On the interior regularity for minimizers of quasilinear functionals

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Functional

The aim of this paper is to study the interior everywhere regularity of functions minimizing variational integrals

$$\mathcal{A}(u;\Omega) = \int\limits_{\Omega} A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx \tag{1}$$

where $u : \Omega \to \mathbb{R}^N$, $N > 1$, $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded open set, $x = (x_1, \ldots, x_n) \in \Omega$, $u(x) = (u^1(x), \ldots, u^N(x))$, $Du = \{D_\alpha u^i\}$, $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, \ldots, n$, $i = 1, \ldots, N$.

Minimizer

We call a function

$$u \in W^{1,2}(\Omega, \mathbb{R}^N)$$

a minimizer of the functional $\mathcal{A}(u; \Omega)$ if and only if

$$\mathcal{A}(u; \Omega) \leq \mathcal{A}(v; \Omega)$$

for every $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $u - v \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Morrey space

Let $p \geq 1$, $\lambda \in [0, n]$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain of type A. A space of functions $u \in L^p(\Omega)$ such that

$$||u||_{L^{p,\lambda}} = \left\{ sup_{x \in \Omega, \rho > 0} \ \rho^{-\lambda} \int_{\Omega_\rho(x)} |u(y)|^p \mathrm{d}y \right\}^{1/p} < +\infty$$

is said to be $L^{p,\lambda}(\Omega)$.

$L^{p,\lambda}(\Omega)$ is a Banach space with the norm $||u||_{L^{p,\lambda}}$.

If $u \in W^{1,2}_{loc}(\Omega)$ and $Du \in L^{2,\lambda}_{loc}(\Omega)$, $n - 2 < \lambda < n$, then

$u \in C^{0,(\lambda+2-n)/2}(\Omega)$.

Basic assumptions

On the functional $\mathcal{A}$ we assume the following conditions:

(i) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, $A_{ij}^{\alpha\beta}$ are continuous functions in $u \in \mathbb{R}^N$ for every $x \in \Omega$ and there exists $M > 0$ such that

$$|A_{ij}^{\alpha\beta}(x,u)| \leq M, \forall\ x \in \Omega, \forall\ u \in \mathbb{R}^N.$$

(ii) (ellipticity) There exists $\nu > 0$ such that

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \geq \nu|\xi|^2, \quad \forall\, x \in \Omega,\ \forall\, u \in \mathbb{R}^N,\ \forall\, \xi \in \mathbb{R}^{nN}. \qquad (2)$$

Basic assumptions

(iii) (oscillation of coefficients) There exists a real function ω continuous on $[0, \infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and u, $v \in \mathbb{R}^N$

$$\left| A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(x, v) \right| \leq \omega\left(|u - v|\right). \tag{3}$$

We set $\omega_\infty = \lim_{t\to\infty} \omega(t) \leq 2M$.

(iv) For all $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$ (uniformly with respect to $u \in \mathbb{R}^N$).

BMO and VMO - spaces

We set for $f \in L^1(\Omega)$, $0 < a < \infty$

$$\mathcal{M}_a(f,\Omega) := \sup_{x\in\Omega, r<a} \fint_{\Omega_r(x)} |f(y) - f_{x,r}|\, dy.$$

A function $f \in L^1(\Omega)$ is said to belong to $BMO(\Omega)$ if

$$\mathcal{M}_{\mathrm{diam}\,\Omega}(f,\Omega) < \infty.$$

A function $f \in L^1(\Omega)$ is said to belong to $VMO(\Omega)$ if

$$\lim_{a\to 0} \mathcal{M}_a(f,\Omega) = 0.$$

Since C^0 is a proper subset of VMO, the continuity of coefficients $A_{ij}^{\alpha\beta}$ with respect to x is not supposed.

Partial regularity

1) M.Giaquinta; E.Giusti, Acta Math. (1982)

$A_{ij}^{\alpha\beta}$ are bounded, uniformly continuous functions in $\Omega \times \mathbb{R}^N$ and (ii) is satisfied for the functional (1). Then for the minimizer $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ it holds:

There exists an open set $\Omega_0 \subset \Omega$, such that $u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^N)$ for every $\alpha < 1$ and $H^{n-q}(\Omega \setminus \Omega_0) = 0$ for some $q > 2$.

Here $\Omega \setminus \Omega_0 = \{x \in \Omega : \liminf_{R\to 0^+} R^{2-n} \int_{B_R(x)} |Du(y)|^2 \, dy > \epsilon_0\}$, $\epsilon_0 > 0$ does not depend on u.

