On boundary integral equations for two–dimensional slow viscous flow past a porous obstacle

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1. Introduction

$\Omega \subset \mathbb{R}^2$ bounded simply connected Lipschitz domain, $\Omega^e = \mathbb{R}^2 \setminus \overline{\Omega}$.

Brinkman in Ω :

$$(\Delta - \chi^2)\mathbf{v}^B - \nabla p^B = \mathbf{0}\,, \ \nabla \cdot \mathbf{v}^B = 0\,;$$

Navier–Stokes in Ω^e :

$$\Delta \mathbf{v}^{NS} - \nabla p^{NS} = \varepsilon \mathbf{v}^{NS} \cdot \nabla \mathbf{v}^{NS}\,, \ \nabla \cdot \mathbf{v}^{NS} = 0\,, \ \mathbf{v}^{NS}(\infty) = \mathbf{i}\,, \ p^{NS}(\infty) = 0\,;$$

Oseen in Ω^e :

$$\Delta \mathbf{v}^O - \nabla p^O = \varepsilon \mathbf{i} \cdot \nabla \mathbf{v}^O\,, \qquad \nabla \cdot \mathbf{v}^O = 0\,, \ \mathbf{v}(\infty) = \mathbf{0}\,, \ p^O(\infty) = 0\,;$$

Stokes in Ω^e :

$$\Delta \mathbf{v}^{ST} - \nabla p^{ST} = 0\,, \qquad \nabla \cdot \mathbf{v}^{ST} = 0\,, \ \mathbf{v}^{ST}(\infty) = \mathbf{i}\,, \ p^{ST}(\infty) = 0\,. \lightning$$

Reynolds number $\varrho\, a U_\infty / \mu = \varepsilon \to 0\,, \ \chi^2 = a^2/k_0$ fixed.

Solid $\boldsymbol{\Omega}$:

$NS - O$: Finn & Smith 1967, Sazonov 2003

$S - O$: Pearson & Proudman 1957, Hsiao–McCamy 1973

$NS - O - ST$: Hsiao 1982, Hsiao–McCamy 1982

Transmission conditions on Γ:

$$\mathbf{v}^{NS} = \mathbf{v}^B \quad \text{and} \quad T\mathbf{v}^{NS} = T\mathbf{v}^B\,;$$

$$\mathbf{v}^{ST} = \mathbf{v}^B \quad \text{and} \quad T\mathbf{v}^{ST} = T\mathbf{v}^B\,;$$

$$\mathbf{v}^{0} = \mathbf{v}^B - \mathbf{i} \quad \text{and} \quad T\mathbf{v}^{0} = T\mathbf{v}^B\,.$$

$ST - B$: Kohr & Raja Sekhar & Wendland 2009, Math. Meth. Appl. Sci.

$ST - B - O$: " submitted

2. Boundary integral equations for the Stokes–Brinkman flow

$$\mathbf{v}^{ST} = -\mathbf{W}\boldsymbol{\phi} + \mathbf{V}\mathbf{h} + \tfrac{1}{4\pi}\boldsymbol{\omega} \quad \text{in } \Omega^e\,, \qquad \mathbf{v}^{B} = -\mathbf{W}_{\chi^2}\boldsymbol{\phi} + \mathbf{V}_{\chi^2}\mathbf{h} \quad \text{in } \Omega\,.$$

$$\mathbf{v}^{ST} = -\tfrac{1}{2}\boldsymbol{\phi} - \mathbf{K}\boldsymbol{\phi} + \mathbf{V}\mathbf{h} + \tfrac{1}{4\pi}\boldsymbol{\omega}\,, \qquad \mathbf{v}^{B} = \tfrac{1}{2}\boldsymbol{\phi} - \mathbf{K}_{\chi^2}\boldsymbol{\phi} + \mathbf{V}_{\chi^2}\mathbf{h}\,;$$

$$T\mathbf{v}^{ST} = \mathbf{D}\boldsymbol{\phi} - \tfrac{1}{2}\mathbf{h} + \mathbf{K}'\mathbf{h}\,, \qquad T\mathbf{v}^{B} = \mathbf{D}_{\chi^2}\boldsymbol{\phi} + \tfrac{1}{2}\mathbf{h} + \mathbf{K}'_{\chi^2}\mathbf{h} \quad \text{on } \Gamma\,.$$

