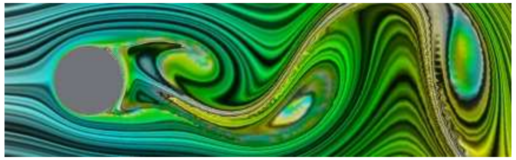
On Generalized Newtonian Fluids

Michael Růžička

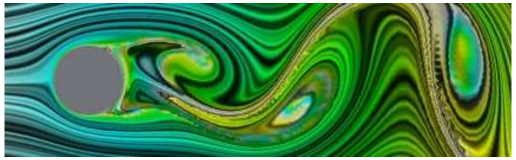
workshop in honour of J. Nečas

Prague, December 2009

ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG



1. Modeling



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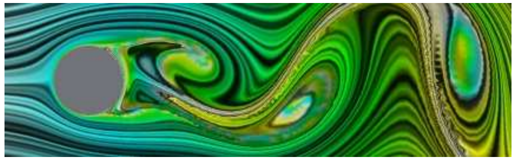
- motion of incompressible viscous fluids is described by

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div} \mathbf{S} + [\nabla \mathbf{u}]\mathbf{u} + \nabla \pi &= \mathbf{f} \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}$$



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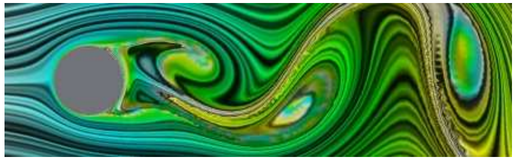

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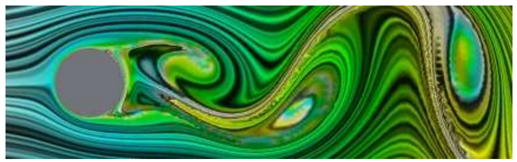
\end{aligned}
$$

- material properties $\Longleftrightarrow$ stress tensor $\mathbf{S}$

- viscous fluids

$$
\mathbf{S} = \mathbf{S}(\mathbf{D}) \,, \qquad \mathbf{D} = \tfrac{1}{2}\Big(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\Big)
$$



Generalized Newtonian Fluids (GNF)

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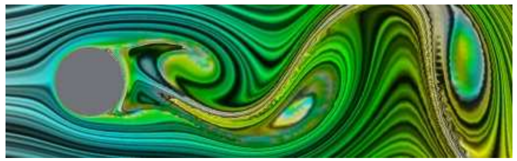
$$\mathbf{S} = \mathbf{S}(\mathbf{D}) = \boldsymbol{\mu}(|\mathbf{D}|)\mathbf{D}$$



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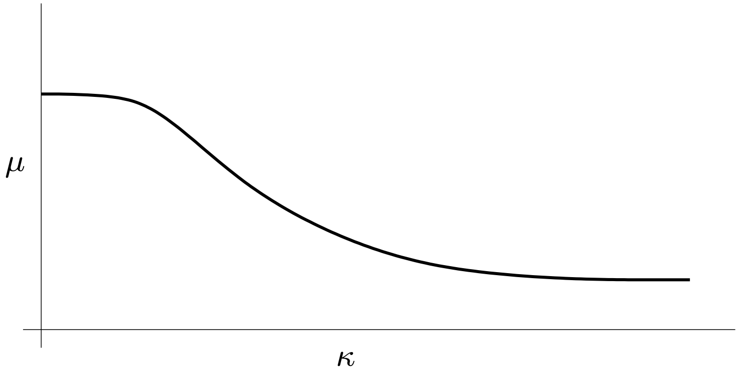

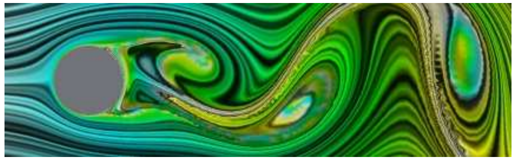

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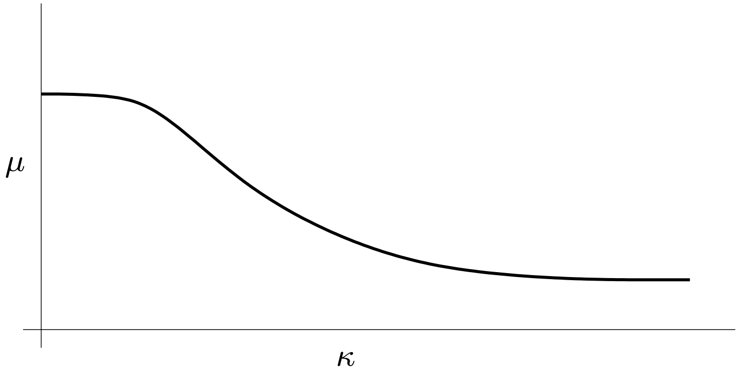
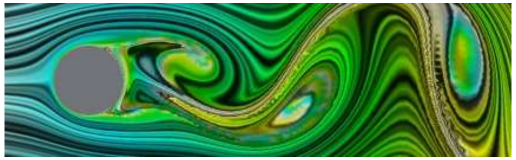


$$\mu(\kappa) = \mu_0 + \mu_1(\delta + \kappa)^{p-2}, \quad 1 < p \leq 2$$



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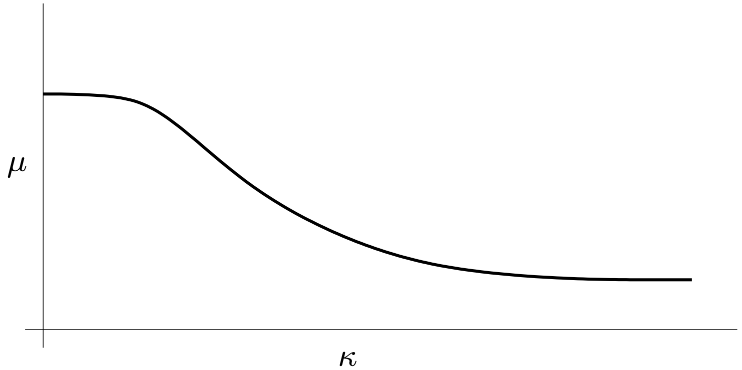

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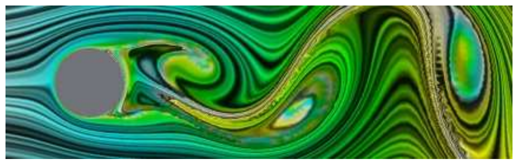


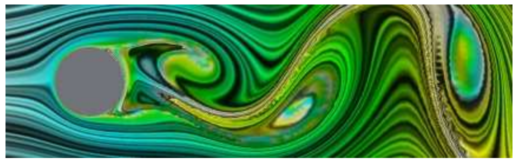
Stress Tensor

∘ we say that $\mathbf{S}$ has $(\boldsymbol{p}, \boldsymbol{\delta})$–structure if

$$\sum_{i,j,k,l=1}^{d} \partial_{kl} S_{ij}(\mathbf{A}) C_{ij} C_{kl} \geq C_0 \big(\boldsymbol{\delta} + |\mathbf{A}|\big)^{\boldsymbol{p}-2} |\mathbf{C}|^2,$$

$$\big|\partial_{kl} S_{ij}(\mathbf{A})\big| \leq C_1 \big(\boldsymbol{\delta} + |\mathbf{A}|\big)^{\boldsymbol{p}-2} |\mathbf{A}|$$

is satisfied for all $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{d \times d}_{\text{sym}}$



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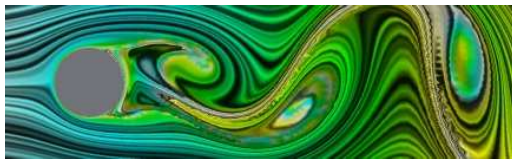
$$\mathbf{F}(\mathbf{A}) := \big(\boldsymbol{\delta} + |\mathbf{A}|\big)^{\frac{\boldsymbol{p}-2}{2}} \mathbf{A}\,,$$



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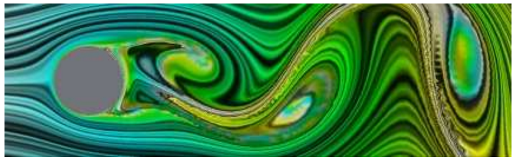
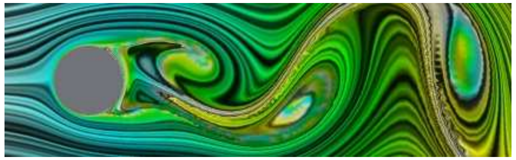
$$\mathbf{F}(\mathbf{A}) := \Big(\boldsymbol{\delta} + |\mathbf{A}|\Big)^{\frac{\boldsymbol{p}-2}{2}} \mathbf{A}\,,$$

- consequently we have

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \simeq |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^{\mathbf{2}}$$



Herschel-Bulkley Fluids (HBF)



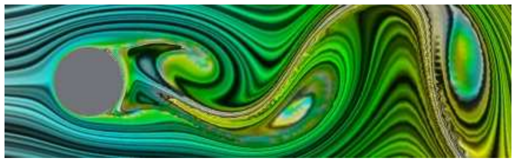
Herschel-Bulkley Fluids (HBF)

- generalization of Bingham fluids

$$\mathbf{S} = \mathbf{T}(\mathbf{D}) + \mathbf{P}\,, \quad \mathbf{P} = \tau^* \frac{\mathbf{D}}{|\mathbf{D}|} \qquad \text{if} \quad \mathbf{D} \neq \mathbf{0}\,,$$

$$|\mathbf{S}| \le \tau^* \qquad \text{if} \quad \mathbf{D} = \mathbf{0}$$

- $\mathbf{T}$ has $(\boldsymbol{p}, \boldsymbol{\delta})$-structure



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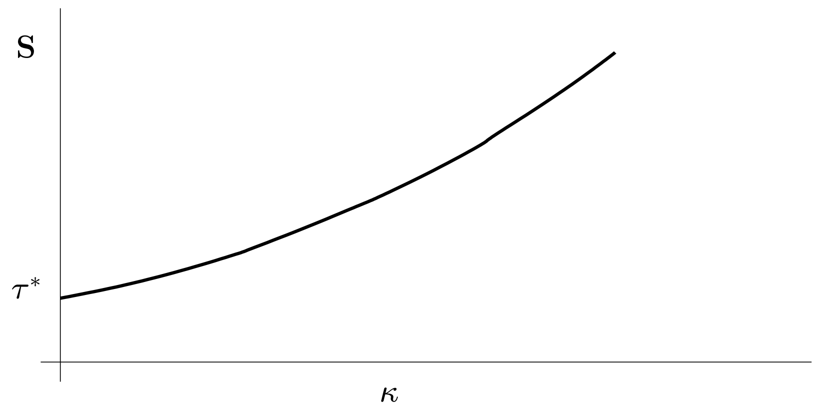
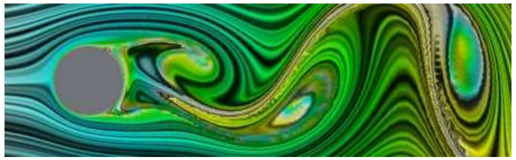
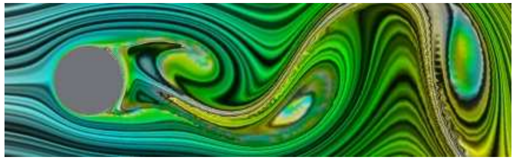
- $\mathbf{T}$ has $(\boldsymbol{p}, \boldsymbol{\delta})$-structure
- typical picture for stress vs. shear rate