Partial regularity

2)P.Di Gironimo; L.Esposito; L.Sgambati, Manusc. Math. (2004)

It is supposed that (i), (ii), (iii) and (iv) are fulfilled for the functional (1).Then for the minimizer $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ it holds:

There exists an open set $\Omega_0 \subset \Omega$, such that $Du \in L^{2,\lambda}_{loc}(\Omega_0, \mathbb{R}^{nN})$, $\forall \lambda < n$ and the Hausdorff dimension $\dim_H(\Omega \setminus \Omega_0) < n - 2$ (this fact implies Hölder continuity of minimizer u on Ω_0).

Here $\Omega \setminus \Omega_0 = \{x \in \Omega : \liminf_{R\to 0^+} R^{2-n} \int_{B_R(x)} |Du(y)|^2 \, dy > 0\}$.

Theorem 1 A.) (D-V, NoDEA 2009.)

Let $\delta \in [n-2, n)$, $P > 1$ and $\overline{\Omega}_0 \subset \Omega$ with $\operatorname{dist}(\Omega_0, \partial\Omega) \geq 2d > 0$. Let u be a minimizer of (1) and $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that $u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.
A.) Let assumptions (i), (ii), (iii), (iv) be satisfied with the constants M, ν such that $M/\nu \leq P$ and suppose that

$$S_{p'} = \sup_{t \in (0,\infty)} \frac{d}{dt}(\omega^{p'})(t) < \infty$$

for $p' = p/(p-1)$, ($2p > 2$ is an exponent of integrability of Du).
Then there exists a constant $\nu_0 = \nu_0(n, \delta, P, S_{p'}, d, \|Dg\|_{L^2(\Omega,\mathbb{R}^{nN})}) > 0$ such that if $\nu \geq \nu_0$ then $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

Theorem 1 B.) (D-V, NoDEA 2009.)

B.) Let assumptions (i), (ii), (iii), (iv) be satisfied with the constants M, ν such that $M/\nu \leq P$ and suppose that

$$S_0 = \sup_{t > t_0} \frac{d}{dt}\omega(t) < \infty$$

where $t_0 = \inf \omega^{-1}(\frac{\omega_\infty}{2KP}\epsilon_0)$, $K = K(n,p,N,P)$ and $\epsilon_0 = \frac{1}{4(2^{n+3}L)^{\frac{\delta}{n-\delta}}}$ $(L = L(n,P))$, then there exists a constant $\nu_0 = \nu_0(n,p,\delta,N,P,S_0,d,\|Dg\|_{L^2(\Omega,\mathbb{R}^{nN})}) > 0$ such that if $\nu \geq \nu_0$ then $Du \in L^{2,\delta}(\Omega_0,\mathbb{R}^{nN})$.

Consequence

Let assumptions of Theorem 1 be satisfied and $\delta \in [n-2, n)$. Then

$$u \in \begin{cases} C^{0,(\delta-n+2)/2}(\overline{\Omega}_0, \mathbb{R}^N) & \text{if } n-2 < \delta < n \\ BMO(\Omega_0, \mathbb{R}^N) & \text{if } \delta = n-2. \end{cases}$$

Remark

As a typical example of modulus of continuity which satisfies the assumptions of Theorem 1 A.) we can consider

$$\omega(t) = \begin{cases} ct^{\alpha} & \text{for} \quad 0 \leq t < t_1, \\ \omega_{\infty} & \text{for} \quad t \geq t_1 \end{cases}$$

where $c > 0$, $\alpha \in (0,1)$, $t_1 = (\omega_{\infty}/c)^{1/\alpha}$. Such a function satisfies the assumptions of Theorem 1 A.) for $p' = 1/\alpha$.

Remark

A typical modulus of continuity which satisfies the assumptions of Theorem 1 B.) and does not satisfy the assumptions of Theorem 1 A.) is the modulus of continuity

$$\omega(t) = \begin{cases} \frac{c_1}{\left(1+\ln(\frac{c_2}{t})\right)^\beta} & \text{for} \quad 0 < t < \frac{c_2}{\mathrm{e}^\beta}, \\ \widetilde{\omega}(t) & \text{for} \quad t \geq \frac{c_2}{\mathrm{e}^\beta} \end{cases}$$

where $0 < c_1 \leq \omega_\infty$, c_2, $\beta > 0$, $\widetilde{\omega}(c_2/\mathrm{e}^\beta) = c_1/(1+\beta)^\beta$ and $\widetilde{\omega}$ is such that ω satisfies the assumption (iii).

