On prolongation of a class of overdetermined semilinear systems of PDE's

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Nonlinear PDE's
Meeting in honour of Jindřich Nečas

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Contents

Formulation of the problem

Main goal:

- to disucss properties of a certain class of linear overdetermined systems D of PDE's

- to construct resolutions starting with D

- to describe the space of solutions of D

- to describe spaces of solutions for semilinear versions of D

Several complex variables

- 60's: Kodaira and Spencer developed a theory of deformation of complex structures
  the main operator was $\bar\partial$ operator:

$$\bar\partial = (\partial_{z_1}, \dots, \partial_{z_1})$$

  the Dolebault complex

$$\mathcal{E}^{0,0} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \dots \xrightarrow{\bar\partial} \mathcal{E}^{0,n-1} \xrightarrow{\bar\partial} \mathcal{E}^{0,n}.$$

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- 70': Spencer and Goldschmidt - a general theory for a large class of overdetermined systems
  - Spencer resolution, Spencer cohomology

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- A trivial model - the gradient
  (a maximally overdetermined case)
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- de Rham sequence in dimension n :

$$\mathcal{E}^0 \stackrel{d}{\to} \mathcal{E}^1 \ldots \stackrel{d}{\to} \mathcal{E}^{n-1} \stackrel{d}{\to} \mathcal{E}^n.$$

Quaternionic analysis

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1.Fueter-Dirac equations

e_1, e_2, e_3 quaternionic units,
$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1;$
$\mathbb{H} = \{q =$
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Quaternionic analysis - comments

- 30's: Fueter in mathematics, Dirac in physics
  but note:
  Fueter: $f; \mathbb{H} \to \mathbb{H}$,
  Dirac: $f : \mathbb{R}^4 \to S$, where $S \simeq \mathbb{C}_2$ is an irreducible Spin(4)-module
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  there is a nice function theory, very similar to complex function theory (elliptic system)
- - C-R equations D_1 for existence for quaternionic derivative are very much overdetermined
  - every solution has a form $f(q) = q\,a + b$; the space of solutions is finite-dimensional!
  - D_1 has a name - the twistor equation

Linear elasticity

Let $u = (u_i) = (u_1, u_2, u_3)$ is a vector field on $\mathbb{R}_3$ (the displacement);

$$D : u \mapsto \partial_{(i}u_{j)},$$

where Du is a map from $\mathbb{R}_3$ to symmetric 3×3 matrices (the strain)
the space $\mathbb{T}$ of solutions of $Du = 0$:

$$\mathbb{T} = \{u_i = a_i + \epsilon_{ijk} b^j x^k\} = \{(a_i, b_j) | a_i, b_j \in \mathbb{R}\},$$

where ϵ_{ijk} is a totally antisymetric tensor with $\epsilon_{123} = 1$.

Linear elasticity complex

- $$\mathcal{E}_i \xrightarrow{D} \mathcal{C}^\infty(\mathbb{R}_3, \mathbb{R}_3) \xrightarrow{D'} \mathcal{C}^\infty(\mathbb{R}_3, \mathbb{R}_3) \xrightarrow{D''} \mathcal{C}^\infty(\mathbb{R}_3, \mathbb{R})$$

  where

  $$D'(v_{ij}) = \epsilon^{ikm}\epsilon^{jln}\partial_k\partial_l v_{mn};$$

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- Applications for a construction of finite element methods for a variety of problems (Hodge Laplacian, Maxwells equations, the equations of elasticity, and elliptic eigenvalue problems).

D. N. Arnold, R. S. Falk, R. Winther:
Finite element exterior calculus, homological techniques, and applications, Acta Numerica, (2006), 1-155.

D. N. Arnold, R. S. Falk, R. Winther:
Finite element exterior calculus: from Hodge theory to numerical stability, accepted for Bull. Am. Math. Society

Higher spin twistor equations

- all representations of $SO(2) \simeq U(1)$ are $\rho_k, k \in \mathbb{Z}$, on $\mathbb{C}$!

$$\rho_k : U(1) \mapsto \mathbb{C} - \{0\}; \rho_k(z) = z^k;$$

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  spin $\frac{1}{2}$: $\mathbb{S} \simeq \mathbb{C}_2$;
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  spin $\frac{1}{2}$: $\mathbb{S} \simeq \mathbb{C}_2$;
  spin $\frac{k}{2}$: $\odot^k(\mathbb{S}) \simeq \mathbb{C}_{k+1}$;
- We can define a quaternionic derivative φ' of a map $\varphi : \mathbb{R}_4 \mapsto \odot^k(\mathbb{S})$
  existence of $\varphi'(q)$ is equivalent to a generalized C-R equations, given by a first order system D_k acting on functions on $\mathbb{R}_4$ with values in $\odot^k(\mathbb{S})$, generalizing the twistor equation
  the space of the solutions $D_k\varphi = 0$ is again finite-dimensional but consist of polynomials of order k;

The Hitchin result

- There is a variant of the Radon trasform (the Penrose transform), mapping the space of all solutions $D_k\varphi = 0$ isomorphically to the space of holomorphic functions of 'homogeneity' k (cohomology groups $H^0(Z, L^k)$) on a certain complex 3-manifold Z
  If we would consider values of Φ in the sum $\mathbb{V} := \oplus_0^\infty \odot^k (\mathbb{S})$, then the space of such solutions is equivalent to the space of all polynomials of three complex variable on Z.

