Higher order estimates for the curvature and nonlinear stability of stationary solutions for the curvature flow with triple junction

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joint work with Harald Garcke and Yoshihito Kohsaka

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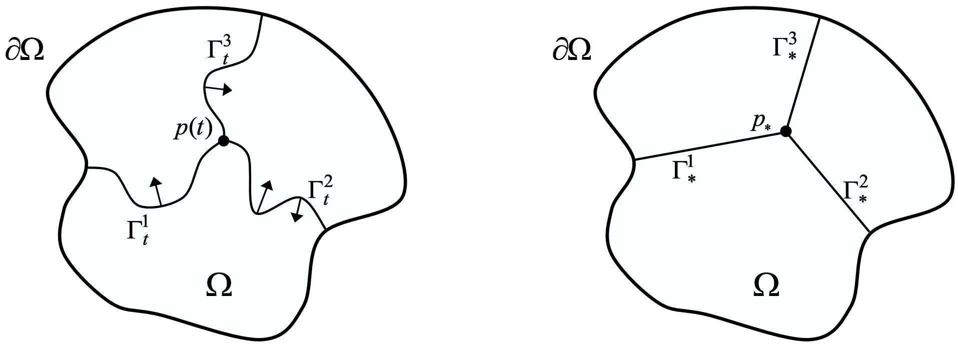
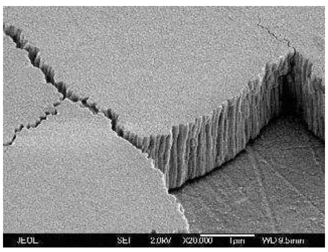
Introduction



Motion of the curvature driven flow Γ_t with the triple junction at $p(t)$ (left) and its steady state Γ_* (right).



$$\beta^i V^i = \gamma^i \kappa^i \quad \text{on} \quad \Gamma_t^i$$
$$\Gamma_t^i \perp \partial\Omega \quad i = 1, 2, 3$$
$$\sum_{i=1}^3 \gamma^i T^i = 0 \quad \text{at } p(t)$$

⇑ Triple junctions in metallurgy (Cu - Fe - Sulfide) ⇑

- The curvature driven flow of is a gradient flow for the total length energy functional

$$E(\Gamma) = \sum_{i=1}^{3} \gamma^i L[\Gamma^i]\,, \qquad L[\Gamma^i] \text{ is the length of } \Gamma^i$$

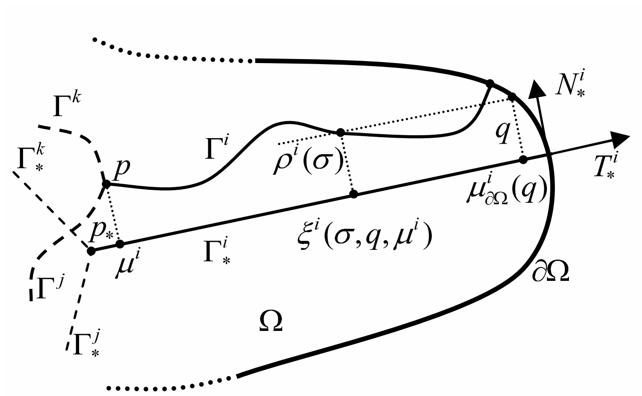
- the angles θ^i between the tangents T_j and T_k fulfill

$$\frac{\sin\theta^1}{\gamma^1} = \frac{\sin\theta^2}{\gamma^2} = \frac{\sin\theta^3}{\gamma^3} \qquad \text{Young's law}$$

- The constants γ^i can be interpreted as surface free energy densities (surface tensions)

Parameterization and local existence

Description of the local parameterization of the curve Γ^i.



Functions Φ^i parameterize the curves Γ^i in the neighbrhd of Γ^i_*

$$\Phi^i(\sigma) = \Psi^i(\sigma, \rho^i(\sigma), \mu^i)\,,\ \ \sigma \in [0, l^i]\,.$$

where $\mu^i = (\boldsymbol{p}, T^i_*)_{\mathbb{R}^2}, \quad \mu^i_{\partial\Omega}(q) = \max\{\, \sigma \mid \Phi^i_*(\sigma) + qN^i_* \in \overline{\Omega}\,\}\,.$
$\Psi^i(\sigma, q, \mu^i) = \Phi^i_*(\xi^i(\sigma, q, \mu^i)) + qN^i_* \quad \Gamma^i_* = \{\Phi^i_*(\sigma) \mid \sigma \in [0, l^i]\}$
$\xi^i(\sigma, q, \mu^i) = \mu^i + \frac{\sigma}{l^i}(\mu^i_{\partial\Omega}(q) - \mu^i)$

