Liouville-type theorems and singularities

Pavol Quittner

Comenius University, Bratislava

Nonlinear PDE's
(to commemorate the work of Jindřich Nečas)
Praha, December 13, 2009

Jindřich Nečas (1979):

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$$\textbf{(L)} \Longleftrightarrow \textbf{(R)}$$

(L) . . . "Liouville property"

(R) . . . Regularity (of weak solutions)

Outline

I. (Variational) problems, where Liouville-type theorems guarantee **regularity** of weak solutions

II. (Evolution) problems, where Liouville-type theorems yield **estimates on** (or nonexistence of) singularities

1. Singularities of backward self-similar solutions
2. Singularity estimates via Liouville for stationary or ancient solutions
3. Singularity estimates via Liouville for entire solutions and doubling

Minimal surfaces (De Giorgi, . . . 1960's):

Σ . . . n-dimensional minimal surface in $\mathbb{R}^{n+1}$

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Liouville: *If C is a minimal cone and $n \leq 6$*

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The result is optimal (Bombieri, De Giorgi, Giusti 1969)

Minimizing harmonic maps

(Schoen, Uhlenbeck 1984, Giaquinta, Souček 1985)

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold, $n \leq 6$,

$$\mathbb{S}^N_+ := \{y \in \mathbb{R}^{N+1} : |y| = 1,\ y_N \geq 0\}.$$

If $u : \mathcal{M} \to \mathbb{S}^N_+$ minimizes the Dirichlet energy $\int |du|^2 \, dv_{\mathcal{M}}$ on each compact domain of $\mathcal{M}$, then u is smooth.

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Every energy minimizing map $u : \mathbb{R}^n \to \mathbb{S}^N_+$ is constant if $n \leq 6$.

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$U : \mathbb{B}^n \to \mathbb{S}^n_+ : x \mapsto \left(\frac{x}{|x|}, 0\right)$ is minimizing iff $n \geq 7$

(Jäger, Kaul 1983)

U is a critical point of the energy if $n \geq 3$

Minimizers of $\quad I(u) = \int_\Omega f(\nabla u(x))\, dx$

f smooth, strongly convex, $D^2 f$ bounded,

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all $W^{1,\infty}$*-solutions of*

$$\frac{\partial}{\partial x_i} a_i^r(\nabla v) = 0 \text{ in } \mathbb{R}^n, \qquad a_i^r := \frac{\partial f}{\partial p_i^r},$$

are polynomials of at most first degree. (L)

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Šverák, Yan 2000-2002: Singular minimizers for $n = 3, 4, \ldots$

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$-\frac{\partial}{\partial x_i}\Big(a_{ij}(|\nabla u|^2)\frac{\partial u_\alpha}{\partial x_j}\Big) = 0$ (P.-L. Lions, Nečas, Netuka 1982)

Boundary regularity (Giaquinta, John, Nečas, Stará 1981,...),

Quasilinear/semilinear elliptic systems (Ivert 1980, Kawohl 1980, Nečas, Olejnik 1980, Wiegner 1982, Meier 1984,...)

Parabolic systems (John, Stará, Daněček 1982, 1984, 1987,...)

Higher-order elliptic systems (Balanda, Viszus 1991-1992)

Fully nonlinear elliptic equations (Huang 2002) ...

Singularities of backward self-similar solutions

Navier-Stokes equations I

$$\left.\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \times (0,T)$$

Leray 1934: Existence of self-similar solutions ?

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$$\left.\begin{aligned} u(x,t) &= \lambda U(\lambda x) \\ p(x,t) &= \lambda^2 P(\lambda x) \end{aligned}\right\} \quad \lambda = \lambda(t) := \frac{1}{2\sqrt{a(T-t)}}, \ a > 0$$

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$U \not\equiv 0 \quad \Rightarrow u$ blows up

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Nonexistence of $U \not\equiv 0$ in the class:

- $L^3(\mathbb{R}^3)$ (Nečas, Růžička, Šverák 1996)
- functions satisfying local energy estimates (Tsai 1998)

Chemotaxis (Herrero, Medina, Velázquez 1998)

$$\left.\begin{aligned} u_t &= \Delta u - \nabla(u\nabla v) \\ 0 &= \Delta v + u \end{aligned}\right\} \quad x \in \mathbb{R}^n,\ t \in (0,T)$$

Consider radial positive solutions with finite mass ($u_0 \in L^1$)

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- $n = 3$: Blow-up in the class of self-similar solutions

$$u(x,t) = \frac{1}{T-t} U\left(\frac{x}{\sqrt{T-t}}\right), \quad v(x,t) = V\left(\frac{x}{\sqrt{T-t}}\right)$$

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  but blow-up in general (Dolbeault, Perthame 2004)

  Type II: $\|u(\cdot,t)\|_\infty (T-t) \to \infty$

  (Herrero, Velázquez 1996; Velázquez 2002)

Nonlinear heat equation I (Giga, Kohn 1985)

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times (0,T) \qquad (u \geq 0)$$