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  If we would consider values of Φ in the sum $\mathbb{V} := \oplus_0^\infty \odot^k (\mathbb{S})$, then the space of such solutions is equivalent to the space of all polynomials of three complex variable on Z.
- There exists a version of the result where X is a selfdual spaces Riemannian manifold and Z its twistor space, which can be still extended to sections of vector bundles and massless equations twisted with a self-dual Yang-Mills field.

M. F. Atiyah, N. J. Hitchin, M. Singer: Self-dual Yang-Mills fields on self-dual spaces, Proc. R. Soc. London **A** , , 1978.

N. J. Hitchin: Linear field equations on self-dual spaces, Proc. R. Soc. London **A 370**, 173-191, 1980.

A prolongation For ordinary differential equations, the n-th order equation

$$\frac{d^3\sigma}{dx^3} = f(x, \sigma, \frac{d\sigma}{dx}, \frac{d^2\sigma}{dx^2})$$

is equivalent to the system of first order equations

$$\frac{d\sigma}{dx} = \sigma_1, \frac{d^2\sigma_1}{dx^2} = \sigma_2, \frac{d^3\sigma_2}{dx^3} = f(x, \sigma, \sigma_1, \sigma_2).$$

For partial differential equations, however, this fails. For overdetermined systems, we want also to introduce new dependent variables for certain higher derivatives in order to achieve a first order 'closed system', i.e., one in which all the first partial derivatives of all the dependent variables are determined in terms of the variables themselves. The introduction of new variables for unknown higher derivatives with the aim of expressing all their derivatives as differential consequences of the original equation is the well-known procedure of prolongation.

The Spencer prolongation Spencer:

$\mathbb{E}, \mathbb{F}$ finite dimensional vector spaces

$D : \mathcal{C}^\infty(\mathbb{R}_n); \mathbb{E}) \mapsto \mathcal{C}^\infty(\mathbb{R}_n); \mathbb{F})$ an overdetermined system of PDE's (possibly of a higher order).

There exists a general construction of vector spaces $\mathbb{V}_i, i = 0, 1, \ldots,$ governing the prolongation of the system.

If it terminates after a finite number of steps, equations are called equations of finite types

Problem: Computation of V_i is too complicated in general.

D.C. Spencer, Overdetermined systems of linear partial differential equations, Bull. Amer. Math. Soc. 75 (1969) 179239.

Symmetry

Theorem

Suppose $D : \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}) \mapsto \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{F})$ *is a* k*th-order semilinear differential operator such that its symbol* $\sigma(D)$ *is the projection of* $\odot^k(\mathbb{R}_n) \otimes \mathbb{E}$ *to the Cartan product* $\mathbb{F}$ *of* $GL(n)$ *modules* $\odot^k(\mathbb{R}_n)$ *and* $\mathbb{E}$.

Then, there is a vector space $\mathbb{V}$ *and a canonically associated connection*

$$\nabla : \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{V}) \mapsto \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{R}_n \otimes \mathbb{V})$$

such that there is a bijection

$$\{\sigma \in \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}) | D\sigma = 0\} \simeq \{\Sigma \in \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{V}) | \nabla\Sigma + \Phi\Sigma = 0\}$$

where $\Phi(x) : \mathbb{V} \mapsto \mathbb{R}_n \otimes \mathbb{V}$ *is a mapping canonically constructed from* D*. If* D *is linear, then so is* Φ*. The bundle* V *is completely determined by* $\mathbb{E}$ *and* k*.*

Identification

The identification is given by an N-th order linear differential operator $L : \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}) \mapsto \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{V})$, where N is easily computable from $\mathbb{E}$ and k.

T. Branson, A. Cap, M. Eastwood, R. Gover: Prolongations of Geometric Overdetermined Systems, Internat. J. Math., 17, No. 6 (2006), 641-664.

M. Hammerl, J. Šilhan, P. Somberg, V. Souček: On normalization of tractor covariant derivatives, preprint

The BGG resolutions

- Klein: geometries are classified by homogeneous spaces G/H, where $H \subset G$ is a Lie subgroup of a Lie subgroup G.
- parabolic case: G semisimple (GL(n,$\mathbb{C}$), SO(n,$\mathbb{C}$), Sp(n,$\mathbb{C}$) or real versions), P parabolic subgroup,
- Consider a couple (G, P) with G semisimple and P parabolic, let $n = \dim G/P$. Then for any irreducible G-module $\mathbb{V}$, there is a resolution

$$\mathbb{V} \mapsto \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}_0) \xrightarrow{D_1} \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}_1) \xrightarrow{D_2} \dots \xrightarrow{D_n} \mathcal{C}^\infty(\mathbb{R}_n, \mathbb{E}_n) \mapsto 0,$$

  where $\mathbb{E}_i$ are suitable irreducible P-modules canonically determined by $\mathbb{V}$.
  The first operator in the resolution is an overdetermined operator (possibly of higher order) of a finite type.

A. Cap, J. Slovak, and V. Soucek, Bernstein-Gelfand-Gelfand sequences, Ann. Math. 154 (2001) 97113

D.M.J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand- Gelfand sequences, Jour. Reine Angew. Math. 537 (2001) 67103.