Stokes potentials:

$$V_{jk}h_k(\mathbf{x}) = \frac{1}{4\pi}\int\limits_\Gamma \left\{\delta_{jk}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_k-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|}\right\} h_k(\mathbf{y})d\Gamma(\mathbf{y})$$

$$W_{jk}\Phi_k(\mathbf{x}) = \frac{1}{\pi}\int\limits_\Gamma \frac{(x_k-y_k)(x_j-y_j)(x_\ell-y_\ell)}{|\mathbf{x}-\mathbf{y}|^4} n_\ell(\mathbf{y})\Phi_k(\mathbf{y})d\Gamma(\mathbf{y})$$

$$\Pi_j h_j(\mathbf{x}) = \frac{1}{2\pi}\int\limits_\Gamma \frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2} h_j(\mathbf{y})d\Gamma(\mathbf{y})$$

$$\Lambda_k\Phi_k(\mathbf{x}) = \frac{1}{\pi}\int\limits_\Gamma \left\{-\frac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|^2} + 2\frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}\right\} n_j(y)\Phi_k(\mathbf{y})d\Gamma(\mathbf{y})$$

Stokes and Brinkman operators:

$$\mathbf{W}_{\chi^2} = \mathbf{W} + \chi^2 \ln \chi PSDO(-3;\mathbb{R}^2)\ ,\ \mathbf{W}PSDO(-1;\mathbb{R}^2)$$

$$\mathbf{V}_{\chi^2} = \mathbf{V} - \left(c_E + \tfrac{1}{2} + \ln \tfrac{\chi}{2}\right) \int_\Gamma \bullet\, d\Gamma + \chi^2 \ln \chi PSDO(-4;\mathbb{R}^2)\ ,\ \mathbf{V}PSDO(-2;\mathbb{R}^2)$$

$$\mathbf{\Lambda}_{\chi^2} = \mathbf{\Lambda} + \chi^2 PSDO(-2;\mathbb{R}^2)\ ,\ \mathbf{\Lambda}PSDO(0;\mathbb{R}^2)$$

$$\mathbf{\Pi}_{\chi^2} = \mathbf{\Pi} + \chi^2 \ln \chi PSDO(-3;\mathbb{R}^2)\ ,\ \mathbf{\Pi}PSDO(-1;\mathbb{R}^2)$$

$$\mathbf{D}_{\chi^2} = \mathbf{D} + \chi^2 \ln \chi PSDO(-2;\mathbb{R}^2)\ ,\ \mathbf{D}PSDO(0;\mathbb{R}^2)$$

$$\boxed{\begin{array}{c} \boldsymbol{\phi} - \mathbf{K}_{\chi^2,0}\boldsymbol{\phi} + \mathbf{V}_{\chi^2,0}\mathbf{h} + \frac{1}{4\pi}\boldsymbol{\omega} = \mathbf{0}\ ,\ \int_\Gamma \mathbf{h}\,d\Gamma = \mathbf{A}\ ; \\ \mathbf{h} + \mathbf{K}'_{\chi^2,0}\mathbf{h} + \mathbf{D}_{\chi^2,0}\boldsymbol{\phi} = \mathbf{0} \end{array}}$$

$$h \in H^{-\frac{1}{2}+\delta}(\Gamma)\ ,\ h \times \delta_\Gamma \in H^{-1+\delta}(\Omega_{2R})\ ,$$

$$\mathbf{V}_{\chi^2,0}(h \times \delta_\Gamma) \in H^{3+\delta}(\Omega_{2R})\ ,\ \gamma_0 \mathbf{V}_{\chi^2,0}(h \times \delta_\Gamma) \in H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}+\delta}(\Gamma)\ ,\ 0 \le \delta < \tfrac{1}{2}.$$

Theorem 1: This is a system of Fredholm integral equations of the second kind

$$\mathfrak{A}\begin{pmatrix}\boldsymbol{\phi}\\ \mathbf{h}\\ \boldsymbol{\omega}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \mathbf{A}\end{pmatrix} \quad \text{for } (\boldsymbol{\phi},\mathbf{h},\boldsymbol{\omega}) \in (H^{\frac{1}{2}+\delta}(\Gamma)\times H^{-\frac{1}{2}+\delta}(\Gamma)\times\mathbb{R}^2)\ ,\ 0\le\delta<\tfrac{1}{2}\,,$$

where

$$\begin{aligned}\mathbf{K}_{\chi^2,0} &: H^{\frac{1}{2}+\delta}(\Gamma) \to H^{\frac{1}{2}+\delta}(\Gamma)\ ,\ \mathbf{V}_{\chi^2,0} : H^{-\frac{1}{2}+\delta}(\Gamma) \to H^{\frac{1}{2}+\delta}(\Gamma)\ ,\\ \mathbf{K}'_{\chi^2,0} &: H^{-\frac{1}{2}+\delta}(\Gamma) \to H^{-\frac{1}{2}+\delta}(\Gamma)\ ,\ \mathbf{D}_{\chi^2,0} : H^{\frac{1}{2}+\delta}(\Gamma) \to H^{-\frac{1}{2}+\delta}(\Gamma)\end{aligned}$$

all are linear compact. It has only the trivial kernel and, hence, it is uniquely solvable for every given $\mathbf{A}\in\mathbb{R}^2$.