Self-similar solutions: $u(x,t) = (T-t)^{-\beta} U\big(\frac{x}{\sqrt{T-t}}\big)$,

$$-\Delta U + \frac{1}{2} y \cdot \nabla U = U^p - \beta U, \qquad \beta := \frac{1}{p-1} \qquad (*)$$

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If $p \leq p_S := \frac{n+2}{n-2}$ then $U \equiv \beta^\beta$ is the only positive bounded solution of $(*)$ $\Rightarrow$ $u(x,t) = \beta^\beta (T-t)^{-\beta}$ is the only self-similar solution with bounded blow-up profile

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- Not true for $p \in (p_S, p_L)$ (Lepin 1988-1990, Budd, Qi 1989)
- Yes for radial solutions if $p > p_L$ (Mizoguchi 2009)

Type II blow-up for $p > p_{JL}$ (Herrero, Velázquez 1994)

Singularity estimates via
Liouville for stationary or ancient solutions

(**ancient** solution = solution defined for $t \in (-\infty, T)$)

Nonlinear heat equation II (Giga, Kohn 1985-1989)

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times (0,T) \qquad \left(u \ge 0,\ 1 < p < \tfrac{n+2}{n-2}\right)$$

$$\Rightarrow \quad \|u(\cdot,t)\|_\infty \le C(T-t)^{-\beta},\ \ \beta = \tfrac{1}{p-1} \quad \text{(type I blow-up)}$$

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Proof:

$$U(y,s) = (T-t)^\beta u(x,t), \quad y = \frac{x}{(T-t)^{1/2}},\ \ s = -\log(T-t)$$

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Assume on the contrary $U(y_k, s_k) \to \infty$. Rescaling U and passing to the limit one obtains (using energy estimates) a positive solution of $-\Delta W = W^p$ in $\mathbb{R}^n$, contradiction with the Liouville theorem by Gidas, Spruck 1981.

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Merle, Zaag 1998: Liouville thm for bounded entire solutions of $(*)$

$\Rightarrow$ Liouville thm for **ancient** solutions of

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times (-\infty, T)$$

(satisfying suitable decay condition as $t \to -\infty$)

Benefits of Merle-Zaag approach:

- More information on the blow-up behavior
- No need of energy

Similar results for:

- vector valued heat equations without gradient structure

  (Merle, Zaag, Nouaili 2000-2010)
- wave equation $u_{tt} = u_{xx} + |u|^{p-1}u$

  (ancient solutions in a cone Merle, Zaag 2008)

Navier-Stokes equations II

(Koch, Nadirashvili, Seregin, Šverák 2009)

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Axi-symmetric case:

$$u(Rx, t) = Ru(x, t), \qquad R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Liouville theorem for ancient solutions:

If u is a bounded axi-symmetric solution in $\mathbb{R}^3 \times (-\infty, 0)$ and $|u(x,t)| \leq C(\sqrt{x_1^2 + x_2^2})^{-1}$ *then $u \equiv 0$.*

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Corollary: *Let u be an axi-symmetric solution in $\mathbb{R}^3 \times (0, T)$ with sufficiently decaying initial data. If* $|u(x,t)| \le C(T-t)^{-1/2}$ *then* $|u| \le M$. (no Type I BU)

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Giga, Miura 2010+: Axisymmetry replaced by unif. continuity of vorticity direction $\zeta = \omega/|\omega|$, $\omega = \operatorname{curl} u$

Scaling and information on stationary/ancient solutions have also been used in the study of singularities of the

- **Ricci flow** (Hamilton 1980-90's, Perelman 2002-3, ...)
- **mean curvature flow** (Huisken, Angenent, Velázquez, ...)

Scaling and nonexistence (or characterization) of non-constant harmonic maps $\mathbb{S}^2 \to \mathcal{N}$ have been used to prove regularity (or behavior of singularities) of

- **heat flow of harmonic maps**

$$u : \mathcal{M} \times (0, \infty) \to \mathcal{N}, \quad \dim \mathcal{M} = 2$$

(Struwe 1985; Chang, Ding, Ye 1992)

- **equivariant wave maps**

$$u : \mathbb{R}^2 \times \mathbb{R} \to \mathcal{N}, \quad \mathcal{N}\text{-surface of revolution}$$

(Struwe 2003; Krieger, Schlag, Tataru 2008)

Singularity estimates
via Liouville for entire solutions
and doubling

$$-\Delta u = u^p \qquad \text{in } \Omega \subset \mathbb{R}^n, \qquad u \geq 0 \qquad (*)$$

Singular solutions of $(*)$ exist iff $p \geq \frac{n}{n-2}$

$$u(x) = C_p |x|^{-2/(p-1)} \text{ if } p > \frac{n}{n-2}$$

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Liouville thm for classical entire solutions of $(*)$ is true if

$p < p_S := \frac{n+2}{n-2}$ (Gidas, Spruck 1981)

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Poláčik, Q., Souplet 2007:

Let $1 < p < p_S$. There exists $C = C(p,n) > 0$ such that any positive classical solution u of $(*)$ satisfies

$$u(x) \leq C(\operatorname{dist}(x, \partial\Omega))^{-2/(p-1)}.$$

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Idea: If $u_k(x_k)\mathrm{dist}(x_k, \partial\Omega_k)^{2/(p-1)} \to \infty$, find $\tilde{x}_k$ which have the same property, and, in addition, $u_k \leq 2u_k(\tilde{x}_k)$ on a sufficiently large neighbourhood of $\tilde{x}_k$. Rescale at $\tilde{x}_k$ and pass to the limit.

