A priori estimates for quasilinear parabolic systems with quadratic nonlinearities in the gradient

Pubblicazioni del Centro di ricerca matematica Ennio De Giorgi, 2009

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Praha, December 13.-15., 2009

System of equations

$$u_t - \mathrm{div}(a(z,u)\nabla u) + b(z,u,\nabla u) = 0; \;\; z \in Q = (0,T) \times \Omega \quad (1)$$

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Weak solution $u \in V_Q = L_2((0,T); W_2^1(\Omega))$:

$$\int_Q (-u\varphi_t + (a(z,u)\nabla u, \nabla\varphi) + b(z,u,\nabla u)\varphi)\, dz = 0$$

for all $\varphi \in C_0^\infty(Q, \mathbb{R}^N)$.

Assumptions

$\Omega \subset \mathbb{R}^n$; $T > 0$, $Q = \Omega \times (0, T)$, $u : Q \to \mathbb{R}^N$,

1. $a(., u) \in L^\infty(Q) \cap VMO(Q)$ satisfies ellipticity condition

$$q(R) = \sup_{z_o \in Q, r \in (0,R), u \in \mathbb{R}^N} \fint_{Q(z_o,r)} |a(z,u) - (a(.,u))_{z_o,r}| dz$$

$$q(R) \to 0 \quad \text{for} \quad R \to 0.$$

where $(a(.,u))_{z_o,r} = \fint_{Q(z_o,r)} |a(z,u)| dz$.

2. $a(z,.)$ are uniformly continuous

$$|a(z,u) - a(z,v)| \leq \omega(|u-v|) \;\; \forall \;\; z \in Q, u, v \in \mathbb{R}^N$$

ω nondecreasing, bounded and concave $\lim_{s\to 0+} \omega(s) = 0$,

3. $b(z,\eta,\xi)$ are measurable in z for all η,ξ and continuous in η,ξ for all $z \in Q$

$$|b(z,\eta,\xi)| \leq b_0|\xi|^2 + b_1; \; (z,\eta,\xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$$

where $b_0, b_1 \in L^p(Q)$ for $p > 1 + n/2$.

Systems with quadratic nonlinearities in gradient on right hand sides

Bounded " small" solutions

M. Giaquinta, M. Struwe (1982) - partial regularity

M. Struwe (1986), O. John, J.S. - counterexamples

Solutions with "small" BMO seminorms

A. Archipova (1997) - partial regularity

Generalization of Archipova (2007)

1. $a(z, u)$ only VMO in z variable (not continuous)
2. without assumptions on existence of u_t

Preliminaries

Definition (A-caloric functions)

Let $\overline{A}$ be a constant positive definite matrix. A function $h \in V(Q)$ is $\overline{A}$-caloric iff for any $\varphi \in C_0^1(Q)$

$$\int_Q \big(h\varphi_t - (\overline{A}\nabla h, \nabla\varphi)\big)\, dx = 0.$$

F. Duzaar, G. Mingione - 2004
M.Giaquinta- 2000

Lemma ($\overline{A}$-caloric approximation)
$0 < \lambda < \Lambda$, $n, N \geq 2$
$\forall \;\; \varepsilon > 0 \;\; \exists \; C(\varepsilon) = C(\varepsilon, n, N, \lambda, \Lambda) > 0$ such that:
for any bilinear form $\overline{A}$ on $\mathbb{R}^{nN}$

$$(\overline{A}\xi, \xi) \geq \lambda|\xi|^2, \quad (2)$$

$$||\overline{A}|| \leq \Lambda, \quad (3)$$

for any $u \in V_{Q(z_o,R)}$ there exist an $\overline{A}$-caloric $h \in V_{Q(z_o,R)})$ and $\varphi \in C^1_0(Q(z_o, R))$ so that

$$\fint_{Q(z_o,R)} (|h-(h)_{z_o,R}|^2 + R^2|\nabla h|^2)\,dz \leq$$

$$\fint_{Q(z_o,R)} (|u-(u)_{z_o,R}|^2 + R^2|\nabla u|^2)\,dz,$$

$$||\nabla\varphi||_{L^\infty} \leq 1,$$

$$\fint_{Q(z_o,R)} |u-h|^2\,dz \leq$$

$$\varepsilon \fint_{Q(z_o,R)} (|u-(u)_{z_o,R}|^2 + R^2|\nabla u|^2)\,dz +$$

$$C(\varepsilon)R^2|\fint_{Q(z_o,R))} \left(u\varphi_t - (\overline{A}\nabla u, \nabla\varphi)\right)\,dz|^2.$$

Higher integrability of gradient and reverse Hölder inequalities

bounded solutions under smallness condition
$2b_o||u||_{L^\infty} < \lambda$ (M. Struwe, M. Giaquinta - 1981, J. Naumann, P. A. Ivert -1988, M. Marino, A. Maugeri 1989)

solutions with small BMO seminorm (A. Archipova-2007).