3. The Stokes expansions

$$\mathbf{v}^{ST} \sim \sum_{k=1}^{\infty} \mathbf{v}^{ST}(\mathbf{x}, \mathbf{A}_k)(\ln \varepsilon)^{-k} \;,\; \mathbf{v}^{B} \sim \sum_{k=1}^{\infty} \mathbf{v}^{B}(\mathbf{x}, \mathbf{A}_k)(\ln \varepsilon)^{-k} \;; \quad \mathbf{A}_k, \boldsymbol{\omega}_k, \boldsymbol{\phi}_k, \mathbf{h}_k \,.$$

Far field: $\mathbf{x} \to \infty$:
$$\mathbf{Vh}(\mathbf{x}) \sim \frac{1}{4\pi}\mathbf{A} \ln \frac{1}{|\mathbf{x}|} + \frac{1}{4\pi}\frac{\mathbf{A}\cdot\mathbf{x}}{|\mathbf{x}|}\,\frac{\mathbf{x}}{|\mathbf{x}|} + O(|\mathbf{x}|^{-1})$$

$$\mathbf{x} = \boldsymbol{\xi}/\varepsilon \;:\; \mathbf{Vh}(\boldsymbol{\xi}/\varepsilon) \sim \frac{1}{4\pi}\mathbf{A}\ln\varepsilon + \frac{1}{4\pi}\mathbf{A}\ln\frac{1}{|\boldsymbol{\xi}|} + \frac{1}{4\pi}\,\frac{\mathbf{A}\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + O(\varepsilon|\boldsymbol{\xi}|)$$

$$\mathbf{v}^{ST} \sim \frac{1}{4\pi}\mathbf{A}_1 + (\ln\varepsilon)^{-1}\frac{1}{4\pi}\Big\{\mathbf{A}_1 \ln\frac{1}{|\boldsymbol{\xi}|} + \frac{1}{4\pi}\,\frac{\mathbf{A}\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2\Big\} + O\big((\ln\varepsilon)^{-2}\big)$$

Oseen in $\Omega^e_\varepsilon := \{\boldsymbol{\xi} \in \mathbb{R}^2 \wedge \boldsymbol{\xi}/\varepsilon \in \Omega^e\}$. For $\varepsilon \to 0$, $\Omega^e_\varepsilon \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

$$\mathbf{Q}(\boldsymbol{\xi}) := \mathbf{v}^O(\boldsymbol{\xi}/\varepsilon) - \mathbf{i} \;,\; P(\boldsymbol{\xi}) = p^O(\boldsymbol{\xi}/\varepsilon)/\varepsilon$$

$$\Delta_\xi \mathbf{Q} - \nabla_\xi \mathbf{P} = \frac{\partial}{\partial \xi_1}\mathbf{Q} \;,\; \nabla_\xi \cdot \mathbf{Q} = 0 \;,\; \mathbf{Q} = -\mathbf{i} \quad \text{for } \boldsymbol{\xi} \to \mathbf{0} \;,\; \mathbf{Q}(\infty) = \mathbf{0}\,.$$

Explicit solutions of Oseen's equation in Ω^e

$$\mathbf{Q}^*(\boldsymbol{\xi}) = -2e^{\xi_1/2}K_0\big(\tfrac{1}{2}|\boldsymbol{\xi}|\big)\mathbf{i} + 2\nabla_\xi\big\{e^{\xi_1/2}K_0\big(\tfrac{1}{2}|\boldsymbol{\xi}|\big) + \ln(|\boldsymbol{\xi}|)\big\}\ ,\ \mathbf{P}^*(\boldsymbol{\xi}) = -2\xi_1|\boldsymbol{\xi}|^{-2}$$

$$\mathbf{v}^O(\boldsymbol{\xi}/\varepsilon) \sim \mathbf{i} + \sum_{k=1}^{\infty}\mathbf{Q}_k(\boldsymbol{\xi})(\ln\varepsilon)^{-k}\ ,\ P^O(\boldsymbol{\xi}) \sim \sum_{k=1}^{\infty}P_k(\boldsymbol{\xi})(\ln\varepsilon)^{-k}$$

$$\Delta_\xi\mathbf{Q}_k - \nabla_\xi P_k = \tfrac{\partial}{\partial\xi_1}\mathbf{Q}_k + \mathbf{R}_k \quad \text{for } \boldsymbol{\xi} \neq 0\ ,\ \nabla_\xi \cdot \mathbf{Q}_k = \mathbf{0}\,,$$

$$\mathbf{Q}_k \to \mathbf{0}\ ,\ P_k \to 0 \quad \text{for } |\boldsymbol{\xi}| \to \infty\,,$$

$$\mathbf{R}_1 = \mathbf{0}\ ,\ \mathbf{R}_k = \sum_{m-1}^{k-1}\mathbf{Q}_m \cdot \nabla_\xi\mathbf{Q}_{k-m}\ ,\ k \geq 2\,.$$