Electrorheological Fluids (ERF)



Electrorheological Fluids (ERF)

- viscosity

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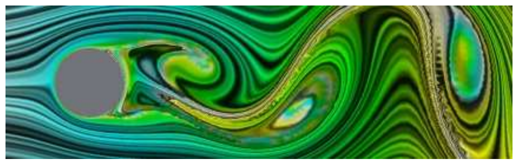
∘ viscosity $\Longleftrightarrow$ electric field



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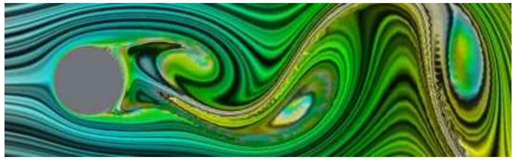
- viscosity $\Longleftrightarrow$ electric field

  electric field $\sim$ 1000 V/mm



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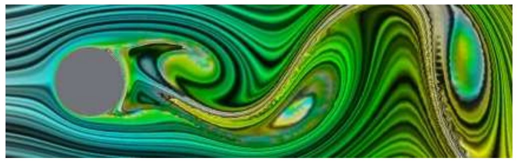

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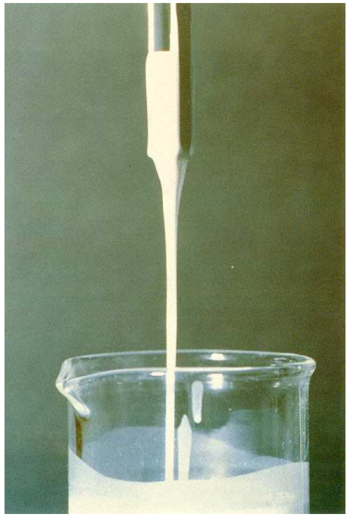

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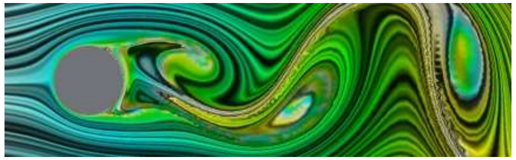

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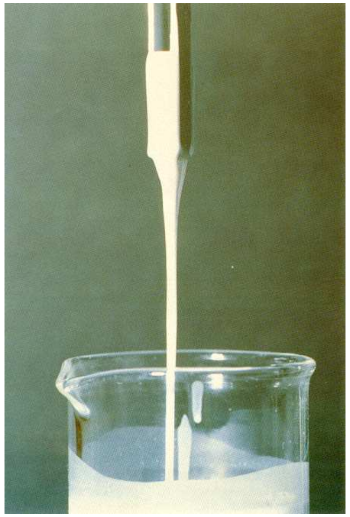

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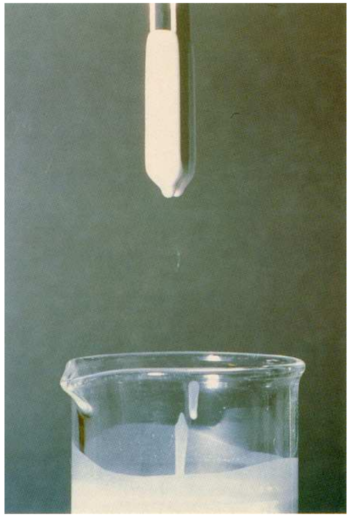

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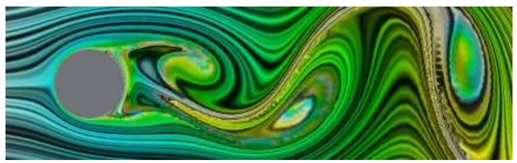

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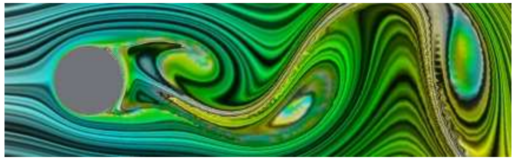

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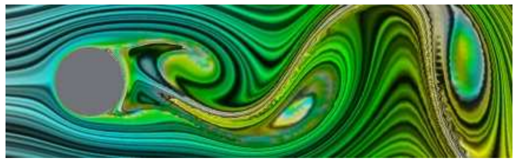

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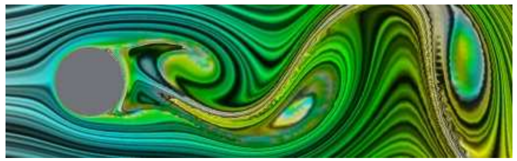

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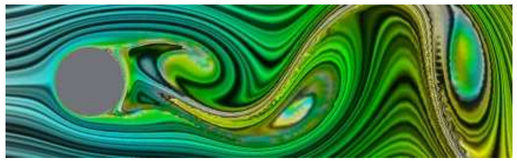

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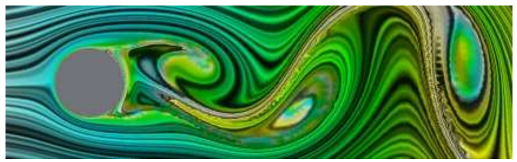

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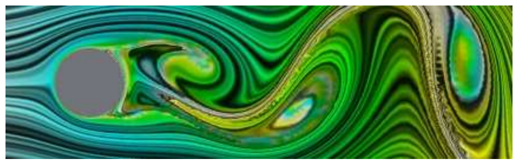

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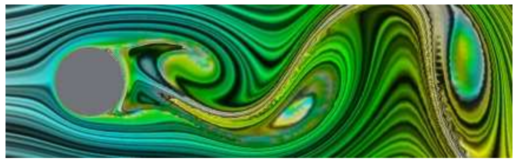

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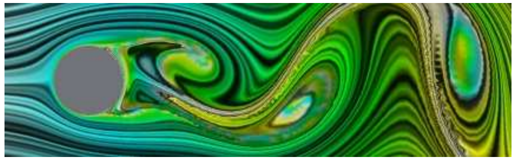

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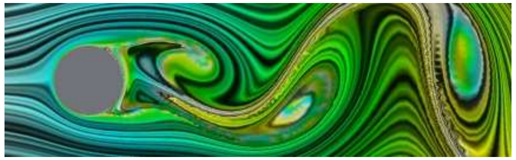

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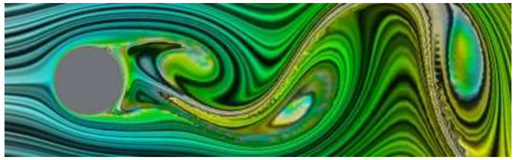
$\boldsymbol{p}$ – structure



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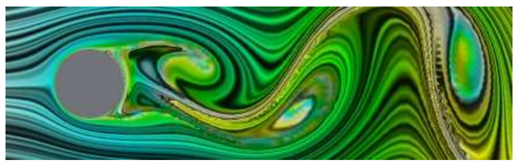
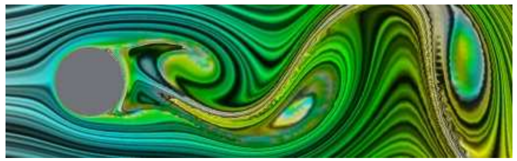
$$\boldsymbol{p} - \text{structure} \quad \Longleftrightarrow \quad \text{convective term } [\nabla \mathbf{u}]\mathbf{u}$$



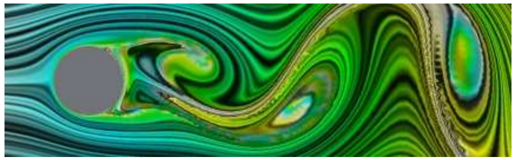
Existence of Solutions



Existence of Solutions

◦ energy estimate for approximate solutions $\mathbf{u}^n$

$$\sup_{t\in(0,T)} \|\mathbf{u}^n(t)\|_2^2 + \int_0^T \int_\Omega |\mathbf{D}(\mathbf{u}^n)|^p \, dx \, dt \le c$$



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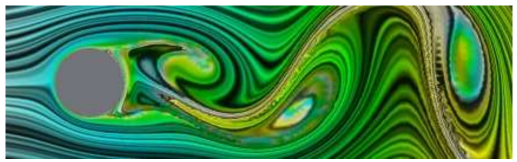
- energy estimate for approximate solutions $\mathbf{u}^n$

$$\sup_{t\in(0,T)} \|\mathbf{u}^n(t)\|_2^2 + \int_0^T \int_\Omega |\mathbf{D}(\mathbf{u}^n)|^p \, dx \, dt \le c$$

- identification of limits in nonlinear terms



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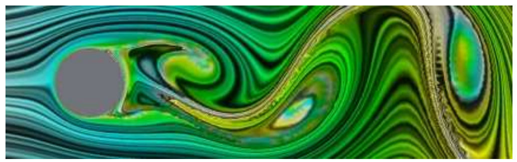

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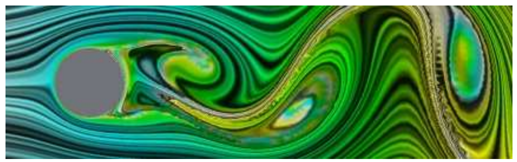

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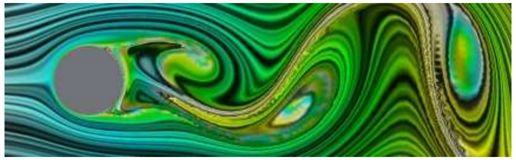

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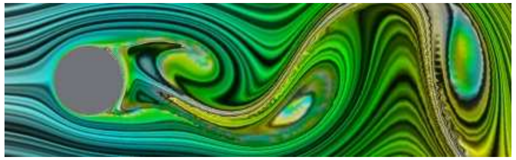
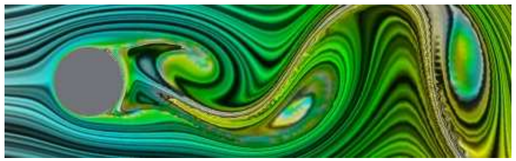
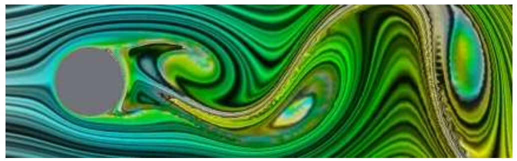