Example

In $\Omega = B_1(0) \subset \mathbb{R}^3$ we consider quasilinear variational integral

$$\mathcal{A}(u;\Omega) = \int_\Omega A_{ij}^{\alpha\beta}(u) D_\alpha u^i D_\beta u^j \, dx$$

where

$$A_{ij}^{\alpha\beta}(u) = a\delta_{ij}\delta_{\alpha\beta} + b\left(\delta_{i\alpha}\arctan(\mu|u^i|^s) + \delta_{j\beta}\arctan(\mu|u^j|^s)\right)$$

for $i,\, j = 1,2,3,\ \alpha,\ \beta = 1,2,3,\ u - g \in W_0^{1,2}(B_1(0),\mathbb{R}^3)$,
$g \in W^{1,2}(B_1(0),\mathbb{R}^3),\ 0 < s \leq 1,\ a,\ b > 0,\ a > 9\pi b,\ \mu > 0.$

Example

In this case we have

$$M = a + \pi b, \quad \nu = a - 9\pi b, \quad \omega_\infty = \pi b$$

and the modulus of continuity is

$$\omega(t) = \begin{cases} 2b\mu t^s, & \text{for} \quad 0 < t < \left(\frac{\pi}{2\mu}\right)^{1/s}, \\ \pi b & \text{for} \quad t \geq \left(\frac{\pi}{2\mu}\right)^{1/s}. \end{cases}$$

Let $B_r(0)$ be a ball with $0 < r < 1$ then for suitable $\mu > 0$ the above functional satisfies the assumptions of Theorem 1 and so the minimizer u is Hölder continuous (in the case $\delta > n - 2$) in $B_r(0)$.

Theorem 2

Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional (1) such that $Du \in L^{2,n-2}(\Omega, \mathbb{R}^{nN})$ and the hypotheses (i), (ii), (iii), (iv) be satisfied.

Assume that there exists $p > 1$ such that

$$Q_p := \min \left\{ \sup_{t \in (0,\infty)} \frac{d}{dt}(\omega^{p/(p-1)})(t), \int_0^\infty \frac{\frac{d}{dt}(\omega^{p/(p-1)})(t)}{t}\, dt \right\} < \infty,$$

and let $\gamma \in (0, 1)$.

Theorem 2

Then the inequality

$$\left(Q_p ||Du||_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\right)^{1-1/p} \leq \nu C$$

implies that $u \in C^{0,\gamma}(\Omega, \mathbb{R}^N)$.

Here

$$C = \frac{2}{3c(n,N,p,M/\nu)(2^{n+3}L)^{\frac{n}{2(1-\gamma)}}},$$

where L is a given constant.

Remarks to proofs

We set $\phi(x_0, r) = \int_{B_r(x_0)} |Du(y)|^2 \, dy$, for $B_r(x_0) \subset \Omega$. Now let x_0 be any fixed point of $\overline{\Omega}_0 \subset \Omega$, $\mathrm{dist}(\Omega_0, \partial\Omega) \geq 2d > 0$, $B_{2R}(x_0) \subset \Omega$ and v be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x_0)) = \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(u_R)_R D_\alpha v^i D_\beta v^j \, dx$$

among all the functions in $W^{1,2}(B_R(x_0), \mathbb{R}^N)$ taking the values u on $\partial B_R(x_0)$ where

$$A_{ij}^{\alpha\beta}(z)_R = \fint_{B_R(x_0)} A_{ij}^{\alpha\beta}(y, z) \, dy.$$

Remarks to proofs

Campanato's estimate for v:

$$\int_{B_\sigma(x_0)} |Dv|^2\,dx \le L\left(\frac{\sigma}{R}\right)^n \int_{B_R(x_0)} |Dv|^2\,dx, \quad \forall\ 0<\sigma\le R$$

Put $w = u - v$. It is clear that $w \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N)$.

