# A Stokes-like system of fourth order

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Let $G \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $H_0^1 := H_0^{1,2}(G)$ and $H_0^2 := H_0^{2,2}(G)$ be the "usual" Sobolev spaces equipped with inner products $\langle \nabla u, \nabla v \rangle$ resp. $\langle \Delta u, \Delta v \rangle$. Let $\underline{H}_0^2 := \left(H_0^2\right)^n$ and let $H_{0,0}^1 := \left\{ p \in H_0^1 : \int\limits_G p dx = 0 \right\}$. Given $\underline{F}^* \in \left(\underline{H}_0^2\right)^*$ we are looking for $\underline{u} \in \underline{H}_0^2$ and $p \in H_{0,0}^1$ such that

(1) $$\langle \Delta \underline{u}, \Delta \underline{\phi} \rangle + \langle \nabla p, \nabla \operatorname{div} \underline{\phi} \rangle = \underline{F}^*(\underline{\phi}) \quad \forall \underline{\phi} \in \underline{H}_0^2$$

and $\operatorname{div} \underline{u} = 0$ a. e. in $G$. The left hand side is the weak form of $\Delta^2 \underline{u} + \nabla \Delta p$. Ch. Amrouche and V. Girault regarded in Portugal. Math **49** (1992), 463 – 503, left hand sides of the type $\Delta^2 \underline{u} + \nabla p$.

$$A := \operatorname{div} : \underline{H}_0^2 \to H_{0,0}^1$$

is a bounded linear operator. For $p \in H_0^1$ and $\underline{\phi} \in \underline{H}_0^2$ let

$$\underline{L}(\underline{\phi}) := \langle \nabla p, \nabla \operatorname{div} \underline{\phi} \rangle = \langle \nabla p, \nabla A \underline{\phi} \rangle$$

Since $\|\nabla \operatorname{div} \underline{\phi}\| \leq \|\Delta \underline{\phi}\| \ \forall \underline{\phi} \in \underline{H}_0^2$ we see $\underline{L} \in \left(\underline{H}_0^2\right)^*$ and by Riesz's theorem $\exists_1 \ \underline{v} \in \underline{H}_0^2$:

(2) $$\langle \Delta \underline{v}, \Delta \underline{\phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\phi} \rangle \qquad \forall \underline{\phi} \in \underline{H}_0^2$$

Let $\underline{A}^* : H^1_{0,0} \to \underline{H}_0^2$, $\underline{A}^* p := \underline{v}$. Then

$$\langle \Delta \underline{A}^* p, \Delta \underline{\phi} \rangle = \langle \nabla p, \nabla A \underline{\phi} \rangle \quad \forall p \in H^1_{0,0}, \quad \forall \underline{\phi} \in \underline{H}_0^2$$

Since $\underline{C}_0^\infty \subset \underline{H}_0^2$ is dense, by partial integration

(3) $$\langle \nabla g, \nabla \operatorname{div} \underline{f} \rangle = \langle \nabla g, \Delta \underline{f} \rangle \quad \forall g \in H_0^1, \quad \forall \underline{f} \in \underline{H}_0^2.$$

**Lemma 1** *(M. Bogovskii; see also G. P. Galdi or H. Sohr). Let $G \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is a constant $C > 0$ such that for every $p_0 \in H^1_{0,0}$ there is (at least) one $\underline{v} \in \underline{H}_0^2$ such that* $\operatorname{div} \underline{v} = p_0$ *and*

$$\|\Delta \underline{v}\| \leq C \|\nabla p_0\|.$$

Application:

**Lemma 2** *There is a constant $C > 0$ such that*

(4) $$\|\nabla p_0\| \leq C \sup_{0 \neq \underline{\phi} \in \underline{H}_0^2} \frac{\langle \nabla p_0, \nabla \operatorname{div} \underline{\phi} \rangle}{\|\Delta \underline{\phi}\|} \quad \forall p \in H^1_{0,0}$$

For $p_0 \in H^1_{0,0}$ let $\underline{A}^* p_0 := \underline{v} \in \underline{H}_0^2$, where

$$\langle \Delta \underline{v}, \Delta \underline{\phi} \rangle = \langle \nabla p, \nabla \operatorname{div} \underline{\phi} \rangle \qquad \forall \underline{\phi} \in \underline{H}_0^2$$

Let $\underline{M}^2 := \left\{\underline{A}^* p_0 : p_0 \in H^1_{0,0}\right\}$. With the help of (4) it is readily proved

**Theorem 1** *Let $p_0 \in H^1_{0,0}$ and let $\underline{v} := \underline{A}^* p_0 \in \underline{H}^2_0$. With $C > 0$ by (4)*

$$C^{-1}\|\nabla p_0\| \le \|\Delta \underline{v}\| \le \|\nabla p_0\| \tag{5}$$

*$\underline{A}^* : H^1_{0,0} \to \underline{H}^2_0$ is bijective $\|\underline{A}^*\| \le 1$, $\|\underline{A}^{*-1}\| \le C$. In addition*

$$\|\nabla \operatorname{div} \underline{v}\| \le \|\Delta \underline{v}\| \le C\|\nabla \operatorname{div} \underline{v}\| \qquad \forall \underline{v} \in \underline{M}^2 \tag{6}$$