Theory of Monotone Operators



Theory of Monotone Operators

- $\mathbf{S}(\mathbf{D})$ defines a coercive, monotone operator



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- $\mathbf{S}(\mathbf{D})$ defines a coercive, monotone operator
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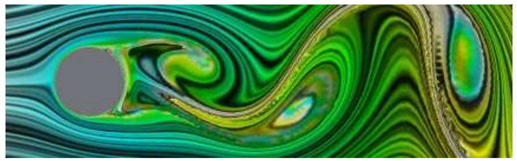
$$-\operatorname{div}\mathbf{S}\colon L^p(I,W^{1,p}(\Omega)) \to L^{p'}(I,W^{-1,p'}(\Omega))$$



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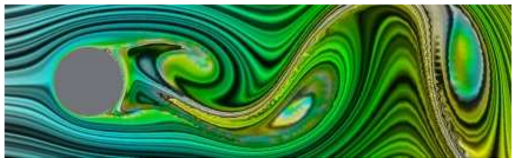

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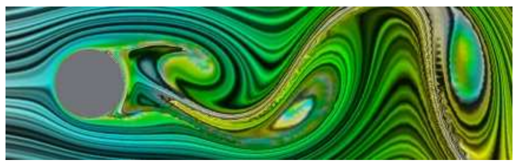

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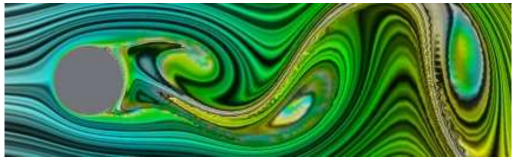
  - existence of global weak solutions for $p \geq \frac{11}{5}$ (unsteady, Dir. b.c.)



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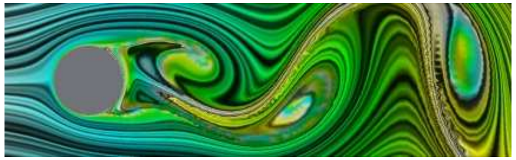

Theorem of Vitali

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◦ identification of the limit

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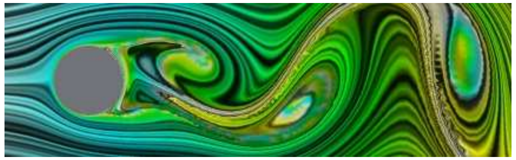

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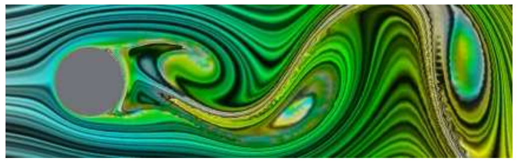

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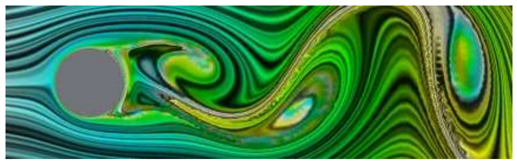
  - integrals are absolutely equicontinuous



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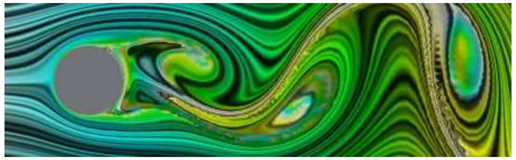

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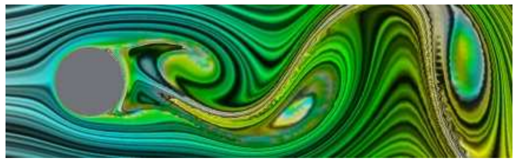

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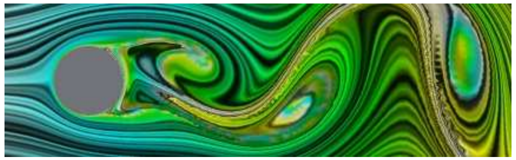
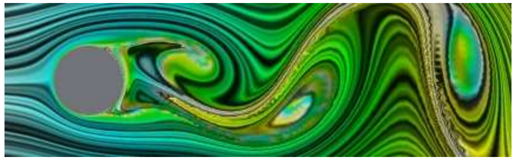


Stress Tensor II

◦ for $\mathbf{S}$ with $(\boldsymbol{p}, \boldsymbol{\delta})$–structure we have

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \simeq |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2$$



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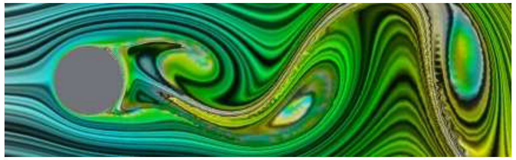

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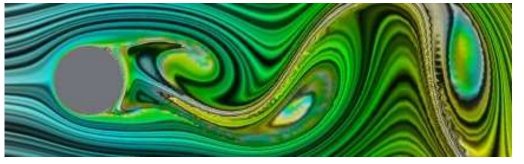

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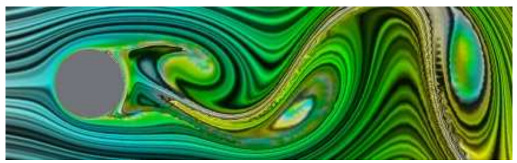

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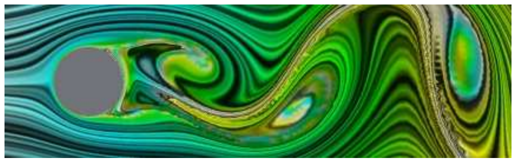
$$

- regularity for problems with $(\boldsymbol{p}, \boldsymbol{\delta})$–structure can be expressed as

$$
\nabla \mathbf{F}(\mathbf{D}(\mathbf{u}))\,,\ \partial_t \mathbf{F}(\mathbf{D}(\mathbf{u})) \in L^{\mathbf{2}}
$$



Testing with $-\Delta u$

Testing with $-\Delta \mathbf{u}$

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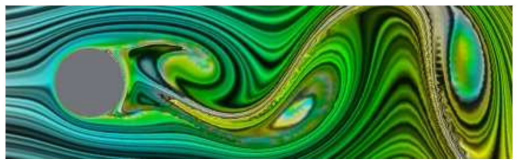

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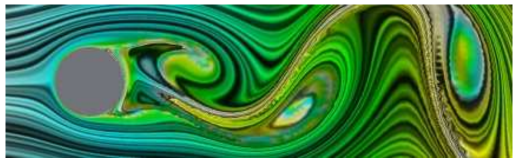

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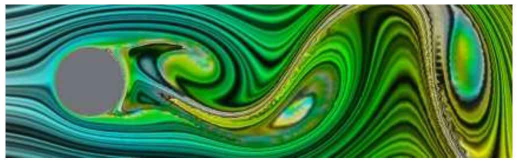

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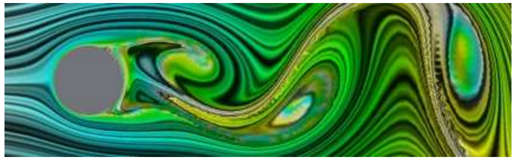

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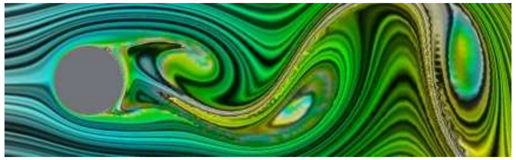

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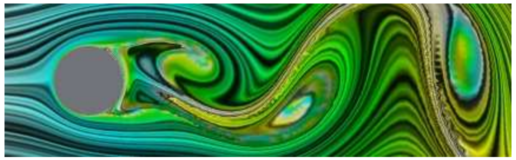

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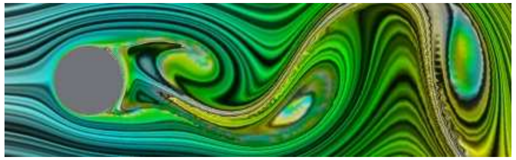
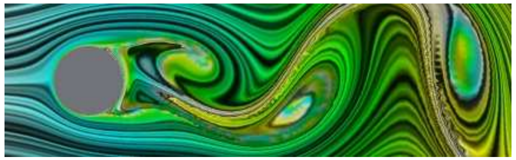


Testing with $-\Delta\mathbf{u}$ and $-\partial_t^2\mathbf{u}$



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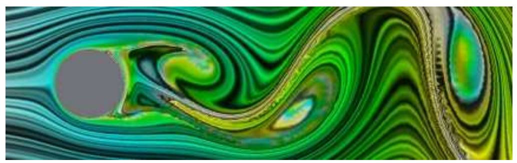

Testing with $-\Delta \mathbf{u}$ and $-\partial_t^2 \mathbf{u}$

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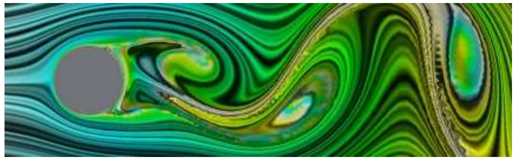

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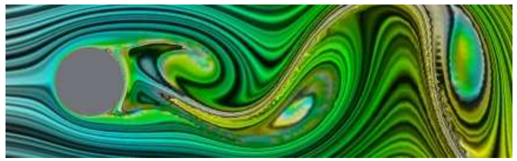
$$\int_0^T \int_\Omega |\nabla \mathbf{u}|^3 \, dx \, dt \,, \qquad \int_0^T \int_\Omega |\nabla \mathbf{u}| \, |\partial_t \mathbf{u}|^2 \, dx \, dt$$



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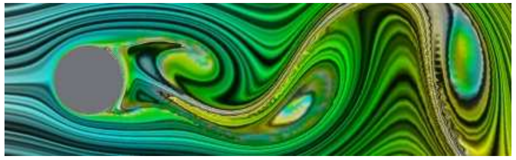

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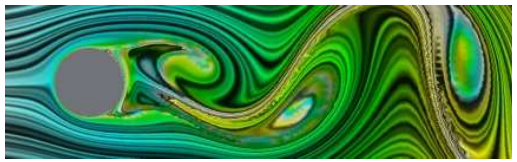
- different approach using semigroup theory (Bothe, Prüss 2007)
  - existence of local strong solutions for $\boldsymbol{p \geq 1}$ (unsteady, Dir. b.c.)
    $\boldsymbol{\delta > 0}$, $\mathbf{u} \in \boldsymbol{W_q^{2,1}(Q_T)}$, $q > d+2$