Then $\mathbf{Q}_1(\boldsymbol{\xi}) = a_1\big\{\mathbf{i}[\ln|\boldsymbol{\xi}| + c_E - \ln 4] - \frac{\xi_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\big\}$,

$$\mathbf{v}^O(\boldsymbol{\xi}/\varepsilon) = \mathbf{i} + a_1\{\mathbf{i}[\ln|\boldsymbol{\xi}| + c_E - \ln 4]\frac{\xi_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\}(\ln\varepsilon)^{-1} + O\big((\ln\varepsilon)^{-2}\big) + O(|\boldsymbol{\xi}|\ln|\boldsymbol{\xi}|)$$

$c_E = 0,5772\ldots$ Euler's constant

Matching:

$$\mathbf{v}^{ST}(\boldsymbol{\xi}/\varepsilon) \sim \mathbf{v}^{O}(\boldsymbol{\xi}/\varepsilon)\ ,\ \varepsilon \to 0\ :\ \mathbf{A} = \sum_{k\geq 1} \mathbf{A}_k (\ln\varepsilon)^{-k}\ ,\ \boldsymbol{\omega} = \sum_{k\geq 1} \boldsymbol{\omega}_k (\ln\varepsilon)^{-k}$$

$$\frac{1}{4\pi}\mathbf{A}_1 + \frac{1}{4\pi}(\ln\varepsilon)^{-1}\Big\{\mathbf{A}_1 \ln\frac{1}{|\boldsymbol{\xi}|} + \frac{\mathbf{A}_1\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2\Big\} + O\big((\ln\varepsilon)^{-2}\big)$$

$$\sim \mathbf{i} + a_1(\ln\varepsilon)^{-1}\Big\{\mathbf{i}[\ln\frac{1}{|\boldsymbol{\xi}|} + c_E - \ln 4] - \frac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big\} + O(|\boldsymbol{\xi}|\ln|\boldsymbol{\xi}|)$$

$$\Rightarrow \mathbf{A}_1 = 4\pi\mathbf{i} \Rightarrow \mathbf{i}\ln\frac{1}{|\boldsymbol{\xi}|} + \frac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2 = a_1\Big\{\mathbf{i}[-\ln\frac{1}{|\boldsymbol{\xi}|} + c_E - \ln 4] - \frac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big\}$$

$$\Rightarrow a_1 = -1 \wedge \mathbf{A}_2 = (\ln 4 - c_E)4\pi\mathbf{i} - \boldsymbol{\omega}_1$$

$k=1$: $\boldsymbol{\phi}_1 - \mathbf{K}_{\chi^2,0}\boldsymbol{\phi}_1 + \mathbf{V}_{\chi^2,0}\mathbf{h}_1 + \frac{1}{4\pi}\boldsymbol{\omega}_1 = \mathbf{0}$,

$\mathbf{h}_1 + \mathbf{K}'_{\chi^2,0}\mathbf{h}_1 + \mathbf{D}_{\chi^2,0}\boldsymbol{\phi}_1 = \mathbf{0}\ ,\ \int_\Gamma \mathbf{h}_1 d\Gamma = \mathbf{A}_1 = 4\pi\mathbf{i}$;

$k=2$: $\boldsymbol{\phi}_2 - \mathbf{K}_{\chi^2,0}\boldsymbol{\phi}_2 + \mathbf{V}_{\chi^2,0}\mathbf{h}_2 + \frac{1}{4\pi}\boldsymbol{\omega}_2 = \mathbf{0}$,

$\mathbf{h}_2 + \mathbf{K}'_{\chi^2,0}\mathbf{h}_2 + \mathbf{D}_{\chi^2,0}\boldsymbol{\phi}_2 = \mathbf{0}\ ,\ \int_\Gamma \mathbf{h}_2 d\Gamma = \mathbf{A}_2 = 4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}_1$

