol{g}}_i := \begin{cases} \boldsymbol{g} & \text{on} \quad \Gamma, \\ \boldsymbol{0} & \text{on} \quad \partial\Omega_i \setminus \Gamma; \end{cases}$$

$\Rightarrow \quad \widetilde{\boldsymbol{g}}_i \in \boldsymbol{W}^{\frac{1}{2},2}(\partial\Omega_i)$.

[without decay: $\widetilde{\boldsymbol{g}}_i \in W^{s,2}(\partial\Omega_i) \quad \forall 0 < s < \frac{1}{2}$].

Define $h_i := G_i(|\widetilde{\boldsymbol{g}}_i|^2)$ a. e. on $\partial\Omega_i$

$\Rightarrow \quad h_i = G_i(|\gamma_1(\boldsymbol{u}_1)|_\Gamma - \gamma_2(\boldsymbol{u}_2)|_\Gamma|^2)$ a. e. on Γ.

Proposition

- $h_i \in \bigcap_{1 \le q < \frac{3}{2}} W^{1-\frac{1}{q},q}(\partial\Omega_i)$
- $\forall q \in \left[1, \frac{3}{2}\right), \forall 0 < s \le \frac{1}{2} \quad \exists c = c(q) > 0$:

$$\|h_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \le c\|\widetilde{\boldsymbol{g}}_i\|^2_{W^{s,2}(\partial\Omega_i)}$$

Corollary $\exists H_i \in W^{1,q}(\Omega_i): \quad \gamma_i(H_i) = h_i$.

3 Solving the bvp for k_i: Lions/Magenes (III), Stampacchia. ■

Theorem 2 Let $\boldsymbol{f}_i \in \boldsymbol{L}^r(\Omega_i)$ $\left(r > \frac{6}{5}\right)$. Let

$$\boldsymbol{u}_i \in \boldsymbol{V}_i, \quad k \in \bigcap_{1 \le q < \frac{3}{2}} W^{1,q}(\Omega_i)$$

satisfy (9), (10) and (11). Then $\exists t > 2$ s. t.

$$\nabla \boldsymbol{u}_i \in \boldsymbol{L}^t(\Omega_i; \mathbb{R}^9), \tag{12}$$

$$k_i \in W^{2,\frac{t}{2}}_{\mathrm{loc}}(\Omega_i). \tag{13}$$

Work in progress:

$$(\boldsymbol{u}_i \cdot \nabla) \cdot \boldsymbol{u}_i - \nabla \cdot (\nu_i(k_i) D(\boldsymbol{u}_i)) + \nabla p_i = \boldsymbol{f}_i, \tag{2'}$$

$$\boldsymbol{u}_i \cdot \nabla k_i - \nabla \cdot (\kappa_i(k_i) \nabla k_i) = \mu_i(k_i) |D(\boldsymbol{u}_i)|^2. \tag{3'}$$

2. Unbounded eddy viscosity

2.1 The model

$\Omega \subset \mathbb{R}^d$ bounded domain ($d = 2$ or $d = 3$),
$\partial\Omega$ Lipschitz.

Differential equations

(1) $\nabla \cdot \boldsymbol{u} = 0,$

(2) $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \cdot ((\nu_0 + \nu_T(k))D(\boldsymbol{u})) - \nabla p - \nabla \cdot \boldsymbol{f},$

(3) $\boldsymbol{u} \cdot \nabla k = \nabla \cdot (\kappa(k)\nabla k) + \nu_T(k)|D(\boldsymbol{u})|^2 - g(k)$

[$\nu_0 =$ const $> 0, \nu_T(k)$ eddy viscosity]

Classical model: $\nu_T(k) = c_0\sqrt{k}$ ($c_0 =$ const > 0)
Kolmogorov (1942), Prandtl (1945);
$g(k) = l_0 k^{\frac{3}{2}}$ ($l_0 =$ const > 0).

Boundary conditions

$$(4) \qquad \boldsymbol{u} = \mathbf{0}, \quad k = k_0 \text{ on } \partial\Omega \quad (k_0 \geq 0).$$

Lederer; Lewandowski (2007)

- space periodic boundary conditions,
- $\nu_T(k) = c_0\sqrt{k}$ is excluded,
- existence of $\boldsymbol{u} \in W^{2,2}(\Omega)$.

2.2 Existence of a weak solution

For notational simplicity, let $k_0 = \text{const} \geq 0, g \equiv 0$

Theorem (DRUET; N. (2009)) *Assume*

$$\begin{cases} \nu_T \in C([0,+\infty)), \\ \exists C_1 > 0 : \quad 0 \leq \nu_T(\xi) \leq C_1 \xi^\alpha \quad \forall \xi \in [0,+\infty), \end{cases}$$

$$\begin{cases} \kappa \in C([0,+\infty)), \\ 0 < \kappa_* \leq \kappa(\xi) \leq \kappa_*(1+\xi^\alpha) \quad \forall \xi \in [0,+\infty), \end{cases}$$

where $0 < \alpha < +\infty$ *if* $d = 2$, $\quad 0 < \alpha < 3$ *if* $d = 3$.

Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$. *Then there exist*

$$\{\mathbf{u}, k\} \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times \Big(\bigcap_{1 \le q < \frac{d}{d-1}} W^{1,q}(\Omega) \Big), \quad \lambda \in \mathcal{M}(\overline{\Omega})$$

such that

- $k \geq 0$ *a. e. in* Ω, $k = k_0$ *a. e. on* $\partial\Omega$,
- $\sqrt{\nu_T(k)} D(\mathbf{u}) \in \mathbf{L}^2$, $\kappa(k) \nabla k \in \bigcap_{1 \le q < \frac{d}{d-1}} W^{1,q}(\Omega)$,
- (2) *is satisfied in the weak sense*

$$\forall \tau > d,$$

$$\int_{\Omega} (\boldsymbol{u} \cdot \nabla k)\varphi + \int_{\Omega} \kappa(k)\nabla k \cdot \nabla \varphi =$$

$$= \int_{\Omega} \nu_T(k)|D(\boldsymbol{u})|^2 \varphi + \int_{\overline{\Omega}} \varphi d\mu \quad \forall \varphi \in W_0^{1,\tau}(\Omega)$$

Let $d = 2$. Let $\boldsymbol{f} \in \boldsymbol{L}^s\,(s > 2)$. If $\|\boldsymbol{f}\|_{\boldsymbol{L}^s}$ is sufficiently small then there exists $t > 2$ s. t.

$$D(\boldsymbol{u}) \in \boldsymbol{L}^t, \quad k \leq \ \mathit{Const} \ \ a.\,e. \ \ in \quad \Omega, \quad \mu = 0.$$

THANK YOU

FOR YOUR ATTENTION !