- We are led to the following nonlinear nonlocal partial differential equations for displacement functions $\rho^i(\sigma, t)$ $(i = 1, 2, 3)$:

$$
\begin{aligned}
\rho^i_t &= \underbrace{a^i(\rho^i, \rho^i_\sigma, \mu^i)\rho^i_{\sigma\sigma}}_{\text{diffusive part}} + \overbrace{\Lambda^i(\rho^i, \rho^i_\sigma, \mu^i) \sum_{j=1}^{3} a_1^{ij}(\mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu})\mathcal{T}^0\rho^j_{\sigma\sigma}}^{\text{nonlocal part}} \\
&+ \underbrace{f^i(\rho^i, \partial_\sigma\rho^i, \mathcal{T}^0\boldsymbol{\rho}, \mathcal{T}^0\boldsymbol{\rho}_\sigma, \boldsymbol{\mu})}_{\text{lower order terms}}, \qquad (\sigma, t) \in (0, l^i) \times (0, T)
\end{aligned}
$$

where $\mathcal{T}^0$ is the trace operator to $\sigma = 0$, i.e. $\mathcal{T}^0 f = f\big|_{\sigma=0}$
$\boldsymbol{\mu}^T = Q\boldsymbol{\rho}^T(0) = Q(\mathcal{T}^0\boldsymbol{\rho})^T$, $\quad Q$ is a rotation marix

- the solution $\boldsymbol{\rho}(., t)$ subject to nonlinear nonlocal boundary (compatibility) conditions at $\sigma = 0$ and $\sigma = l^i$.

Parameterization and local existence

Theorem (Garcke, Kohsaka, Ševčovič, 2009)

Let $\alpha \in (0, 1)$ and let us assume that $\rho^i_0 \in C^{2+\alpha}(\mathcal{I}^i)$ $(i = 1, 2, 3)$ with sufficiently small $\|\rho^i_0\|_{C^{2+\alpha}(\mathcal{I}^i)}$ fulfill the compatibility conditions. Then there exists a

$$T_0 = T_0\Big(1/\|\boldsymbol{\rho}_0\|_{C^{2+\alpha}}\Big) > 0$$

such that the problem with $\rho^i(\cdot, 0) = \rho^i_0$ $(i = 1, 2, 3)$ has a unique solution $\boldsymbol{\rho} \in C^{2+\alpha,1}(\overline{\mathcal{Q}^1_{0,T_0}})$

1. linearization of around the initial data $\rho^i_0 \in C^{2+\alpha}(\mathcal{I}^i)$ $(i = 1, 2, 3)$.
2. verification of the complementing conditions for the linearized system
3. existence and uniqueness of a solution to the linearized system via optimal regularity theory on C^β spaces due to A.Lunardi
4. contraction mapping principle *a la* S. Angenent idea for nonlinear semiflows

L^2 norm of curvature κ controls just H^2 norm of $\boldsymbol{\rho}$
$\Rightarrow$ we need to control H^1 norm of curvature

Theorem (Yanagida & Ikota, 2003)

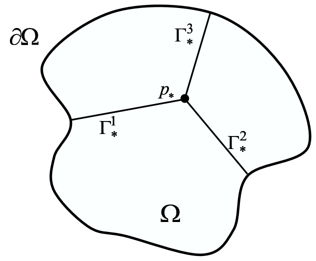
The maximal eigenvalue of the linearized problem at $\boldsymbol{\rho} = 0$ is negative and the stationary solution is linearly stable if one of the following conditions is satisfied:

a) *either all $h_*^1, h_*^2, h_*^3 > 0$ are positive,*

b) *or, at most one of them is nonpositive, and*

$$\gamma^1(1 + l^1 h_*^1)h_*^2 h_*^3 + \gamma^2(1 + l^2 h_*^2)h_*^1 h_*^3 + \gamma^3(1 + l^3 h_*^3)h_*^1 h_*^2 > 0$$

Case b) where $h_*^1, h_*^2 > 0$ but $h_*^3 < 0$



h_*^i is the curvature of the outer boundary $\partial\Omega$ at the contact point of Γ_*^i with $\partial\Omega$

Ikota R. and Yanagida E.:
A stability criterion for stationary curves to the curvature-driven motion with a triple junction, Differential Integral Equations **16** (2003), 707–726.