Problem Revisited

- consider for simplicity steady case with $\mathbf{f} = \mathbf{0}$

$$\int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{u}^n)) \cdot \mathbf{D}(\boldsymbol{\varphi})\,dx = -\int_\Omega [\nabla \mathbf{u}^n]\mathbf{u}^n \cdot \boldsymbol{\varphi}\,dx$$

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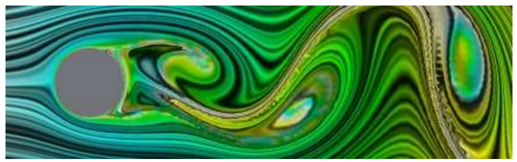

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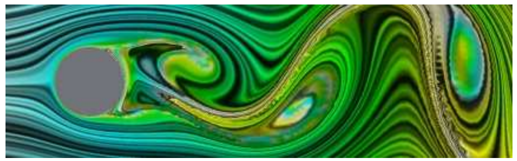
$$[\nabla \mathbf{u}]\mathbf{u} \in \left(W_0^{1,p}(\Omega)\right)^* \;\&\; \text{compact} \qquad \Longleftrightarrow \qquad p \ge \frac{9}{5}$$



Problem Revisited

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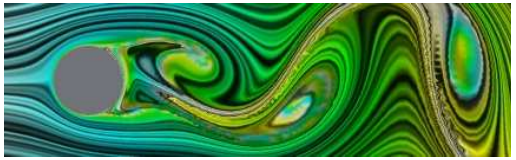

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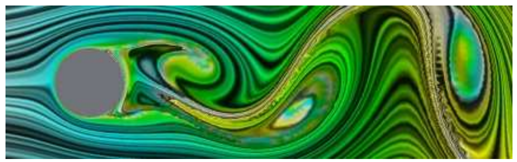

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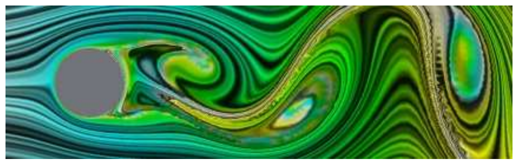

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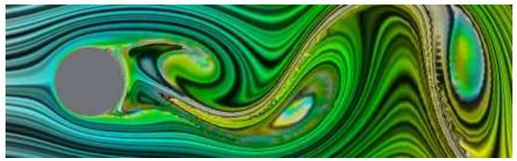
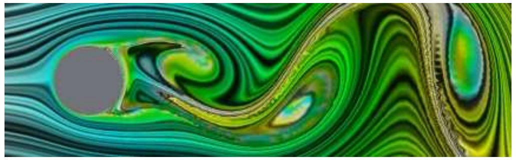
- strict monotonicity implies

$$\mathbf{D}(\mathbf{u}^n) \to \mathbf{D}(\mathbf{u}) \qquad \text{a.e.}$$



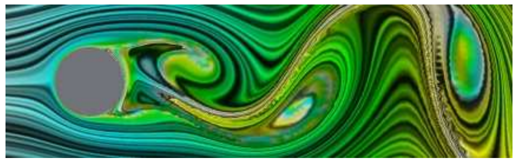
Problem Revisited II



Problem Revisited II

(B) $L^\infty-$ test functions

$$[\nabla \mathbf{u}^n]\mathbf{u}^n \in L^1(\Omega) \ \ \& \ \ \text{compact} \qquad \Longleftrightarrow \qquad p \geq \frac{3}{2}$$



Problem Revisited II

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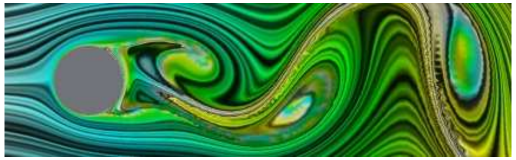

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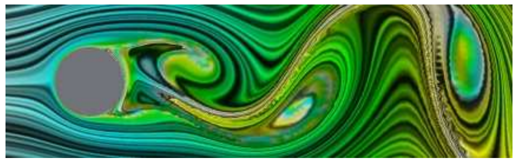

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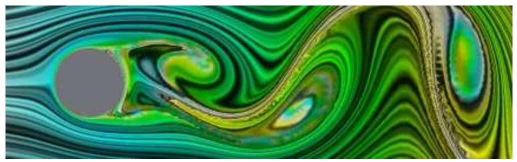

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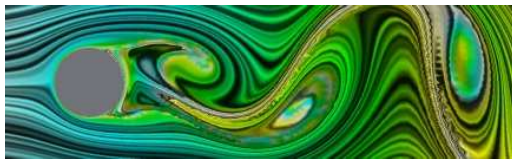

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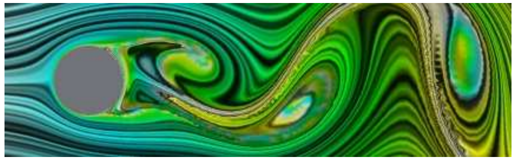

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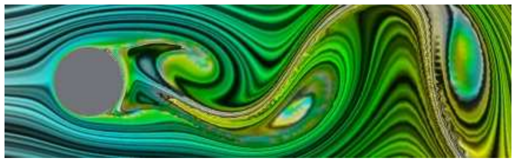

Problem Revisited II

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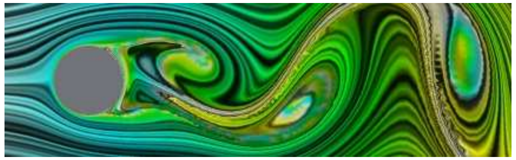
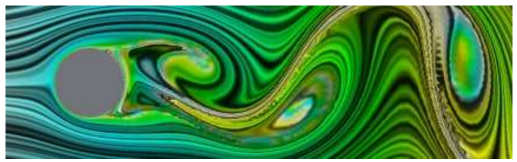


Problem Revisited II

- we need divergence–free test function



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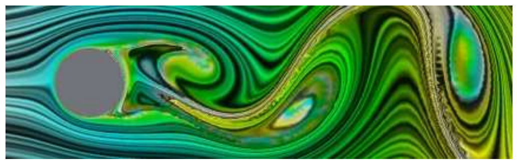

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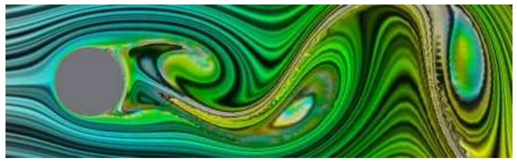
- testing equation of motion with $\boldsymbol{\psi_n(\mathbf{u}^n - \mathbf{u}) - \nabla \mathbf{h}_n}$



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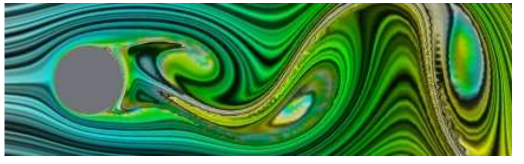
- testing equation of motion with $\psi_n(\mathbf{u}^n - \mathbf{u}) - \nabla \mathbf{h}_n$
- results (Frehse, Málek, Steinhauer, Rů. 1997, 1999)
  - existence of global weak solutions for $p > \frac{8}{5}$ (unsteady, per. b.c.)



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- results (Frehse, Málek, Steinhauer, Rů. 1997, 1999)
  - existence of global weak solutions for $\boldsymbol{p > \frac{8}{5}}$ (unsteady, per. b.c.)
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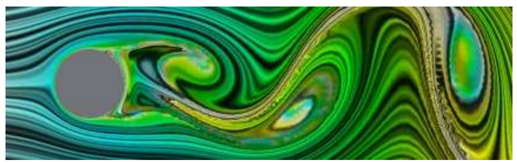

Problem Revisited II

- we need divergence–free test function

$$-\Delta \mathbf{h}_n = \text{div}\left(\psi_n(\mathbf{u}^n - \mathbf{u})\right) \quad \text{in } L^p(\Omega)$$

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- local pressure method (Wolf 2007)
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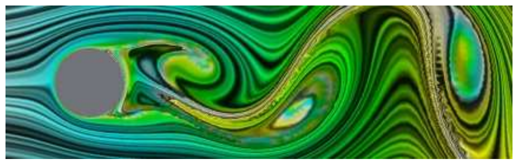
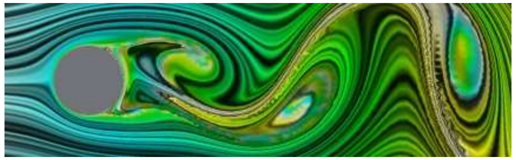


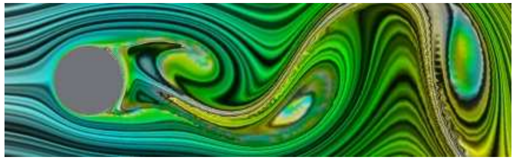
Problem Revisited III



Problem Revisited III

(C) $W^{1,\infty}$–test functions

$$\int_\Omega [\nabla \mathbf{u}]\mathbf{u} \cdot \boldsymbol{\varphi}\, dx = -\int_\Omega \mathbf{u} \otimes \mathbf{u} \cdot \nabla \boldsymbol{\varphi}\, dx$$



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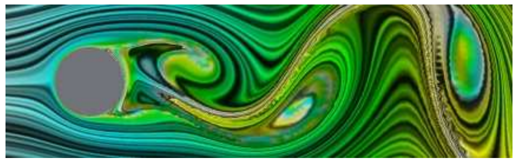

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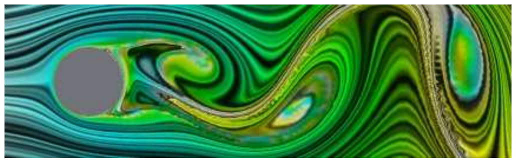

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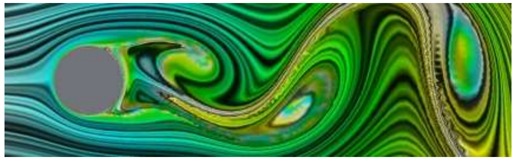

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$$|A_{j,n}| := \Big| \{ \mathbf{T}_{\lambda_j}(\mathbf{u}^n - \mathbf{u}) \neq \mathbf{u}^n - \mathbf{u} \} \Big| \leq 2^{-j} \quad \forall j, n$$

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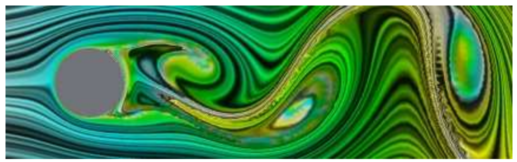