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Definition

$\theta > 0, R > 0$. u satisfies condition $S_{\theta,R}$ if

$$\nabla u \in L^m(Q(z_o, R))$$

with exponent $m > 2$ and

$$[u]_{\mathcal{L}^{2,n+2}(Q(z_o,R))} \leq \theta.$$

Theorem
Let u solve (1), assumptions on coefficients hold.
Then there exist θ, R so that if u satisfies $S_{\theta,R}$ then

$$u \in C^{0,\alpha}(Q(z_o, r))$$

for sufficiently small r and there is a positive constant C such that

$$||u||_{C^{0,\alpha}(Q(z_o,r))} \leq C(||u||_{V_Q}, \alpha, 1/r).$$

Idea of the proof

Part 1. - Caccioppoli inequality
Step 1.
Testing (1) by $u-(u)_{z_o,r}$ gives ($\rho \le ar$, $a<1$ fixed)

$$\int_{Q(z_o,\rho)} |\nabla u|^2 dz \le C(\frac{1}{r^2}\int_{Q(z_o,r)} |u-(u)_{z_o,r}|^2 dz + \theta \int_{Q(z_o,r)} |\nabla u|^2 dz + r^{n+4})$$

Step 2.
Use M. Giaquinta, G. Modica-1982

$$\int_{Q(z_o,\rho)} |\nabla u|^2 dz \le C(\frac{1}{r^2}\int_{Q(z_o,r)} |u-(u)_{z_o,r}|^2 dz + r^{n+4}).$$

Step 3.

$$\frac{1}{r^n}\int_{Q(z_o,\rho)} |\nabla u|^2 dz \le C\left(⨍_{Q(z_o,r)} |u-(u)_{z_o,r}|^2 dz + r^2\right)$$
$$\le C(\theta^2 + r^2) = \kappa(\theta, r) \to 0$$

for $\theta, r \to 0$.

Step 3.

$$\frac{1}{r^n}\int_{Q(z_o,\rho)}|\nabla u|^2 dz \leq C(\fint_{Q(z_o,r)}|u-(u)_{z_o,r}|^2 dz + r^2)$$
$$\leq C(\theta^2 + r^2) = \kappa(\theta, r) \to 0$$

for $\theta, r \to 0$.

Part 2. - Decomposition of solution
Set $\overline{A} = (a(z,u_0))_{Q(z_o,r)}$ with $u_o = (u)_{Q(z_o,r)}$,
find $\overline{A}$-harmonic approximation h
and decompose $u = h + (u-h)$ on $Q_(z_o, r)$

Step 4.

A. Campanato inequalities and approximation lemma for h:

$$\fint_{Q(z_o,\rho)} |h-(h)_\rho|^2 dz \leq C\left(\frac{\rho}{r}\right)^2 \fint_{Q(z_o,r)} |h-(h)_r|^2 dz \leq C\left(\frac{\rho}{r}\right)^2 \fint_{Q(z_o,r)} (|u-(u)_r|^2 + r^2|\nabla u|^2) dz$$

B. Approximation lemma for $u-h$:

$$\fint_{Q(z_o,r)} |u-h|^2\, dz \leq \varepsilon \fint_{Q(z_o,r)} (|u-(u)_{z^o,r}|^2 + r^2|\nabla u|^2)\, dz + C(\varepsilon) r^2 \left| \fint_{Q(z_o,r)} \left(u\varphi_t - (\overline{A}\nabla u, \nabla\varphi)\right)\, dz\right|^2.$$

Step 5.
Estimate of the last term in (B.):

$||\nabla\varphi||_{L^\infty(Q(z_o,r))} \leq 1$ implies $||\varphi||_{C(\overline{Q(z_o,r)})} \leq r$

$$L = | ⨍_{Q(z_o,r)} \left(u\varphi_t - (\overline{A}\nabla u, \nabla\varphi)\right) \, dz|$$

$$\begin{aligned} & L \leq ⨍_{Q(z_o,r)} |a(z,u) - \overline{A}||\nabla u| + \\ & b_o r ⨍_{Q(z_o,r)} |\nabla u|^2 dz + b_1 r \\ \leq \quad & C(\omega^{1/2}(\theta) + q(r) + b_o\kappa(\theta,r))(⨍_{Q(z_o,r)} |\nabla u|^2 dz)^{1/2} + b_1 r \end{aligned}$$

Step 6.

Set $\Phi_r(u) = \fint_{Q(z_o,r)}(|u-(u)_{z^o,r}|^2 dz$.

Caccioppoli + (step 4.A.) + (step 4.B.)

$$\Phi_\rho(u) \leq C((\frac{\rho}{r})^2 + \epsilon + C(\epsilon)(\omega^{1/2}(\theta) + q(r) + b_o\kappa(\theta,r))^2)\Phi_r(u) + Dr^2$$

+ algebraic lemma and sufficiently small ϵ, r, θ conclude

$$u \in \mathcal{L}^{2,\beta}(Q(z_o,\tilde{r})) \subset C^{0,\alpha}(Q(z_o,\tilde{r}))$$

for any $\beta \in (0,2), \alpha \in (0,1)$.