Linearization

Bilinear form:

$$I_*[\boldsymbol{w}, \boldsymbol{w}] = \sum_{i=1}^{3} \gamma^i \left\{ \int_0^{l^i} |w_s^i|^2 \, ds + h_*^i |w^i|^2 |_{s=l^i} \right\}$$

for all $\boldsymbol{w} = (w^1, w^2, w^3)^T$ with H^1-functions $w^i, i = 1, 2, 3$ defined on the curve Γ_*^i and such that $\sum_{i=1}^{3} \gamma^i w^i(0) = 0$.

Lemma

Let λ be the maximal eigenvalue of the linearized system. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$I_*[\boldsymbol{w}, \boldsymbol{w}] > (-\lambda - \varepsilon)\|\boldsymbol{w}\|_{L^2}^2 + \delta \sum_{i=1}^{3} \gamma^i \|w_s^i\|_{L^2}^2.$$

Let $X := \{\boldsymbol{\rho} \in H^2 \text{ with } \gamma^1\rho^1(0) + \gamma^2\rho^2(0) + \gamma^3\rho^3(0) = 0\}$.

Lemma

Let I_* be positive definite. Then there exists a H^2-neighborhood of $\boldsymbol{\rho} \equiv 0$ in X, such that $\boldsymbol{\rho} \equiv 0$ is the only solution of the problem

$$\begin{aligned} \kappa^i &= 0\,, \sphericalangle(\partial\Omega, \Gamma_t^i) = \pi/2\,, \\ \sphericalangle(\Gamma^i(t), \Gamma^j(t)) &= \cos\theta^k \quad \textit{for} \quad i,j,k \in \{1,2,3\} \quad \textit{mut. diff.} \end{aligned}$$

Furthermore, there exist a constant $C > 0$ and an

L^2–neighborhood $\{\kappa,\ \|\kappa\|_{L^2} < \delta\}$ of $\kappa = 0$ with $\delta > 0$ sufficiently small and such that

$$\|\rho\|_{H^2} \le C\|\kappa\|_{L^2} \quad \textit{for any} \|\kappa\|_{L^2} < \delta,$$

Proof. The idea of the proof is to use the local inverse mapping theorem for the curvature operator with appropriate boundary conditions.

Equation for the curvature

Mean curvature flow $V^i = \kappa^i$ fulfills the curvature equation:

$$\kappa_t^i = \kappa_{ss}^i + (\kappa^i)^3 + \kappa_s^i v^i, \qquad s \in (0, l^i)$$

- We choose the tangential velocity v^i such that $v_s^i = |\kappa_i|^2$
- At triple junction $p(t)$: $\sum_{i=1}^3 \gamma^i\kappa^i = 0$,

$$\kappa_s^1 + \kappa^1 v^1 = \kappa_s^2 + \kappa^2 v^2 = \kappa_s^3 + \kappa^3 v^1$$

- At $\Gamma_t^i \cap \partial\Omega$: $(\partial_s + h^i)\kappa^i = 0$.

Here h^i is the curvature of $\partial\Omega$ at the points $X^i(r^i(t), t) \in \Gamma_t^i \cap \partial\Omega$ and $v^i = (X_t^i, T^i)_{\mathbb{R}^2}$ is the tangential velocity. $s \in [0, r^i(t)]$ where $r^i(t) = L(\Gamma_t^i)$ is the length of Γ_t^i.

Lemma

A solution κ fulfills

$$\frac{d}{dt}E[\Gamma_t] + \sum_{i=1}^{3} \gamma^i \int_{\Gamma_t^i} (\kappa^i)^2 \, ds = 0,$$

$$\frac{d}{dt}\sum_{i=1}^{3} \gamma^i \int_{\Gamma_t} |\kappa^i|^2 \, ds = -2\sum_{i=1}^{3} \gamma^i \left\{ \int_{\Gamma_t^i} |\kappa_s^i|^2 ds + h^i |\kappa^i(r^i,t)|^2 \right\} + \sum_{i=1}^{3} \gamma^i \int_{\Gamma_t^i} |\kappa^i|^4 ds + \sum_{i=1}^{3} \gamma^i (\kappa^i)^2 v^i \big|_{s=0}$$

where h^i is curvature of $\partial\Omega$ evaluated at $X^i(r^i(t),t) \in \partial\Omega$.