Stokes approximation:

$$
\begin{aligned}
\mathbf{\Phi}_\varepsilon &:= \boldsymbol{\phi}_1(\ln\varepsilon)^{-1} + \boldsymbol{\phi}_2(\ln\varepsilon)^{-2}\ ,\ \boldsymbol{\omega}_\varepsilon := \boldsymbol{\omega}_1(\ln\varepsilon)^{-1} + \boldsymbol{\omega}_2(\ln\varepsilon)^{-2} \\
\mathbf{H}_\varepsilon &:= \mathbf{h}_1(\ln\varepsilon)^{-1} + \mathbf{h}_2(\ln\varepsilon)^{-2} \\
\mathbf{v}_\varepsilon^{ST} &:= \mathbf{W}\mathbf{\Phi}_\varepsilon + \mathbf{V}\mathbf{H}_\varepsilon + \tfrac{1}{4\pi}\boldsymbol{\omega}_\varepsilon\ ,\ p_\varepsilon^{ST} = \Lambda\mathbf{\Phi}_\varepsilon + \Pi\mathbf{H}_\varepsilon \\
\mathbf{v}_\varepsilon^{B} &:= \mathbf{W}_{\chi^2}\mathbf{\Phi}_\varepsilon + \mathbf{V}_{\chi^2}\mathbf{H}_\varepsilon\ ,\ p_\varepsilon^{B} = \Lambda_{\chi^2}\mathbf{\Phi}_\varepsilon + \Pi_{\chi^2}\mathbf{H}_\varepsilon
\end{aligned}
$$

Question:

$$
\|\mathbf{v}^{NS} - \mathbf{v}_\varepsilon^{ST}\|_{H^{1+\delta}(\Omega_{2R}^e)} + \|\mathbf{v}^{NSB} - \mathbf{v}_\varepsilon^{B}\|_{H^{1+\delta}(\Omega)}
$$

$$
+ \|p^{NS} - p_\varepsilon^{ST}\|_{H^{\delta}(\Omega_{2R}^e)} + \|p^{NSB} - p_\varepsilon^{B}\|_{H^{\delta}(\Omega)} \le c(R,\delta)|\ln\varepsilon|^{-3}
$$

$$
\Omega_{2R}^e := \{\mathbf{x} \in \Omega^e \,|\, |\mathbf{x}| \le 2R\}
$$

4. Stokes approximation of Oseen flow

Theorem: Γ Lipschitz. There exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$:

$$\|\mathbf{v}^O - \mathbf{v}_\varepsilon^{ST}\|_{H^{1+\delta}(\Omega^e_{2R})} + \|\mathbf{v}^{OB} - \mathbf{v}_\varepsilon^B\|_{H^{1+\delta}(\Omega)} + \|p^O - p_\varepsilon^{ST}\|_{H^\delta(\Omega^e_{2R})} + \|p^{OB} - p_\varepsilon^B\|_{H^\delta(\Omega)} \le c(R,\delta)|\ln\varepsilon|^{-3}$$

Proof: Follow [Hsiao 1982 and Hsiao & MacCamy 1982].

$$\begin{aligned}
\mathbf{v}^O &= -\mathbf{W}^\varepsilon \boldsymbol{\phi}_\varepsilon + \mathbf{V}^\varepsilon \mathbf{h}_\varepsilon && \text{in } \Omega^e\,,\\
\mathbf{v}^B &= -\mathbf{W}_{\chi^2} \boldsymbol{\phi}_\varepsilon + \mathbf{V}_{\chi^2}\, \mathbf{h}_\varepsilon && \text{in } \Omega\,,\\
\mathbf{v}^O &= \mathbf{v}^B - \mathbf{i} \quad \text{and } T\mathbf{v}^O = T\mathbf{v}^B && \text{on } \Gamma\,.
\end{aligned}$$

Lemma [Sazonov 2003]

$$\mathbf{V}^\varepsilon\mathbf{h} = \mathbf{V}\mathbf{h} + \tfrac{1}{4\pi}(\ln\varepsilon\mathbf{I} - \mathbf{J})\int_\Gamma \mathbf{h}d\Gamma + \varepsilon\ln\varepsilon\mathbf{V}_\varepsilon^R\mathbf{h}\,,$$

$$\mathbf{W}^\varepsilon\boldsymbol{\phi} = \mathbf{W}\boldsymbol{\phi} + \varepsilon\ln\varepsilon\mathbf{W}_\varepsilon^R\boldsymbol{\phi}\,,\ \mathbf{J} = \begin{pmatrix}\ln 4 - c_E & 0\\ 0 & \ln 4 - c_E - 1\end{pmatrix},$$

$$\mathbf{V}_\varepsilon^R PSDO(-2\,;\,\mathbb{R}^2)\,,\ \mathbf{W}_\varepsilon^R PSDO(-1\,;\,\mathbb{R}^2)\,;\ \exists c\,:$$

$$\forall\varepsilon\in[0,1]\,:\ \|\mathbf{V}_\varepsilon^R\mathbf{h}\|_{H^{\frac12+\delta}(\Gamma)} \le c\|\mathbf{h}\|_{H^{-\frac12+\delta}(\Gamma)}\,,\ \|\mathbf{W}_\varepsilon^R\boldsymbol{\phi}\|_{H^{\frac12+\delta}(\Gamma)} \le c\|\boldsymbol{\phi}\|_{H^{\frac12+\delta}(\Gamma)}\,.$$