Remarks to proofs

$$
\begin{aligned}
\int_{B_R(x_0)} |Dw|^2\,dx \le \frac{2}{\nu}\Bigg\{ & \int_{B_R(x_0)} \Big(A_{ij}^{\alpha\beta}(u_R)_R - A_{ij}^{\alpha\beta}(x,u_R)\Big)\, D_\alpha u^i D_\beta u^j\,dx \\
& + \int_{B_R(x_0)} \Big(A_{ij}^{\alpha\beta}(x,u_R) - A_{ij}^{\alpha\beta}(x,u)\Big)\, D_\alpha u^i D_\beta u^j\,dx \\
& + \int_{B_R(x_0)} \Big(A_{ij}^{\alpha\beta}(x,u_R) - A_{ij}^{\alpha\beta}(u_R)_R\Big)\, D_\alpha v^i D_\beta v^j\,dx \\
& + \int_{B_R(x_0)} \Big(A_{ij}^{\alpha\beta}(x,v) - A_{ij}^{\alpha\beta}(x,u_R)\Big)\, D_\alpha v^i D_\beta v^j\,dx \Bigg\} \\
& = \frac{2}{\nu}\left(I + II + III + IV\right).
\end{aligned}
$$

Remarks to proofs - 1 A.)

$$|II| \leq \int\limits_{B_R(x_0)} \omega\left(|u - u_R|\right)|Du|^2\,dx$$

$$\leq \left(\int\limits_{B_R(x_0)} \omega^{p'}\left(|u - u_R|\right)\,dx\right)^{1/p'}\left(\int\limits_{B_R(x_0)} |Du|^{2p}\,dx\right)^{1/p}.$$

Remarks to proofs - 1 A.)

$$\left(\int_{B_R(x_0)} \omega^{p'} \left(|u - u_R| \right) \, dx \right)^{1/p'}$$

$$\leq \left(\sup_{t>0} \frac{d}{dt} \left(\omega^{p'}(t) \right) \right)^{1/p'} \left(\int_{B_R(x_0)} |u(x) - u_R| \, dx \right)^{1/p'}$$

$$\leq c(n) S_{p'}^{1/p'} R^{(n+2)/2p'} \left(\int_{B_R(x_0)} |Du|^2 \, dx \right)^{1/2p'} .$$

Remarks to proofs - 1 B.)

$$|II| \leq \int_{B_R(x_0)} \omega\left(|u-u_R|\right)|Du|^2\,dx$$
$$\leq \int_{B_R(x_0)} \Phi\left(\varepsilon|Du|^2\right)\,dx + \int_{B_R(x_0)} \Psi\left(\frac{1}{\varepsilon}\omega\left(|u-u_R|\right)\right)\,dx = I_1 + I_2.$$

Here Φ, Ψ are complementary Young functions

$$\Phi(t) = t\ln_+(at) \quad \text{for} \quad t \geq 0, \qquad \Psi(t) = \begin{cases} \frac{t}{a} & \text{for} \quad 0 \leq t < 1, \\ \frac{e^{t-1}}{a} & \text{for} \quad t \geq 1 \end{cases}$$

where $a > 0$ is a constant,

Remarks to proofs - 1 B.)

$$I_1 = \varepsilon \int_{B_R(x_0)} |Du|^2 \ln_+ \left(a\varepsilon|Du|^2\right)\, dx$$

$$\leq \varepsilon \left(\int_{B_R(x_0)} |Du|^{2p}\, dx \right)^{1/p} \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)} \left(a\varepsilon|Du|^2\right)\, dx \right)^{1-1/p}$$

$$\leq c(p, M/\nu)\varepsilon \frac{p}{\mathrm{e}(p-1)} \left(a\varepsilon \fint_{B_R(x_0)} |Du|^2\, dx \right)^{1-1/p} \phi(2R).$$

Remarks to proofs - 1 B.)

$$
\begin{aligned}
I_2 &\le \frac{1}{a}\left(\kappa_n R^n + \frac{\mathrm{e}^{\omega_\infty/\varepsilon}}{\mathrm{e}\varepsilon} S_0 \int\limits_{B_R(x_0)} |u - u_R|\, dx\right) \\
&\le \frac{1}{a}\left(\kappa_n R^n + c(n)\frac{\mathrm{e}^{\omega_\infty/\varepsilon}}{\mathrm{e}\varepsilon} S_0 R^{1+n/2}\phi^{1/2}(2R)\right)
\end{aligned}
$$

where $S_0 = \sup_{t>t_0} \omega'(t)$ $(\omega(t_0) = \varepsilon)$.

Remarks to proofs - 1 B.)