$$-\Delta u = u^p \qquad \text{in } \Omega \subset \mathbb{R}^n, \qquad u \geq 0 \qquad\qquad (*)$$

Singular solutions of $(*)$ exist iff $p \geq \frac{n}{n-2}$

$$u(x) = C_p|x|^{-2/(p-1)} \text{ if } p > \frac{n}{n-2}$$

Liouville thm for classical entire solutions of $(*)$ is true if
$p < p_S := \frac{n+2}{n-2}$ (Gidas, Spruck 1981)

Poláčik, Q., Souplet 2007:

Let $1 < p < p_S$. There exists $C = C(p,n) > 0$ such that any positive classical solution u of $(*)$ satisfies

$$u(x) \leq C\big(\mathrm{dist}(x, \partial\Omega)\big)^{-2/(p-1)}.$$

Similar results obtained for $-\Delta u = f(x, u, \nabla u)$,

problems with Δ_p, elliptic systems, estimates of ∇u, ...

Nonlinear heat equation III

$$u_t - \Delta u = u^p \quad \text{in } \Omega \times (0,T), \qquad u = 0 \;\text{ on } \partial\Omega \times (0,T) \quad (**)$$

$\Omega \subset \mathbb{R}^n$ smooth, $\quad 1 < p < p_S = \frac{n+2}{n-2}, \quad u \geq 0$

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If Liouville thms for entire solutions in $\mathbb{R}^n \times \mathbb{R}$ and $\mathbb{R}^n_+ \times \mathbb{R}$ are true then all classical solutions of $(**)$ satisfy

$$u(x,t) \leq C(\Omega, p)(t^{-1/(p-1)} + (T-t)^{-1/(p-1)})$$

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Giga, Kohn 1980's, Merle, Zaag 1998:

Estimates on the (final) blow-up rate only

- for Ω convex (to avoid blow-up on the boundary)
- with $C = C(u)$

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Liouville for classical entire solutions in $\mathbb{R}^n \times \mathbb{R}$ is true for

(i) $p < \frac{n(n+2)}{(n-1)^2}$ (Bidaut-Véron 1998)

(ii) $p < p_S$ if $u = u(r,t)$ (Poláčik, Q., Souplet 2006-2007)

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(ii) remains true for nodal solutions with bounded zero-number (Bartsch, Poláčik, Q. 2010+; elliptic case: Ni 1983)

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Related: $u_t - \Delta u = h(x_1)u^p$ (Poláčik, Q. 2005)

$u_t - \Delta(u^m) = u^p$ (Souplet 2009, Ammar, Souplet 2010)

Liouville theorems, scaling and doubling also used in the

- study of boundary singularities for the Dirichlet problem for $-\Delta u = u^p$ (Bidaut-Véron, Ponce, Véron 2007-2010)

- proof of new Liouville theorems for semilinear elliptic systems (Souplet 2009, Chen, Li 2009, Dancer, Wei, Weth 2010+)

- study of various elliptic equations involving p-Laplacian (Ávila, Brock 2008, Vétois 2009)

- study of singularities for conformal k-Hessian equation (Huang, Xu 2009)

Liouville, scaling and doubling also imply universal bounds for global solutions and have been used in the proofs of existence of nontrivial (e.g. periodic) solutions or the study of the threshold between global existence and blow-up.

A simple version of doubling appeared in Hu 1996.

Miscellaneous

Nonlinear boundary conditions

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = u^p \ \text{ on } \partial\Omega$$

$\Omega \subset \mathbb{R}^n, \quad p > 1, \quad u \geq 0$

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$p^* = \frac{n-1}{n-2}$ (Q., Reichel 2008)

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$\Rightarrow$ estimates on the blow-up rate for time-increasing solutions of the parabolic problem

No "parabolic" Liouville known: only Fujita-type results (nonexistence of solutions defined for $t \in (0, \infty)$)

requiring $p \leq \frac{n+1}{n}$ (Deng, Fila, Levine 1994)

Scaling and Liouville/rigidity theorems also used in the study of the asymptotic behavior of solutions of

- critical KdV equation

  $u_t + (u_{xx} + u^5)_x = 0$ (Martel, Merle 2000)

- critical Schrödinger equation

  $iu_t + \Delta u \pm |u|^{4/(n-2)} u = 0$ (Kenig, Merle 2006)

- 4D Yang-Mills equations / 2D corotational wave maps

  $u_{tt} - u_{rr} - \frac{1}{r} u_r = -\frac{f(u)}{r^2}$ (Côte, Kenig, Merle 2008)

- 3D Navier-Stokes equations (Kenig, Koch 2010+):

  mild solutions bounded in $\dot{H}^{1/2}$ are regular;

  similarly for $iu_t + \Delta u - |u|^2 u = 0$ in $\mathbb{R}^3$

  (Kenig, Merle 2010)