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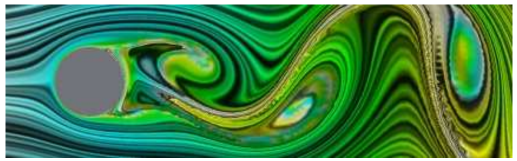
- maximal operator, Whitney decomposition and extension



Problem Revisited III

- recall $A_{j,n} := \{\mathbf{T}_{\lambda_j}(\mathbf{u}^n - \mathbf{u}) \neq \mathbf{u}^n - \mathbf{u}\}$

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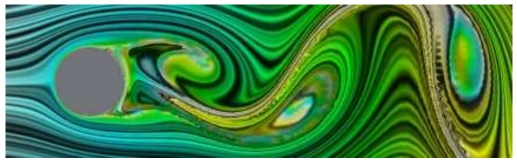

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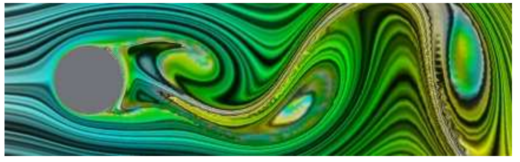

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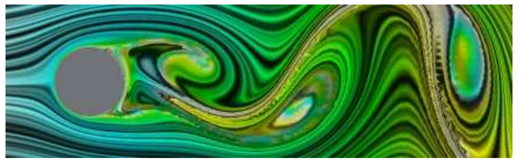

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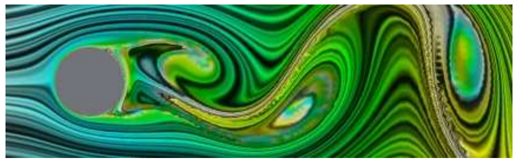

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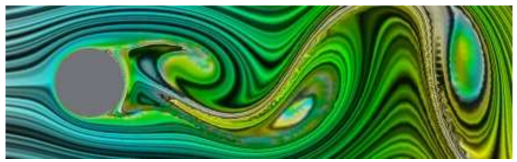
- use Bogovskii operator to get divergence–free test functions



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- use Bogovskii operator to get divergence–free test functions
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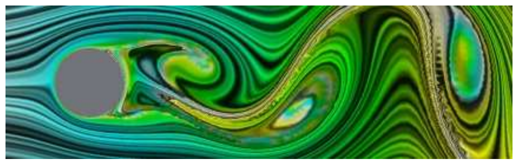
  - existence of global weak solutions for $\boldsymbol{p} > \frac{6}{5}$ (steady, Dir. b.c.)



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- localized version for arbitrary open domains (Diening, Rů., Schumacher 2009)
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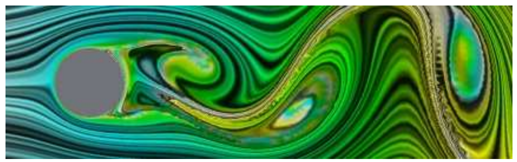

Problem Revisited III - Unsteady Problem

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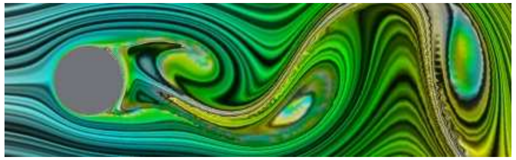
◦ different regularity in space and time

$$\begin{aligned} \mathbf{u}^n &\in L^{\infty}(I; L^{2}(\Omega)) \cap L^{p}(I; W_0^{1,p}(\Omega)) \\ \partial_t \mathbf{u}^n &\in L^p(I; (W_0^{1,p}(\Omega))^{*}) \end{aligned}$$



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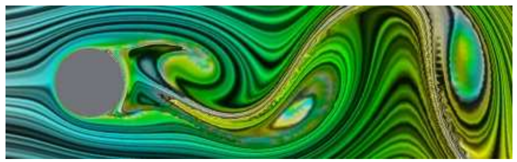
- adapt construction to scaled parabolic metric

$$d_\lambda\Big((t,x),(s,y)\Big) := \max\Big\{|x-y|, |\lambda^{p-2}(s-t)|^{1/2}\Big\}$$



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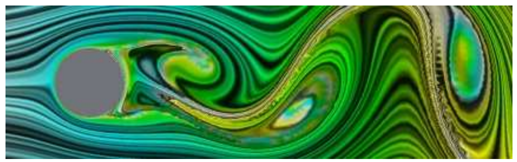
$\Rightarrow$ $\mathbf{T}_\lambda(\mathbf{v})$ is Lipschitz with respect to metric d_λ



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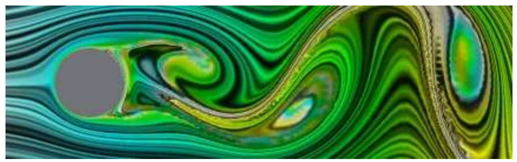

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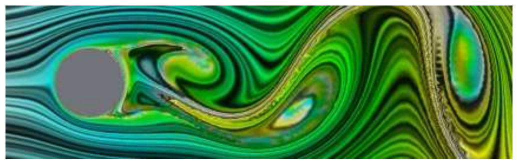
- handle partial integration with respect to time
- use local pressure method to handle divergence constraint



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- handle partial integration with respect to time
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- result (Diening, Rů., Wolf 2008)
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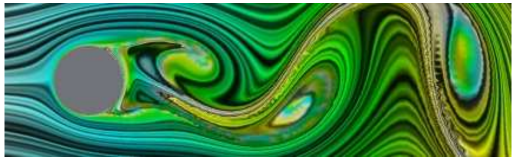

Analysis of Herschel-Bulkley Fluids (HBF)

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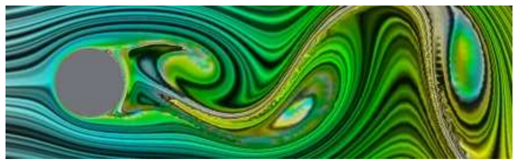
◦ extra stress tensor is given by

$$\mathbf{S} = \mathbf{T}(\mathbf{D}) + \mathbf{P}\,, \qquad \mathbf{P} = \tau^* \frac{\mathbf{D}}{|\mathbf{D}|} \qquad \text{if} \quad \mathbf{D} \neq \mathbf{0}\,,$$

$$|\mathbf{S}| \leq \tau^* \qquad \text{if} \quad \mathbf{D} = \mathbf{0}\,,$$

where $\mathbf{T}$ has $\boldsymbol{p}$-structure



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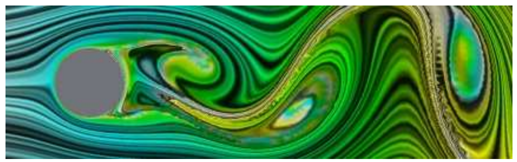
where $\mathbf{T}$ has $\boldsymbol{p}$-structure

◦ $\mathbf{S}$ is a coercive, maximal monotone operator with "jump" at $\mathbf{0}$



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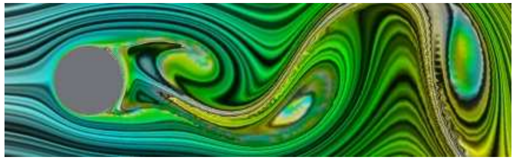
where $\mathbf{T}$ has $\boldsymbol{p}$-structure

- $\mathbf{S}$ is a coercive, maximal monotone operator with "jump" at $\mathbf{0}$
- key problem: identification of the limit

$$\lim_{n\to\infty} \int_0^T \int_\Omega \mathbf{S}^{\boldsymbol{n}}(\mathbf{D}(\mathbf{u}^{\boldsymbol{n}})) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx\, dt$$

for appropriate approximations $\mathbf{S}^{\boldsymbol{n}}$ and $\mathbf{u}^{\boldsymbol{n}}$



Existence Results for HBFs

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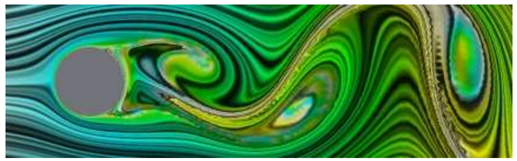
  - existence of global weak solutions for $p = 2$ (unsteady, Dir. b.c.)



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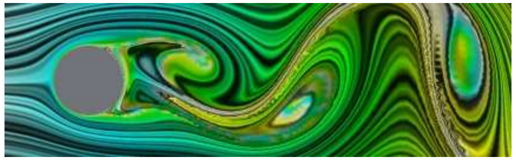

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- variational equality (Málek, Růžička, Shelukhin 2004)
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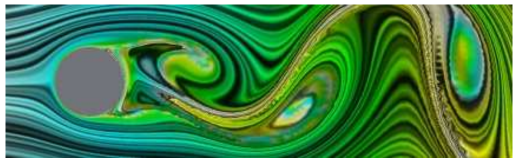

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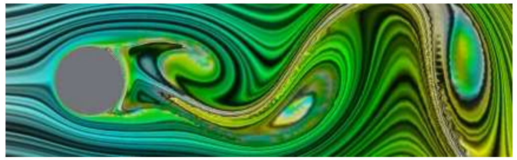
  - existence of a global weak solution couple $(\mathbf{u}, \mathbf{S})$ for $p > \frac{6}{5}$ (steady, Dir. b.c.), uses Lipschitz truncation method



Analysis of ERFs

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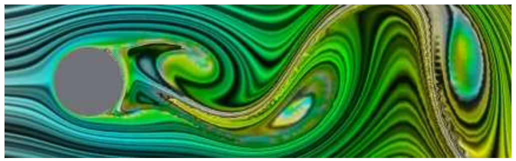
- motion of incompressible ERFs is described by

$$\partial_t \mathbf{u} - \operatorname{div} \mathbf{S} + [\nabla \mathbf{u}]\mathbf{u} + \nabla \pi = \mathbf{f} + \chi^E [\nabla \mathbf{E}]\mathbf{E}$$
$$\operatorname{div} \mathbf{u} = 0$$



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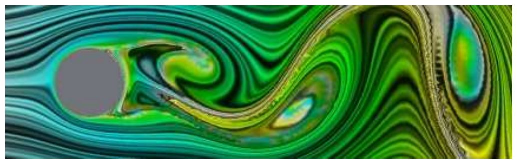
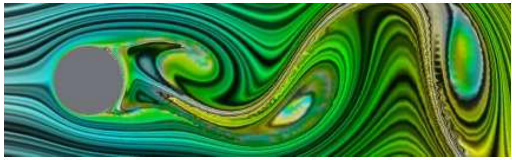
- stress tensor is given through

$$\mathbf{S} = \mathbf{S}(\mathbf{D}, \mathbf{E}) = \mu(1 + |\mathbf{E}|^2)\big(1 + |\mathbf{D}|\big)^{\boldsymbol{p}(|\mathbf{E}|^2)-2}\,\mathbf{D}\,,$$
$$\mu > 0\,, \quad \boldsymbol{p}(|\mathbf{E}|^2) \in [p^-, p^+]$$