Last two estimates lead to

$$|II| \leq c(p, M/\nu)\varepsilon \left(a\varepsilon \fint_{B_R(x_0)} |Du|^2 \, dx \right)^{1-1/p} \phi(2R)$$
$$+ \frac{1}{a} \left(\kappa_n R^n + \frac{c(n)}{\varepsilon} \mathrm{e}^{\omega_\infty/\varepsilon} S_0 R^{1+n/2} \phi^{1/2}(2R) \right).$$

Concluding remark

Let $\delta \in [n-2, n)$ and $\overline{\Omega}_0 \subset \Omega$ with $\mathrm{dist}(\Omega_0, \partial\Omega) \geq 2d > 0$ ($d \leq 1$). Let u be a minimizer of (1), where $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$ and $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that $u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N)$. Let assumptions (i), (ii), (iii) be satisfied with the constants M, ν and suppose that

$$\frac{\left[1 + 2\theta\left(U_{2d}(x)\right)\mathcal{M}\right]\Theta\left(U_{2d}(x)\right)}{K\left(\frac{p\mu}{\mathrm{e}(p-1)}\right)^{\mu}} \leq 1, \quad \forall\ x \in \Omega_0$$

$$U_r(x) = r^{2-n}\int_{B_r(x)} |Du|^2\, dy, \qquad \mathcal{M} = \min\{\mathcal{P}, \mathcal{R}\}$$

Concluding remark

$$\mathcal{P} = \sup_{t_0<t<\infty} \frac{\frac{\omega(t)}{E\omega_\infty}\,\mathrm{e}^{\left(\frac{\omega(t)}{\mu E\omega_\infty}\right)^{1/(\mu-1)}} - \mathrm{e}^{\left(\frac{1}{\mu}\right)^{1/(\mu-1)}}}{t-t_0}$$

$$\mathcal{R} = \int_{t_0}^{\infty} \frac{1}{t}\frac{d}{dt}\left(\frac{\omega(t)}{E\omega_\infty}\,\mathrm{e}^{\left(\frac{\omega(t)}{\mu E\omega_\infty}\right)^{1/(\mu-1)}}\right) dt$$

$$\theta(t) = \begin{cases} \sqrt{\frac{t}{\mathrm{e}}} \\ 1 \end{cases} \qquad \Theta(t) = \begin{cases} \frac{1}{\sqrt{n\mathrm{e}}} & \text{for} \quad 0 < t \le \mathrm{e} \\ \frac{\left(\ln\, t^{1/n}\right)^{\ln\,\sqrt{t}}}{\sqrt{t}} & \text{for} \quad t > \mathrm{e} \end{cases}$$

Concluding remark

Then $u \in C^{0,(\delta-n+2)/2}(\Omega_0,\mathbb{R}^N)$ in the case $\delta > n-2$ and $u \in BMO(\Omega_0,\mathbb{R}^N)$ for $\delta = n-2$. Here $K = K(n)$, $\mu \geq 2$, $\epsilon_0 = \frac{1}{4(2^{n+3}L)^{\delta/(n-\delta)}}$ (L is the Campanato's constant),

$$E = \min\left\{1, \frac{\epsilon_0}{24CK\mu c_P^{1/2}\left(\frac{p\mu}{\mathrm{e}(p-1)}\right)^{\mu}\frac{\omega_\infty}{\nu}}\right\}, \; C = C(p,n,N,M/\nu)$$

($2p > 2$ is the exponent of integrability, c_P is the constant from Poincare inequality), $\omega(t_0) = E\,\omega_\infty$, $t_0 > 0$.

Concluding remark

1) For the constant K we have of the following estimates

$$K \le 10\mathrm{e}^n, \qquad n = 3,$$
$$K \le 20\mathrm{e}^n, \qquad n = 4.$$

2) It is clear that for u and g from the above theorem the following estimate holds

$$\|Du\|_{L^2(\Omega,\mathbb{R}^{nN})} \le c_D(n, N, M/\nu)\|Dg\|_{L^2(\Omega,\mathbb{R}^{nN})}$$

Concluding remark

3) We can replace the term $\Theta\left(U_{2d}(x)\right)$ with $\Theta\left(\|Du\|^2_{L^2}(\Omega,\mathbb{R}^{nN})/(2d)^{n-2}\right)$ or in the case of Dirichlet problem for (1) with $\Theta\left(c_D\|Dg\|_{L^2(\Omega,\mathbb{R}^{nN})}/(2d)^{n-2}\right)$ where c_D is from 2).

Thank you for your attention.