*Furthermore $\underline{M}^2$ is closed and the orthogonal decomposition*

$$\underline{H}^2_0 = \underline{M}^2 \oplus \underline{N}(A) \tag{7}$$

*holds true, where $\underline{N}(A) = \left\{\underline{u} \in \underline{H}^2_0 : \operatorname{div} \underline{u} = 0 \text{ a. e.}\right\}$*

**Theorem 2** *1. $\left\{\operatorname{div} \underline{v} : \underline{v} \in \underline{M}^2\right\} \subset H^1_{0,0}$ is closed and*

$$R(\underline{A}^*) = \left\{\underline{A}^* g : g \in H^1_{0,0}\right\} = \underline{M}^2$$

*is closed too. $\underline{A}^* : H^1_{0,0} \to \underline{H}^2_0$ is bijective, $\underline{A}^*$, $\underline{A}^{*-1}$ are continuous.*

*2. $\left\{\operatorname{div} \underline{v} : \underline{v} \in \underline{M}^2\right\} = H^1_{0,0}$ and $\operatorname{div} : \underline{M}^2 \to H^1_{0,0}$ is bijective and continuous and the inverse $(\operatorname{div})^{-1} : H^1_{0,0} \to \underline{M}^2$ is continuous too.*

**Theorem 3** *With $C > 0$ by Lemma 2 for $p_0 \in H^1_{0,0}$*

$$\|\nabla p_0\| \leq C \sup_{0 \neq \underline{v} \in \underline{M}^2} \frac{\langle \nabla p_0, \nabla \operatorname{div} \underline{v} \rangle}{\|\Delta \underline{v}\|} \leq$$

$$\leq C \sup_{0 \neq \underline{v} \in \underline{M}^2} \frac{\langle \nabla p_0, \nabla \operatorname{div} \underline{v} \rangle}{\|\nabla \operatorname{div} \underline{v}\|}$$

*Solution of system (1):*

**Theorem 4** *Let $\underline{F}^* \in \left(\underline{H}_0^2\right)^*$ be given. Then there is a unique pair $(\underline{u}, p_0) \in \underline{H}_0^2 \times H_{0,0}^1$ such that*

$$\langle \Delta \underline{u}, \Delta \underline{\phi} \rangle + \langle \nabla p_0, \nabla \operatorname{div} \underline{\phi} \rangle = \underline{F}^*(\underline{\phi}) \quad \forall \underline{\phi} \in \underline{H}_0^2$$

*and* $\operatorname{div} \underline{u} = 0$ *a.e. in $G$. Further*

(8) $$\|\Delta \underline{u}\| + \|\nabla p_0\| \leq (1 + C) \|\underline{F}^*\|_{\left(H_0^{1,2}\right)^*}$$

**Proof.** Uniqueness is trivial. Existence: There is a unique $\underline{w} \in \underline{H}_0^2$ such that

$$\langle \Delta \underline{w}, \Delta \underline{\phi} \rangle = \underline{F}^*(\underline{\phi}) \qquad \forall \underline{\phi} \in \underline{H}_0^2$$

and $\|\Delta \underline{w}\| = \|\underline{F}^*\|_{\left(\underline{H}_0^{1,2}\right)^*}$. By (7) there is $\underline{u} \in \underline{N}(A)$ and $\underline{v} \in \underline{M}^2$ with

$$\underline{w} = \underline{u} + \underline{v}, \qquad \|\Delta \underline{w}\|^2 = \|\Delta \underline{u}\|^2 + \|\Delta \underline{v}\|^2.$$

Since there is a unique $p_0 \in H_{0,0}^1$ such that

$$\langle \Delta \underline{v}, \Delta \underline{\phi} \rangle = \langle \nabla p_0, \nabla \operatorname{div} \underline{\phi} \rangle \quad \forall \underline{\phi} \in \underline{H}_0^2$$

$\|\nabla p_0\| \le C\|\Delta \underline{v}\|$ (by Theorem 3). Since $\|\Delta \underline{u}\| \le \|\Delta \underline{w}\|$ and $\|\Delta \underline{v}\| \le \|\Delta \underline{w}\|$, (8) follows. ■

A refined decomposition of $H^1_{0,0}$:

**Theorem 5** *Let*

$$\tilde{A} := \left\{ \Delta s : s \in H_0^3 \right\} \subset H^1_{0,0}$$

*and*

$$\tilde{B} := \left\{ p_b \in H^1_{0,0} \cap C^\infty : \int_G p_b \Delta^2 \phi = 0 \quad \forall \phi \in C_0^\infty \right\}$$

*Then in the sense of an orthogonal decomposition*

$$H^1_{0,0} = \tilde{A} \oplus \tilde{B}, \quad p_0 = \Delta s + p_b$$
$$\|\nabla \Delta s\|^2 + \|\nabla p_b\|^2 = \|\nabla p_0\|^2$$