$I[\boldsymbol{w},\boldsymbol{w}] = \sum_{i=1}^{3} \gamma^i \left\{ \int_{\Gamma_t^i} |\boldsymbol{w}_s^i|^2 ds + h^i |\boldsymbol{w}^i(r^i)|^2 \right\}$ is the same bilinear form as in the Ikota & Yannagida linearized stability theorem

First order estimates for curvature

Lemma

Let λ be the maximal eigenvalue of the linearized problem. For $\varepsilon > 0$ there exist $\delta > 0$ and $\mu > 0$ such that, for any perturbation satisfying $\|\rho^i\|_{C^0} < \delta$ we have

$$I[\boldsymbol{w},\boldsymbol{w}] > (-\lambda - \varepsilon)\sum_{i=1}^{3} (w^i, w^i)_{L^2} + \mu \sum_{i=1}^{3} \gamma^i \|w_s^i\|_{L^2}^2$$

for $\boldsymbol{w} = (w^1, w^2, w^3)^T$ with H^1-functions $w^i, i = 1,2,3,$ defined on the curve Γ^i and such that $\sum_{i=1}^{3} \gamma^i w^i(0) = 0$.

Notice that $\|\rho^i\|_{C^0} \ll 1$ implies: $|h^i - h_*^i| \ll 1$ and $\sum_{i=1}^{3} |L[\Gamma^i] - L[\Gamma_*^i]| \ll 1$

The first order energy functional $\Lambda(t) := \sum_{i=1}^{3} \gamma^i \|\kappa^i(.,t)\|_{L^2}^2$ satisfies, for small $\Lambda(0) \ll 1$, :

$$\frac{d}{dt}\Lambda(t) + \frac{(-\lambda)}{2\gamma}\Lambda(t) + \nu_* \gamma \sum_{i=1}^{3} \|\kappa_s^i(.,t)\|_{L^2}^2 \leq 0.$$

- $$\sum_{i=1}^{3} \gamma^i \|\kappa^i(.,t)\|_{L^2}^2 \leq e^{-t\frac{-\lambda}{2\gamma}} \sum_{i=1}^{3} \gamma^i \|\kappa^i(.,0)\|_{L^2}^2 \, ,$$

  for any $t \in [0,T]$, and, moreover,
- $$\sum_{i=1}^{3} \int_0^T \|\kappa_s(.,\tau)\|_{L^2}^2 \, d\tau \leq \frac{1}{\nu_* \gamma}\Lambda(0)$$

As $\|\boldsymbol{\rho}\|_{H^2} \leq C\|\boldsymbol{\kappa}\|_{L^2}$ it implies bound of the $C^{1+\alpha}$ norm of the displacement $\boldsymbol{\rho}$.
Still not enough to get global existence of smooth solutions

Higher order estimates for curvature

We shall derive a priori bound for the time derivative $w^i = \kappa_t^i$.

Equations for the time derivative of the curvatures

$$w_t^i = w_{ss}^i + 3(\kappa^i)^2 w^i + v^i w_s^i + v_t^i \kappa_s^i$$

- At the triple junction: $\sum_{i=1}^{3} \gamma^i w^i = 0$ and

  $$w_s^i + w^i v^i = G'(t) - \kappa^i \frac{d}{dt} v^i \, .$$

  where $G = G(t) \equiv \kappa_s^1 + \kappa^1 v^1 = \kappa_s^2 + \kappa^2 v^2 = \kappa_s^3 + \kappa^3 v^1$
- At the outer boundary contact with $\partial\Omega$

$$w_s^i + h^i w^i = d^i \equiv (w^i - (\kappa^i)^3 - (h^i - v^i) h^i \kappa^i) v^i - |\kappa^i|^2 (\nabla h^i, N^i)$$

Multiplying the equation for w by itself:

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\gamma^i\int_{\Gamma^i}|w^i|^2 ds \quad + \quad I(w,w)$$

$$= \sum_{i=1}^{3}\gamma^i w^i d^i|_{s=r^i} - \sum_{i=1}^{3}\gamma^i w^i w^i_s|_{s=0}$$

$$+ 3\int_{\Gamma^i}|\kappa^i|^2|w^i|^2 ds + \int_{\Gamma^i}\left(v^i w^i w^i_s + v^i_t w^i \kappa^i_s\right) ds$$