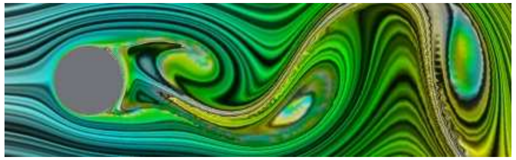


Existence of Solutions



Existence of Solutions

- $\mathbf{S}$ defines a monotone, coercive operator with $\boldsymbol{p}(|\mathbf{E}|^2)$-structure



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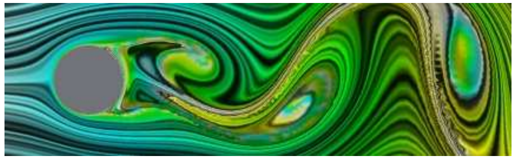

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$\Rightarrow$ consider spaces with variable exponents, e.g. $L^{p(\cdot)}(G)$ consisting of functions f with

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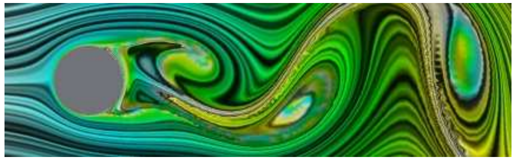

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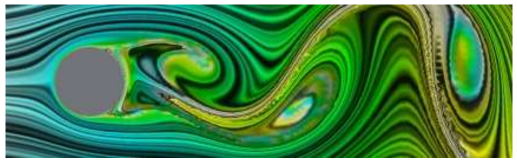
- key problem: identification of the limit

$$\lim_{n \to \infty} \int_0^T \int_\Omega \mathbf{S}(\mathbf{D}(\mathbf{u}^n), \mathbf{E}) \cdot \mathbf{D}(\varphi) \, dx \, dt$$



Monotone Operators: Steady Problem

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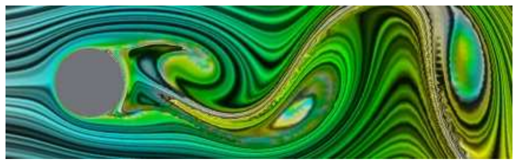
◦ $W^{1,p(\cdot)}(\Omega)$ is separable, reflexive Banach space



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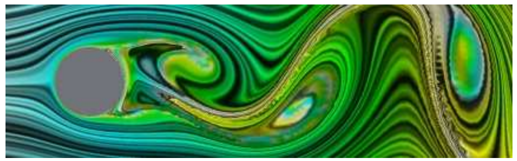

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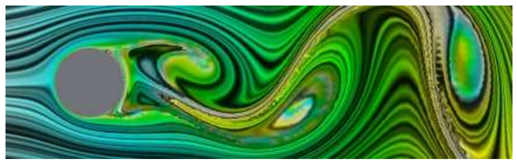

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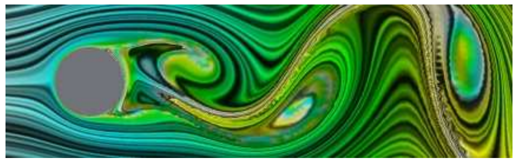

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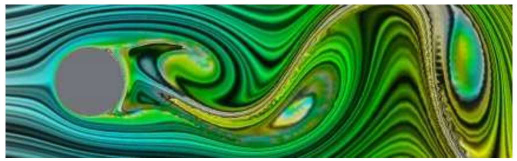

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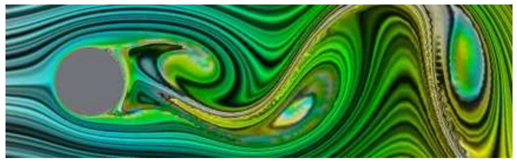
$\Rightarrow$ existence of global weak solutions for $p^- \ge \frac{9}{5}$ (steady, Dir. b.c.) (Rů., Diening 2000–2003)



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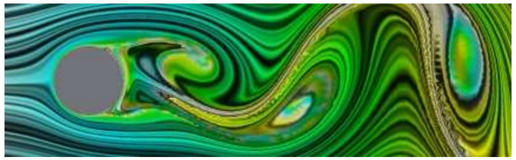

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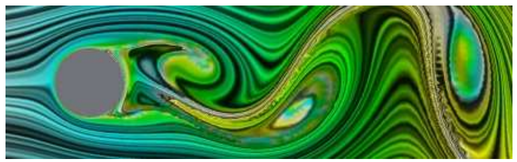

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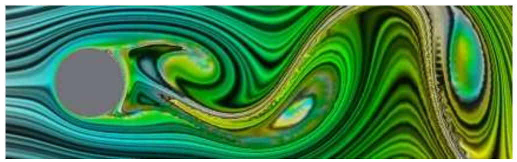

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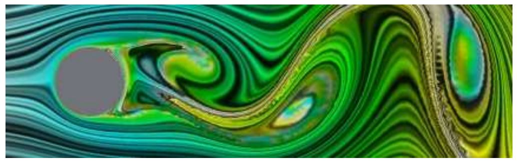

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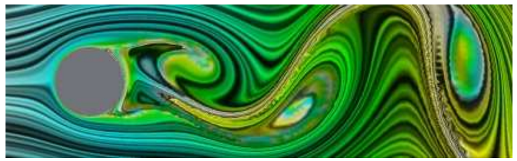

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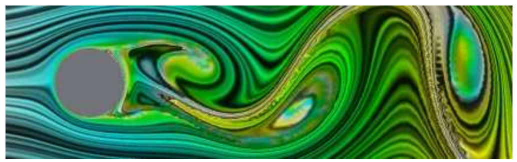

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- not clear if estimate for divergence equation $\operatorname{div} \mathbf{u} = \mathbf{f}$ holds, i.e.

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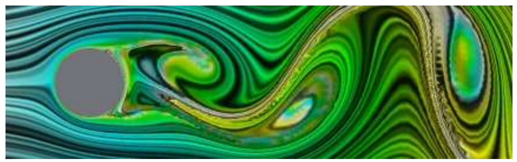
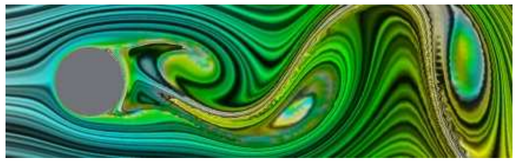


Parabolic System with $p(\cdot)$-Structure



Parabolic System with $p(\cdot)$-Structure

- for $\mathbf{S}(\nabla \mathbf{u})$ with $p(\cdot)$-structure we consider

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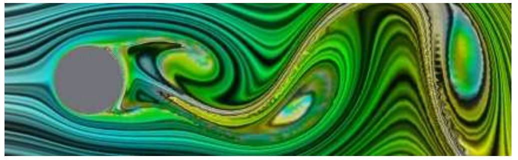

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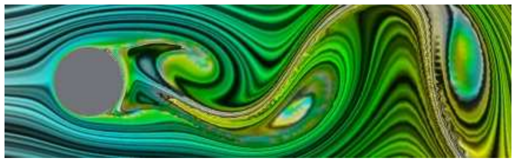

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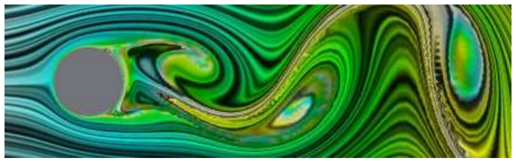

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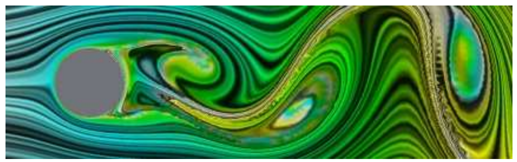

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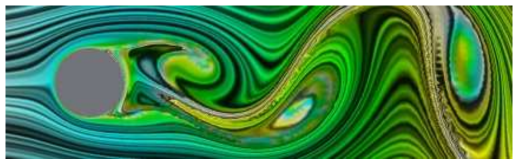
- results (Antontsev, Shmarev 2006, Nägele 2009)
  - existence of global weak solutions for $p^- > 1$ (unsteady, Dir. b.c.)



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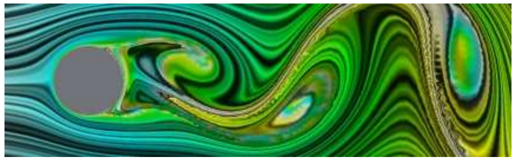
- results (Antontsev, Shmarev 2006, Nägele 2009)
  - existence of global weak solutions for $p^- > 1$ (unsteady, Dir. b.c.)
  - works also for pseudomonotone problems



ERFs: Unsteady Problem

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- testing with $-\boldsymbol{\Delta}\mathbf{u}$ yields:

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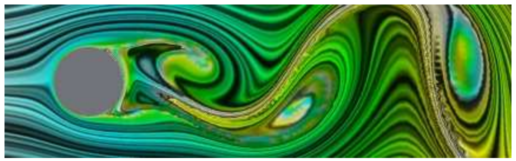

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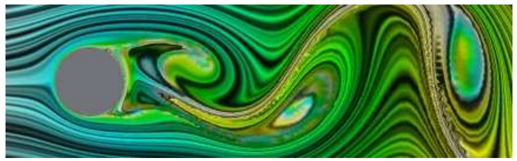

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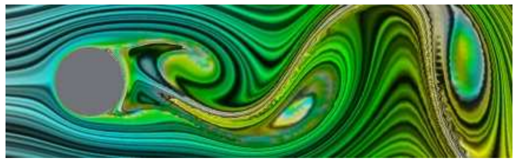
- we have to estimate

$$\int_0^T \int_\Omega |\nabla \mathbf{u}|^3 \, dx \, dt$$



ERFs: Unsteady Problem

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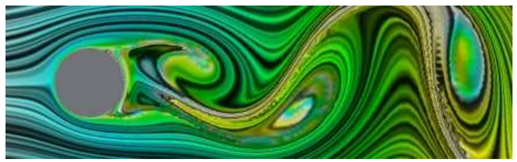

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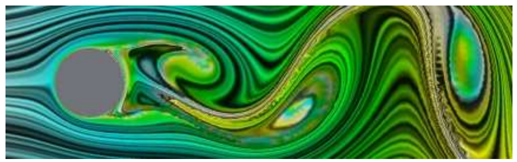