Boundary integral equations

$$\boldsymbol{\Phi}_\varepsilon - \mathbf{K}_{\chi^2,0}\boldsymbol{\Phi}_\varepsilon + \mathbf{V}_{\chi^2,0}\mathbf{h}_\varepsilon + (\varepsilon\ln\varepsilon)\{\mathbf{K}_\varepsilon^R\boldsymbol{\Phi}_\varepsilon + \mathbf{V}_\varepsilon^R\mathbf{h}_\varepsilon\}$$
$$+\tfrac{1}{4\pi}\ln\varepsilon\big(\mathbf{I} - (\ln\varepsilon)^{-1}\mathbf{J}\big)\int_\Gamma \mathbf{h}_\varepsilon d\Gamma + \mathbf{i} = \mathbf{0}\,,$$

$$\mathbf{h}_\varepsilon + \mathbf{K}'_{\chi^2,0}\mathbf{h}_\varepsilon + \mathbf{D}_{\chi^2,0}\boldsymbol{\Phi}_\varepsilon + \varepsilon\ln\varepsilon(\mathbf{D}_\varepsilon^R\boldsymbol{\Phi}_\varepsilon - \mathbf{K}_\varepsilon^{R\prime}\mathbf{h}_\varepsilon) = \mathbf{0}\,.$$

Perturbation of Stokes–Brinkman with

$$\tfrac{1}{4\pi}\boldsymbol{\omega}_\varepsilon = \tfrac{1}{4\pi}(\ln\varepsilon)\big(\mathbf{I} - (\ln\varepsilon)^{-1}\mathbf{J}\big)\int_\Gamma \mathbf{h}_\varepsilon d\Gamma + \mathbf{i}$$

$$\mathfrak{A}_\varepsilon \begin{pmatrix} \mathbf{\Phi}_\varepsilon \\ \mathbf{h}_\varepsilon \\ \boldsymbol{\omega}_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{A}_\varepsilon \end{pmatrix} , \ \mathfrak{A}_\varepsilon = \mathfrak{A} + (\varepsilon \ln \varepsilon)\mathfrak{B}_\varepsilon \ , \ \boldsymbol{\omega}_\varepsilon = \tfrac{1}{4\pi}(\ln \varepsilon)\big(\mathbf{I} - (\ln\varepsilon)^{-1}\mathbf{J}\big)\mathbf{A}_\varepsilon + \mathbf{i}$$

$\mathfrak{A}_\varepsilon$ is a regular perturbation of $\mathfrak{A}$. Hence, there exists $\varepsilon_0 > 0$ such that

$$\mathfrak{A}_\varepsilon^{-1} = \big(\mathbf{I} - (\varepsilon \ln \varepsilon)\mathfrak{A}^{-1}\mathfrak{B}_\varepsilon\big)^{-1}\mathfrak{A}^{-1} = \textstyle\sum_{k=0}^{\infty}(\varepsilon\ln\varepsilon)^k(-\mathfrak{A}^{-1}\mathfrak{B}_\varepsilon)^k\mathfrak{A}^{-1}$$

$$= \mathfrak{A}^{-1} + (\ln\varepsilon)^{-5}\mathfrak{C}_\varepsilon\mathfrak{A}^{-1} \ : \ \mathcal{H}^\delta(\Gamma) \to \mathcal{H}^\delta(\Gamma)$$

exixts for $0 \le \varepsilon \le \varepsilon_0$ where $\mathcal{H}^\delta(\Gamma) := H^{\frac{1}{2}+\delta}(\Gamma) \times H^{-\frac{1}{2}+\delta}(\Gamma) \times \mathbb{R}^2$ and $0 \le \delta < \frac{1}{2}$.

$$\mathbf{A}_\varepsilon = \int_\Gamma \mathbf{h}_\varepsilon d\Gamma = (\ln\varepsilon)^{-1}\Big(\mathbf{I} + \textstyle\sum_{k=1}^{\infty}(\ln\varepsilon)^{-k}\mathbf{J}^k\Big)(4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}_\varepsilon) \quad (A)$$

$$\mathbf{A}_\varepsilon = \mathbf{A}_0 + (\ln\varepsilon)^{-1}\mathbf{A}_1 + (\ln\varepsilon)^{-2}\mathbf{A}_2 + (\ln\varepsilon)^{-3}\mathbf{A}_R$$

$$\boldsymbol{\omega}_\varepsilon = \boldsymbol{\omega}_0 + (\ln\varepsilon)^{-1}\boldsymbol{\omega}_1 + (\ln\varepsilon)^{-2}\boldsymbol{\omega}_2 + (\ln\varepsilon)^{3}\boldsymbol{\omega}_R$$