Here we have used the identity: $w^i_s + w^i v^i = G'(t) - \kappa^i v^i_t$ at $s = 0$ and the fact that, in the triple junction we have

$$0 = \frac{d}{dt}\sum_{i=1}^{3}\gamma^i\kappa^i = \sum_{i=1}^{3}\gamma^i w^i$$

Higher order estimates for curvature

Recall that, for any perturbation satisfying $\|\rho^i\|_{C^0} < \delta$ we have

$$I[\boldsymbol{w},\boldsymbol{w}] > (-\lambda - \varepsilon)\sum_{i=1}^{3}(w^i, w^i)_{L^2} + \mu\sum_{i=1}^{3}\gamma^i\|w^i_s\|^2_{L^2}$$

where λ is the maximal eigenvalue of the linearized problem. and, consequently, the estimate

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + I(w,w) \leq C\Big((\|\kappa\|^2_\infty + \|\kappa\|^3_\infty + \|\kappa\|^4_\infty)\|w\|_\infty + \|\kappa\|_\infty\|w\|^2_\infty$$

$$+\|\kappa\|^2_\infty\|w\|^2_2 + \|\kappa\|_\infty\|w\|_2\|w_s\|_2 + \|w\|_\infty\|w\|_2\|\kappa_s\|_2\Big)$$

Using Gagliardo-Nirenberg interpolation inequalities:

$$\|\kappa\|_\infty \le C_0 \|\kappa\|_{2,2}^{\frac{1}{4}} \|\kappa\|_2^{\frac{3}{4}}, \quad \|\kappa_s\|_2 \le C_0 \|\kappa\|_{2,2}^{\frac{1}{2}} \|\kappa\|_2^{\frac{1}{2}}$$

$$\|w\|_\infty \le C_0 \|w\|_{1,2}^{\frac{1}{2}} \|w\|_2^{\frac{1}{2}} .$$

and the Young inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}(.,t)\|_2^2 \ \le C_1 + C_2\eta(t)\|\boldsymbol{w}(.,t)\|_2^2$$

where $\eta(t) = 1 + \|\kappa(.,t)\|_2^2 + \|\kappa_s(.,t)\|_2^2$ is such that $\int_0^T \eta(\tau)d\tau < \infty$

Higher order estimates for curvature

For any finite $T < \infty$ $\sup_{0\le t<T} \|w(t)\|_2^2 < \infty$.
Since $\|\kappa_{ss}\|_2 \le C\left(\|\kappa\|_2 + \|w\|_2\right)$ and $\|\kappa\|_2$ was already shown to be small and bounded and the norm $\|\rho\|_{C^{2+\alpha}}$ can be estimated by the H^2 norm of κ we just have shown the following conclusion:

Theorem (Garcke, Kohsaka, Ševčovič, 2009)

The maximal time of existence of a solution $\rho(.,t) \in C^{2+\alpha}$ is infinite, $T = +\infty$ and hence it exists globally in time.

H. Garcke, Y. Kohsaka, and D. Ševčovič:
Nonlinear stability of stationary solutions for curvature flow with triple junction, Hokkaido Mathematical Journal, 38(4), 2009, s. 721-769
www.arxiv.org/abs/0802.3036 www.iam.fmph.uniba.sk/institute/sevcovic

Theorem (Garcke, Kohsaka, Ševčovič, 2009)

Let Γ_* be such that I_* is positive definite, i.e. the maximal eigenvalue of the linearized problem is negative. Then there exist constants $C, \omega, \delta > 0$ such that

$$\sum_{i=1}^{3} \|\rho^i(.,t)\|_{H^2} \leq Ce^{-\omega t} \sum_{i=1}^{3} \|\kappa^i(.,0)\|_{L^2}$$

for any $t \geq 0$ and $\sum_{i=1}^{3} \|\kappa^i(.,0)\|_{L^2} < \delta$. Moreover,

$$\sum_{i=1}^{3} \|\rho^i(.,t)\|_{C^{1+\alpha}} \leq Ce^{-\omega t} \sum_{i=1}^{3} \|\rho^i(.,0)\|_{C^{2+\alpha}}$$

for any $t \geq 0$ and $\sum_{i=1}^{3} \|\rho^i(.,0)\|_{C^{2+\alpha}} < \delta$

References

[1] H. Garcke, Y. Kohsaka, and D. Ševčovič: Nonlinear stability of stationary solutions for curvature flow with triple junction, Hokkaido Mathematical Journal, 38(4), 2009, s. 721-769

[2] Garcke, H., Ito, K., and Kohsaka, Y.: Nonlinear stability of stationary solutions for surface diffusion with boundary conditions, SIAM J. Math. Anal. Volume 40, Issue 2, pp. 491-515 (2008)

[3] Ikota R. and Yanagida E., A stability criterion for stationary curves to the curvature-driven motion with a triple junction, Differential Integral Equations **16** (2003), 707–726.