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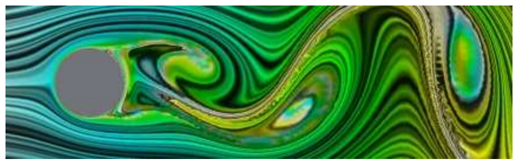
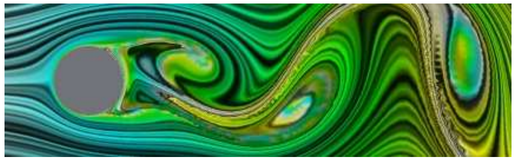


ERFs: Steady Problem



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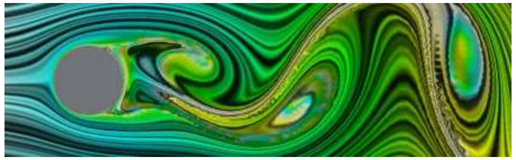

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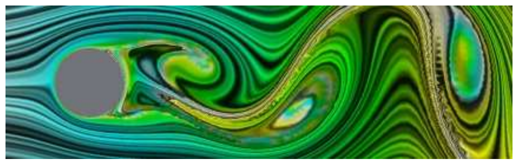

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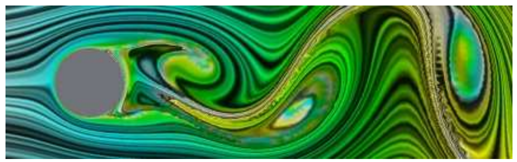

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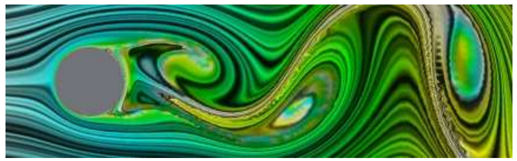

3. Numerical Analysis

- semi–implicit time discretization of governing equations read:

$$\begin{aligned}
\frac{1}{k}\left(\mathbf{u}^m - \mathbf{u}^{m-1}\right) - \operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u}^m)) + [\nabla \mathbf{u}^m]\mathbf{u}^{m-1} + \nabla \pi^m &= \mathbf{f}^m \\
\operatorname{div} \mathbf{u}^m &= 0 \qquad (\mathsf{GN}_k) \\
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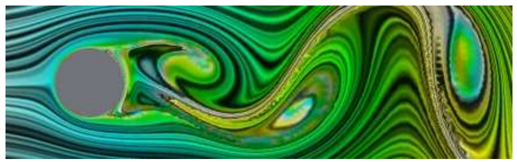

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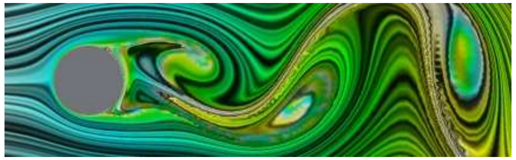

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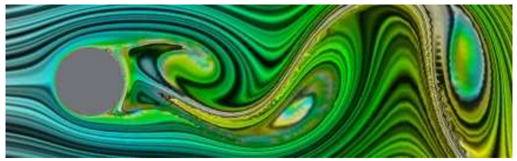
- special case $p = 2$ (NSE)

$$\max_{1\le m\le M}\|\mathbf{e}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}(\mathbf{e}^m)\|_2^2 \le c\,k^{\mathbf{2}}$$



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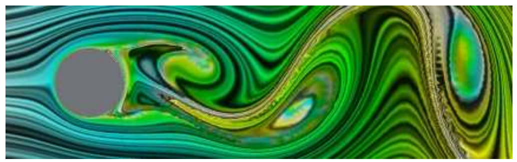

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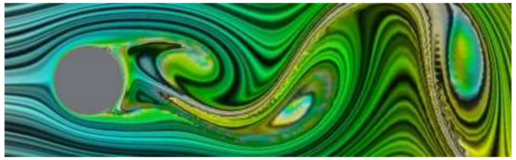

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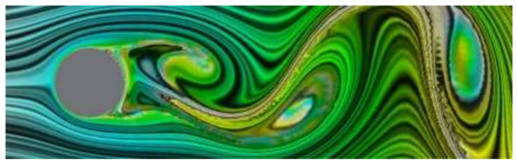
- convective term similarly



Convergence Results II (Berselli, Diening, Rů. 08)

∘ alltogether we get for $\mathbf{3/2 < p \leq 2}$

$$
\begin{aligned}
\max_{1\le m\le M} \|\mathbf{u}(t_m) - \mathbf{u}^m\|_2^2 + k \sum_{m=1}^{M} \Big\|\mathbf{F}(\mathbf{D}(\mathbf{u}(t_m))) - \mathbf{F}(\mathbf{D}(\mathbf{u}^m))\Big\|_2^2 \\
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+ c \sum_{m=1}^{M} \Big(\int_{\Omega} \Big| \fint_{I_m} \mathbf{u}\,ds - \mathbf{u}(t_m)\Big|^{p'} dx\Big)^{2/p'}
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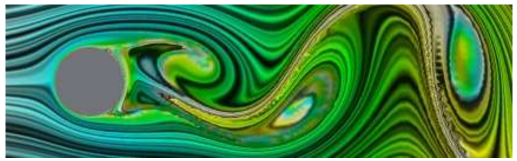

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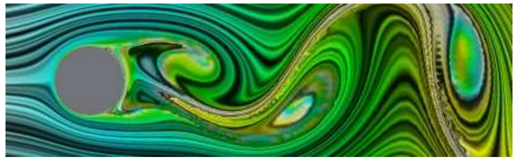
$$



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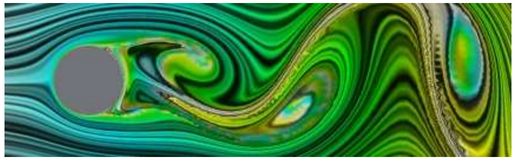

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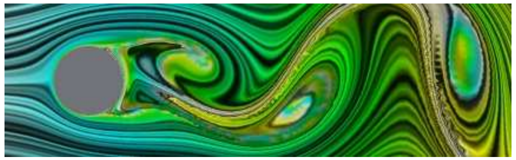

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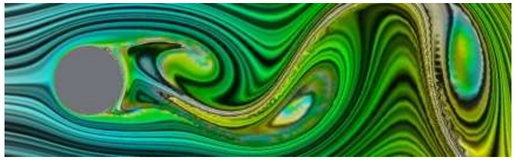
if

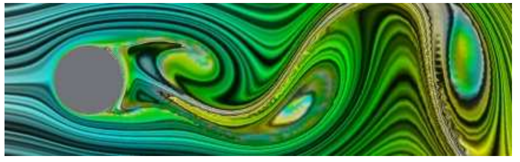
- $p > d/2$, Lagrange operator (Ebmeyer, Liu 03)
- $p > 1$ Scott–Zhang operator (Diening, Rů. 06)
  Orlicz spaces included



Convergence Result for p–Laplacian

◦ FEM approximation $\mathbf{u}_h$ with linear, conforming elements of

$$-\operatorname{div} \mathbf{S}(\nabla \mathbf{u}) = \mathbf{f}$$



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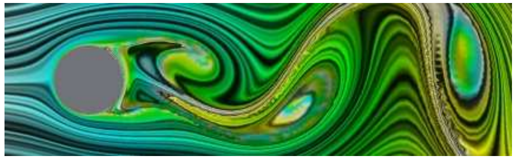

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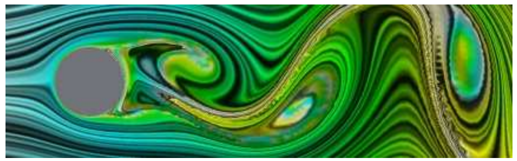

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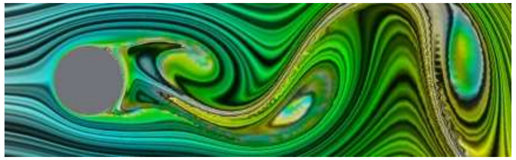
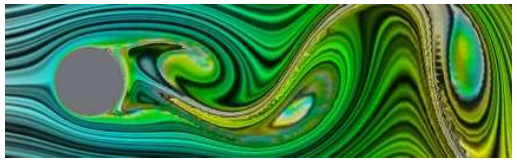


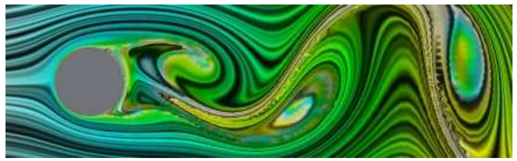
Fully Space–Time Discretization (Diening, Ebmeyer, Rů. 06)



Fully Space–Time Discretization (Diening, Ebmeyer, Rů. 06)

- for a parabolic system with p–structure

$$\partial_t \mathbf{u} - \operatorname{div} \mathbf{S}(\nabla \mathbf{u}) = \mathbf{f}$$



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we get for $p > \frac{6}{5}$:

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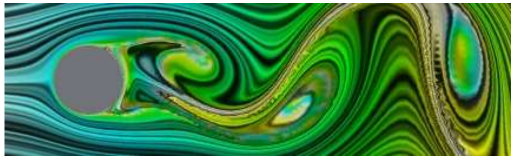
$$\le c \left(k^2 + h^2 \right)$$

provided that $h^{\beta(p)} \le c\, k$



p–Stokes System

- Stokes system with $\boldsymbol{p}$–structure

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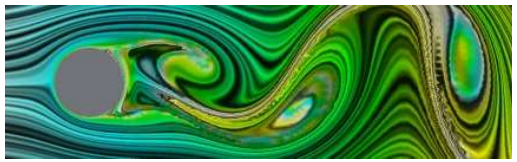
\begin{aligned}
-\operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u})) + \nabla \pi &= \mathbf{f} \\
\operatorname{div} \mathbf{u} &= 0
\end{aligned}
$$



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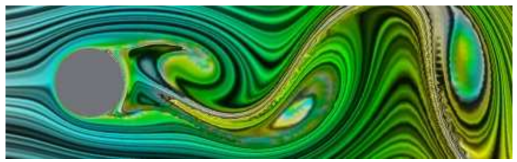
- FEM approximation $\mathbf{u}_h$ with linear, conforming, LBB-stable elements



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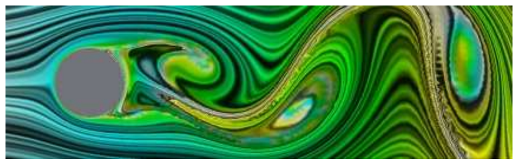
$$\|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h))\|_2 \le c \min_{\boldsymbol{\omega}\in V_h} \|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\boldsymbol{\omega}))\|_2$$



p–Stokes System

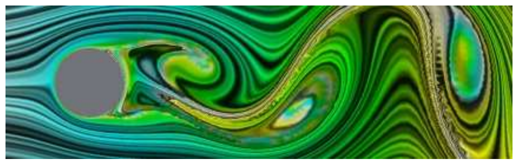
- Stokes system with $\boldsymbol{p}$–structure

$$
\begin{aligned}
-\operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u})) + \nabla \pi &= \mathbf{f} \\
\operatorname{div} \mathbf{u} &= 0
\end{aligned}
$$