Assumption: $|\mathbf{A}_\varepsilon| < \infty \quad \text{for } 0 \le \varepsilon \le 1\,.$

$$\Rightarrow \|\boldsymbol{\Phi}_\varepsilon\|_{H^{\frac{1}{2}+\delta}(\Gamma)}\,,\ \|\mathbf{h}_\varepsilon\|_{H^{-\frac{1}{2}+\delta}(\Gamma)}\,,\ |\boldsymbol{\omega}_\varepsilon| \quad \text{uniformly bounded}$$

$$(A) \Rightarrow \mathbf{A}_0 = 0 \Rightarrow \boldsymbol{\phi}_0 = 0\,,\ \mathbf{h}_0 = 0\,,\ \boldsymbol{\omega}_0 = 0$$

$$(A) \Rightarrow \mathbf{A}_1 = 4\pi\mathbf{i} \Rightarrow \boldsymbol{\phi}_1\,,\ \mathbf{h}_1\,,\ \boldsymbol{\omega}_1$$

$$(A) \Rightarrow \mathbf{A}_2 = 4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}_1 \Rightarrow \boldsymbol{\phi}_2\,,\ \mathbf{h}_2\,,\ \boldsymbol{\omega}_2$$

$$\begin{aligned}(A) \Rightarrow \mathbf{A}_R = 4\pi\sum_{k=0}^{\infty}(\ln\varepsilon)^{-k}\mathbf{J}^{k+2}\mathbf{i} - \sum_{k=0}^{\infty}(\ln\varepsilon)^{-k}\mathbf{J}^{k+1}\big(\mathbf{I} + (\ln\varepsilon)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_1 \\ -\big\{\mathbf{I} + \sum_{k=0}^{\infty}(\ln\varepsilon)^{-k}\mathbf{J}^{k}\big(\mathbf{I} + (\ln\varepsilon)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_2 \\ + (\ln\varepsilon)^{-1}\big\{\mathbf{I} - \sum_{k=0}^{\infty}(\ln\varepsilon)^{-k}\mathbf{J}^{k}\big(\mathbf{I} + (\ln\varepsilon)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_R\end{aligned}$$

$$\Rightarrow \quad |\mathbf{A}_R| \le c_1 + |\ln\varepsilon|^{-1}c_2|\boldsymbol{\omega}_R|$$

$$(\boldsymbol{\Phi}_R, \mathbf{h}_R, \boldsymbol{\omega}_R)^\top = \mathfrak{A}_\varepsilon^{-1}\mathbf{A}_R + (\ln\varepsilon)^{-2}\mathfrak{C}_\varepsilon\mathfrak{A}^{-1}\big\{(\ln\varepsilon)^{-1}\mathbf{A}_1 + (\ln\varepsilon)^{-2}\mathbf{A}_2\big\}$$

$$\Rightarrow |\boldsymbol{\omega}_R| \le c_3|\mathbf{A}_R| + c_4 \le (c_1c_3 + c_4) + c_2c_3|\ln\varepsilon|^{-1}|\boldsymbol{\omega}_R|$$

$\Rightarrow$ Ex. c_5 and $\varepsilon_1 > 0$, $\varepsilon_1 \le \varepsilon_0$ such that for all $\varepsilon \in [0, \varepsilon_1]$:

$|\boldsymbol{\omega}_R| \le c_5$ and $\|\boldsymbol{\Phi}_R\|_{H^{\frac{1}{2}+\delta}(\Gamma)} \le c_5$, $\|\mathbf{h}_R\|_{H^{-\frac{1}{2}+\delta}(\Gamma)} \le c_5$.