**Proof.** For $p_0 \in H^1_{0,0}$ and $\phi \in H_0^3$

$$|\langle \nabla p_0, \nabla \Delta \phi \rangle| \le \|\nabla p_0\| \|\nabla \Delta \phi\|.$$

By $\langle \nabla \Delta s, \nabla \Delta \phi \rangle$ an inner product is defined on $H_0^3$ and by the Riesz theorem there exists a unique $s \in H_0^3$ such that

$$\langle \nabla \Delta s, \nabla \Delta \phi \rangle = \langle \nabla p_0, \nabla \Delta \phi \rangle \qquad \forall \phi \in H_0^3$$

Let $p_b := p_0 - \Delta s$. For $\phi \in C_0^\infty$

$$0 = \langle \nabla p_b, \nabla \Delta \phi \rangle = -\langle p_b, \Delta^2 \phi \rangle$$

and by Weyl's lemma for $\Delta^2$ follows $p_b \in C^\infty$ and $\Delta^2 p_b = 0$. Since $C_0^\infty$ is dense in $H_0^3$ with respect to $\|\nabla \Delta .\|$–norm it follows

$$\langle \nabla p_b, \nabla \Delta s \rangle = 0 \quad \forall p_b \in \tilde{B}, \forall \Delta s \in \tilde{A}.$$

■

Let $s \in H_0^3$ and put $\underline{v} := \nabla s \in \underline{H}_0^2$. Then $\operatorname{div} \underline{v} = \Delta s$ and first for $\underline{\phi} \in C_0^\infty$

$$\begin{aligned}(9)\quad \langle \Delta \underline{v}, \Delta \underline{\phi} \rangle &= \langle \Delta \nabla s, \Delta \phi \rangle = \\ &= -\langle \Delta s, \Delta \operatorname{div} \underline{\phi} \rangle = 1 \cdot \langle \nabla \operatorname{div} \underline{v}, \nabla \operatorname{div} \underline{\phi} \rangle\end{aligned}$$

and finally by approximation for all $\underline{\phi} \in \underline{H}_0^2$. By (9) $\lambda = 1$ is an eigenvalue of infinite multiplicity and $\left\{\underline{v} = \nabla s : s \in H_0^3\right\}$ belongs to the eigenspace.

Thorsten Riedl proved the analogous theory for $1 < q < \infty$, where in (1) $\underline{u} \in \underline{H}_0^{2,q}(G)$, $p \in H_{0,0}^{1,q}(G)$, $\underline{F}^* \in \left(\underline{H}_0^{2,q'}(G)\right)^*$, $\left(q' = \frac{q}{q-1}\right)$ and (1) holds for all $\underline{\phi} \in \underline{H}_0^{2,q'}(G)$. If

$$p_b \in \tilde{B}^q := \left\{ p_b \in H^{1,q}_{0,0}(G) \cap C^\infty(G) : \int\limits_G p_b \Delta^2 \phi = 0 \right.$$
$$\left. \forall \phi \in C^\infty_0(G) \right\}$$

and $\underline{v} \in \underline{H}^{2,q}_0(G)$ satisfies

$$\text{(10)} \qquad \langle \Delta \underline{v}, \Delta \underline{\phi} \rangle = \langle \nabla p_b, \nabla \operatorname{div} \underline{\phi} \rangle \qquad \forall \underline{\phi} \in \underline{H}^{2,q'}_0(G)$$

then it is readily seen that $\operatorname{div} \underline{v} \in \tilde{B}^q$. We write $Z_q : \tilde{B}^q \to \tilde{B}^q$, $Z_q p_b := \operatorname{div} \underline{v}$ with $\underline{v}$ by (10). For $p_b \in \tilde{B}^q$ Riedl proved $Z_q p_b - \frac{1}{2} p_b \in H^{2,q}(G) \cap H^{1,q}_{0,0}(G)$ and with $C > 0$

$$\text{(11)} \qquad \left\| Z_q p_b - \frac{1}{2} p_b \right\|_{H^{2,q}(G)} \leq C \| \nabla p_b \|_q \qquad \forall p_b \in \tilde{B}^q.$$

From this follows (since the imbedding from $H^{2,q}(G) \cap H^{1,q}_{0,0}(G)$ in $H^{1,q}_{0,0}(G)$ is compact) compactness of the operator $Z_q - \frac{1}{2} I : \tilde{B}^q \to \tilde{B}^q$. By the spectral theorem and some easy calculations we get an infinite series $(\lambda_k) \subset \mathbb{R}$ and corresponding $\underline{v}_k \in \bigcap\limits_{1<t<\infty} \underline{H}^{2,t}_0(G)$ such that

$$\langle \Delta \underline{v}_k, \Delta \underline{\phi} \rangle = \lambda_k \langle \nabla \operatorname{div} \underline{v}_k, \nabla \operatorname{div} \underline{\phi} \rangle$$
$$\forall \underline{\phi} \in \underline{H}^{2,t'}_0(G), 1 < t' < \infty$$

and $\lambda_k \to 2$ $(k \to \infty)$.