- FEM approximation $\mathbf{u}_h$ with linear, conforming, LBB-stable elements

- best approximation estimate

$$
\begin{aligned}
\|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h))\|_2 &\leq c \min_{\boldsymbol{\omega} \in V_h} \|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\boldsymbol{\omega}))\|_2 \\
&\leq c \, \|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\pi_h \mathbf{u}))\|_2
\end{aligned}
$$



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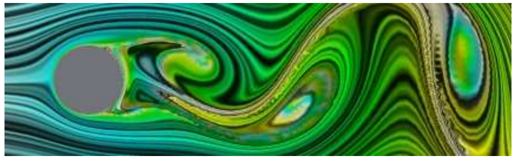
&\le c\,\|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\pi_h\mathbf{u}))\|_2 \\
&\le c\,\boldsymbol{h}\,\|\nabla\mathbf{F}(\mathbf{D}(\mathbf{u}))\|_2
\end{aligned}
$$



Results (Berselli, Diening, Rů. 09)

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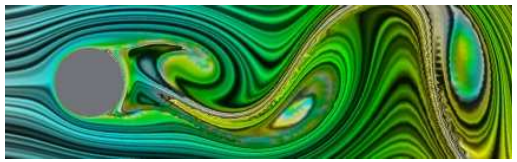
◦ in the case $p \le 2$ we have:

$$\Big\|\mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h))\Big\|_2^2 + \Big\|\pi - \pi_h\Big\|_{p'}^{p'}$$
$$\le c\, h^2 \left(\Big\|\nabla \pi\Big\|_{p'}^{p'} + \Big\|\mathbf{F}(\mathbf{D}(\mathbf{u}))\Big\|_2^2 + \Big\|\nabla \mathbf{F}(\mathbf{D}(\mathbf{u}))\Big\|_2^2 \right)$$



Results (Berselli, Diening, Rů. 09)

- in the case $p \le 2$ we have:

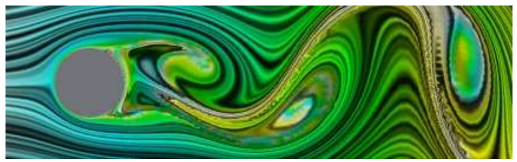
$$\Big\| \mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h)) \Big\|_2^2 + \Big\| \pi - \pi_h \Big\|_{p'}^{p'}$$

$$\le c\, h^2 \left(\Big\| \nabla \pi \Big\|_{p'}^{p'} + \Big\| \mathbf{F}(\mathbf{D}(\mathbf{u})) \Big\|_2^2 + \Big\| \nabla \mathbf{F}(\mathbf{D}(\mathbf{u})) \Big\|_2^2 \right)$$

- similarly we get for $p \ge 2$

$$\Big\| \mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h)) \Big\|_2^2 + \Big\| \pi - \pi_h \Big\|_{p'}^{2}$$

$$\le c\, h^{p'} \left(\Big\| \nabla \pi \Big\|_{p'}^{p'} + \Big\| \mathbf{F}(\mathbf{D}(\mathbf{u})) \Big\|_2^2 + \Big\| \nabla \mathbf{F}(\mathbf{D}(\mathbf{u})) \Big\|_2^2 \right)$$



Results (Berselli, Diening, Rů. 09)

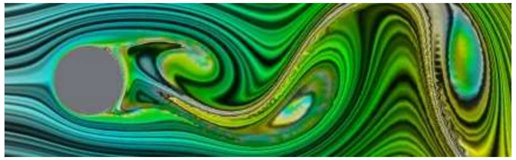
- in the case $p \leq 2$ we have:

$$\left\| \mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h)) \right\|_2^2 + \left\| \pi - \pi_h \right\|_{p'}^{p'} \leq c\, h^2 \left(\left\| \nabla \pi \right\|_{p'}^{p'} + \left\| \mathbf{F}(\mathbf{D}(\mathbf{u})) \right\|_2^2 + \left\| \nabla \mathbf{F}(\mathbf{D}(\mathbf{u})) \right\|_2^2 \right)$$

- similarly we get for $p \geq 2$

$$\left\| \mathbf{F}(\mathbf{D}(\mathbf{u})) - \mathbf{F}(\mathbf{D}(\mathbf{u}_h)) \right\|_2^2 + \left\| \pi - \pi_h \right\|_{p'}^{2} \leq c\, h^{p'} \left(\left\| \nabla \pi \right\|_{p'}^{p'} + \left\| \mathbf{F}(\mathbf{D}(\mathbf{u})) \right\|_2^2 + \left\| \nabla \mathbf{F}(\mathbf{D}(\mathbf{u})) \right\|_2^2 \right)$$

- similar results for local projection pressure stabilization (Hirn 09)

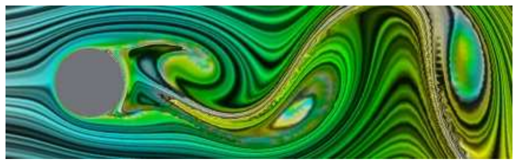


Numerical Tests for p-Laplace (P) with ALBERTA

$$\mathbf{u}_{sol} = |\mathbf{x}|^{\alpha-1} \begin{bmatrix} x_1 \\ -x_0 \end{bmatrix}, \delta = 0.00001, \Omega = (-1,1) \times (-1,1)$$

$$\mathbf{F}(\mathbf{D}(\mathbf{u}_{sol})) \in W^{1,2}(\Omega)$$

h \ p	1.1	1.2	$1.\overline{3}$	1.5	1.8	2.0	3.0	4.0	5.0	6.0
$1.76e-01$	0.84	0.85	0.84	0.84	0.84	0.84	0.86	0.88	0.90	0.93
$8.83e-02$	0.88	0.87	0.87	0.87	0.87	0.87	0.89	0.91	0.92	0.92
$4.41e-02$	0.90	0.89	0.89	0.89	0.89	0.89	0.91	0.92	0.94	0.95
$2.20e-02$	0.92	0.91	0.90	0.90	0.91	0.91	0.92	0.93	0.95	0.96
$1.10e-02$	0.93	0.92	0.92	0.92	0.92	0.92	0.93	0.94	0.95	0.89
$5.52e-03$	0.93	0.93	0.93	0.93	0.93	0.93	0.94	0.95	0.96	0.95
Theory	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

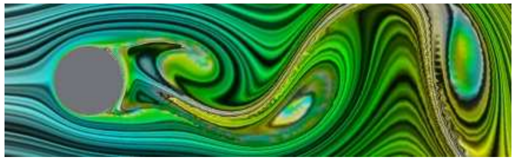


Numerical Tests for p-Stokes (S) with ALBERTA

$$\mathbf{u}_{sol} = |\mathbf{x}|^{\alpha-1} \begin{bmatrix} x_1 \\ -x_0 \end{bmatrix}, \pi_{sol} = |\mathbf{x}|^{\gamma}, \delta = 0.00001, \Omega = (-1,1) \times (-1,1)$$

$$\mathbf{F}(\mathbf{D}(\mathbf{u}_{sol})) \in W^{1,2}(\Omega), \pi_{sol} \in W^{1,p'}(\Omega)$$

p	1.1		1.2		1.5		1.8		2.0	
h	$\mathrm{eoc}_{\mathbf{u}}$	eoc_{π}	$\mathrm{eoc}_{\mathbf{u}}$	eoc_{π}	$\mathrm{eoc}_{\mathbf{u}}$	eoc_{π}	$\mathrm{eoc}_{\mathbf{u}}$	eoc_{π}	$\mathrm{eoc}_{\mathbf{u}}$	eoc_{π}
$1.76e-01$	0.85	0.89	0.86	1.00	0.85	1.06	0.86	0.98	0.85	1.24
$8.83e-02$	0.89	0.27	0.88	0.51	0.87	0.86	0.88	0.99	0.88	1.07
$4.41e-02$	0.91	0.21	0.90	0.41	0.89	0.82	0.90	0.99	0.90	1.05
$2.20e-02$	0.92	0.19	0.91	0.37	0.91	0.78	0.91	0.99	0.91	1.03
$1.10e-02$	0.93	0.19	0.92	0.35	0.92	0.75	0.92	0.99	0.92	1.02
$5.52e-03$	0.93	0.19	–	–	0.93	0.73	0.93	0.99	0.93	1.02
Theory	**1.0**	**0.18**	**1.0**	**0.33**	**1.0**	**0.66**	**1.0**	**0.88**	**1.0**	**1.0**

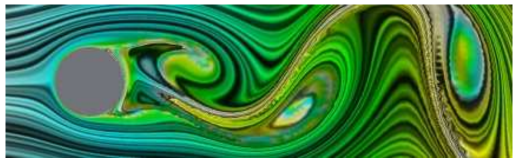


Numerical Tests for p-Stokes (S) with Gascoigne

$$\mathbf{u}_{sol} = |\mathbf{x}|^{\alpha-1} \begin{bmatrix} x_1 \\ -x_0 \end{bmatrix}, \pi_{sol} = |\mathbf{x}|^{\gamma}, \delta = 0.00001, \Omega = (-1,1) \times (-1,1)$$

$$\mathbf{F}(\mathbf{D}(\mathbf{u}_{sol})) \in W^{1,2}(\Omega), \pi_{sol} \in W^{1,p'}(\Omega)$$

p	2.0		3.0		4.0		5.0		6.0	
h	eoc_u	eoc_π	eoc_u	eoc_π	eoc_u	eoc_π	eoc_u	eoc_π	eoc_u	eoc_π
$1.76e-01$	0.89	1.01	0.50	1.01	0.35	1.01	0.26	1.00	0.20	0.98
$8.83e-02$	0.90	1.01	0.51	1.01	0.34	1.01	0.26	1.01	0.20	0.99
$4.41e-02$	0.91	1.01	0.51	1.01	0.34	1.01	0.26	1.02	0.20	1.00
$2.20e-02$	0.92	1.01	0.51	1.01	0.34	1.01	0.26	1.03	0.20	1.00
$1.10e-02$	0.93	1.01	0.51	1.01	0.34	1.01	0.27	1.04	0.20	1.01
$5.52e-03$	0.94	1.01	0.51	1.01	–	–	–	–	–	–
Theory	1.00	1.00	0.50	0.75	$0.\overline{3}$	$0.\overline{6}$	0.25	0.62	0.20	0.60



THANK YOU !