$$\begin{pmatrix}\boldsymbol{\Phi}_\varepsilon\\ \mathbf{h}_\varepsilon\\ \boldsymbol{\omega}_\varepsilon\end{pmatrix} - (\ln\varepsilon)^{-1}\begin{pmatrix}\boldsymbol{\phi}_1\\ \mathbf{h}_1\\ \boldsymbol{\omega}_1\end{pmatrix} - (\ln\varepsilon)^{-2}\begin{pmatrix}\boldsymbol{\phi}_2\\ \mathbf{h}_2\\ \boldsymbol{\omega}_2\end{pmatrix} = (\ln\varepsilon)^{-3}\begin{pmatrix}\boldsymbol{\phi}_R\\ \mathbf{h}_R\\ \boldsymbol{\omega}_R\end{pmatrix}$$

$$\begin{aligned}&\|\mathbf{v}^O - \mathbf{v}_\varepsilon^{ST}\|_{H^{1+\delta}(\Omega^e_{2R})} + \|\mathbf{v}^{OB} - \mathbf{v}_\varepsilon^{B}\|_{H^{1+\delta}(\Omega)}\\ &\le |\ln\varepsilon|^{-3}c(R,\delta)\Big\{\|\boldsymbol{\Phi}_R\|_{H^{\frac{1}{2}+\delta}(\Gamma)} + \|\mathbf{h}_R\|_{H^{-\frac{1}{2}+\delta}(\Gamma)} + |\boldsymbol{\omega}_R|\Big\}\end{aligned}$$

5. Boundary elments on $\mathbf{\Gamma}$:

Γ smooth: $\mathbf{\Phi} \in H^2(\Gamma)$, $\mathbf{h} \in H^1(\Gamma)$

Δ periodic partition , knots t_j , $j = 1, \ldots, N$, $\tau_j := t_j + h_j/2$
$\mathbf{\Phi} \sim \mathbf{\Phi}_h \in \mathcal{S}^1(\Delta_h)$, $\mathbf{h} \sim \mathbf{h}_h \in \mathcal{S}^0(\Delta_h)$

Spline collocation: Equations in $H^0(\Gamma)$

$$\mathbf{\Phi}_h(t_j) - (\mathbf{K}_{\chi^2,0}\mathbf{\Phi}_h)(t_j) + (\mathbf{V}_{\chi^2,0}\mathbf{h}_h)(t_j) + \frac{1}{4\pi}\boldsymbol{\omega} = 0\,,$$

$$\mathbf{h}_h(t_j) + (\mathbf{K}'_{\chi^2,0}\mathbf{h}_h)(\tau_j) + (\mathbf{D}_{\chi^2,0}\mathbf{\Phi}_h)(\tau_j) = 0\,,\ \int\limits_\Gamma \mathbf{h}_h d\Gamma = \mathbf{A}$$

Prößdorf & Schmidt 1981, 1984

$$\|\mathbf{\Phi} - \mathbf{\Phi}_h\|_{H^{\frac{1}{2}+\delta}(\Gamma)} + \|\mathbf{h} - \mathbf{h}_h\|_{H^{-\frac{1}{2}+\delta}(\Gamma)} \le ch^{3/2-\delta}\big(\|\mathbf{\Phi}\|_{H^2(\Gamma)} + \|\mathbf{h}\|_{H^1(\Gamma)}\big)$$

$$\|p - p_h\|_{H^\delta} + \|\mathbf{v} - \mathbf{v}_h\|_{H^{1+\delta}} \le ch^{3/2-\delta}\big(\|\mathbf{\Phi}\|_{H^2(\Gamma)} + \|\mathbf{h}\|_{H^1(\Gamma)}\big)$$

Γ Lipschitz: $\mathbf{\Phi} \in H^1(\Gamma)$, $\mathbf{h} \in H^0(\Gamma)$

Spline – Galerkin:

$$\|\mathbf{\Phi} - \mathbf{\Phi}_h\|_{H^{\frac{1}{2}+\delta}(\Gamma)} + \|\mathbf{h} - \mathbf{h}_h\|_{H^{-\frac{1}{2}+\delta}(\Gamma)} \leq ch^{\frac{1}{2}-\delta}\big(\|\mathbf{\Phi}\|_{H^1(\Gamma)} + \|\mathbf{h}\|_{H^0(\Gamma)}\big)$$

$$\|p - p_h\|_{H^\delta} + \|\mathbf{v} - \mathbf{v}_h\|_{H^{1+\delta}} \leq ch^{\frac{1}{2}-\delta}\big(\|\mathbf{\Phi}\|_{H^1(\Gamma)} + \|\mathbf{h}\|_{H^0(\Gamma)}\big)$$

Conclusions:

- **Existence of the weak solution to the transmission problem with $\mathbf{v}^{NS}$ in $H^{1+\delta}(\Omega^e_{2R}) \cap L_{q_1 q_2}(\Omega^c_R)$ and $\mathbf{v}^{NSB}$ in $H^{1+\delta}(\Omega)$ is not yet known.**
- **The Finn–Smith–Sazonov approximation $\mathbf{v}^{NS} - \mathbf{v}^{O}$, $\mathbf{v}^{NSB} - \mathbf{v}^{OB}$ is an open problem.**

Thank you for your attention