BI-SOBOLEV HOMEOMORPHISM WITH ZERO JACOBIAN ALMOST EVERYWHERE

LUIGI D'ONOFRIO, STANISLAV HENCL AND ROBERTA SCHIATTARELLA

Abstract. Let $N \geq 3$. We construct a homeomorphism f in the Sobolev space $W^{1,1}((0,1)^N,(0,1)^N)$ such that $f^{-1} \in W^{1,1}((0,1)^N,(0,1)^N)$, $J_f = 0$ a.e. and $J_{f^{-1}} = 0$ a.e.. It follows that f maps a set of full measure to a null set and a remaining null set to a set of full measure. We also show that such a pathological homeomorphism cannot exist in dimension $N = 2$ or with higher regularity $f \in W^{1,N-1}$.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^N$ is an open set and $f : \Omega \to \mathbb{R}^N$ is a mapping of the Sobolev class $W^{1,p}(\Omega,\mathbb{R}^N)$, $p \geq 1$. Here $W^{1,p}(\Omega,\mathbb{R}^N)$ consists of all p-integrable mappings of Ω into $\mathbb{R}^N$ whose coordinate functions have p-integrable distributional derivatives. In geometric function theory we study mappings f and their properties. One of the most important properties are the fact that f maps sets of measure zero to sets of measure zero and that preimages of sets of measure zero have zero measure. If we imagine our mapping f as the deformation of the body in the space than these properties have the following physical interpretation: new material cannot be created from 'nothing' and no material can be 'lost' during our transformation. From the mathematical point of view these properties are strongly connected with the validity of change of variables formula which is crucial in the development of the theory. For an overview of the field, discussion of interdisciplinary links and further references see [11].

It was known already to Ponomarev [17] that it is possible to construct a Sobolev homeomorphism which maps a null set to a set of positive measure (see also [15], [12]). On the other hand under suitable assumptions (see e.g. [16], [13] and references given there) we know that a Sobolev homeomorphism f satisfies the Lusin (N) condition, i.e. maps sets of measure zero to sets of measure zero. Using Cantor type construction similar to [17] one can show that there are Lipschitz mappings which map a set of positive measure to a null set and thus $J_f = 0$ on this set of positive measure while such examples cannot exist for reasonable mappings f (see [15] and [14]). For an overview on this subject, detailed proofs and counterexamples we recommend [8].

Motivated by these results and also by some results about the sign of the Jacobian [9] it was recently shown in [6] that it is possible to construct even homeomorphism in the the Sobolev space $W^{1,p}((0,1)^N,(0,1)^N)$, $1 \leq p < N$, such that $J_f = 0$ a.e. This mapping cannot be obtained as a simple iteration of known counterexamples and it requires several new ideas and a novel construction. Let us mention some strange consequences of the existence of a mapping such that $J_f = 0$ a.e. The area formula for Sobolev mappings (see e.g. [5]) holds up to a set of measure zero Z, i.e.

$$0 = \int_{\Omega\setminus Z} J_f(x) = \int_{f(\Omega\setminus Z)} 1 = \mathcal{L}_N\big(f(\Omega \setminus Z)\big),$$

but $\mathcal{L}_N(\Omega \setminus Z) = \mathcal{L}_N(\Omega)$. It also follows that

$$\mathcal{L}_N(Z) = 0 \quad \text{but} \quad \mathcal{L}_N\big(f(Z)\big) = \mathcal{L}_N\big(f(\Omega)\big).$$

It means that such a mapping simultaneously sends a null set to a set of full measure and a set of full measure to a null set. On the other hand each homeomorphism in the Sobolev space $W^{1,N}((0,1)^N,\mathbb{R}^N)$ satisfies the Lusin (N) condition [16] and therefore the image of each null set is a

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null set, in particular there is no homeomorphism in $W^{1,N}$ such that $J_f = 0$ a.e. Similar construction with finer choice of parameters and estimates was later used by Černý [1] to obtain such a mapping with the sharp integrability of the derivative slightly below $W^{1,N}$.

In this paper we address the issue of the possible Sobolev regularity of the inverse of this pathological homeomorphism. In particular we would like to know if there is such a bi-Sobolev homeomorphism, i.e. homeomorphism with $f \in W^{1,1}$ and $f^{-1} \in W^{1,1}$. We recommend [10] for basic properties and applications of bi-Sobolev mappings.

Theorem 1.1. *Let $N \geq 3$. There is a bi-Sobolev homeomorphism $f : (0,1)^N \to (0,1)^N$ such that $J_f(x) = 0$ and $J_{f^{-1}}(y) = 0$ almost everywhere.*

We combine some nontrivial known results to show that such a pathological homeomorphism cannot exist in dimension $N = 2$ or in higher dimension with $W^{1,N-1}$ regularity of f. Let us recall that the construction in [6] and [1] gives a Sobolev homeomorphism (but not bi-Sobolev) in $W^{1,p}$ for every $p < N$.

Theorem 1.2. *Let $N \geq 2$ and let $f \in W^{1,N-1}((0,1)^N, \mathbb{R}^N)$ be a bi-Sobolev homeomorphism. Then $J_f(x) \neq 0$ on a set of positive measure.*

To construct a mapping f in Theorem 1.1 we use some ideas and notations from the previous paper [6] but our construction is essentially more complicated and it requires several new ideas and improvements. We know from [10] that each bi-Sobolev mapping satisfies $J_f(x) = 0 \Rightarrow \operatorname{adj} Df(x) = 0$ a.e. and thus our mapping must satisfy also $\operatorname{adj} Df(x) = 0$ a.e. in $(0,1)^N$. Here $\operatorname{adj} Df(x)$ denotes the adjugate matrix (matrix of all $(N-1) \times (N-1)$ subdeterminants of $Df(x)$). To obtain a map with zero Jacobian in the previous constructions it was enough to squeeze certain Cantor type set only in one direction, but we have to squeeze these sets in two directions to obtain mapping with $\operatorname{adj} Df = 0$ a.e.

The mappings are constructed as a composition of many mappings and the derivative is computed using chain rule as a product of derivatives of corresponding functions. In [6] it was essential that all the matrices involved are almost diagonal and thus we can make better estimates of the norm of their product than simply estimate norm of each matrix. After squeezing in two directions our mappings are no longer almost diagonal but we repair this by choosing different coordinate systems in different steps of our construction. This linear transformation allows us to make some of the matrices almost upper triangular which will be sufficient for our estimates.

Of course we need to estimate also the derivatives of the inverse mappings in these constructions. After applying all the improvements described above we would get a mapping f whose inverse does have an integrable derivative. The main new ingredient is the following which makes the properties of f and f^{-1} somewhat similar. We will construct a sequence of homeomorphisms F_j which will eventually converge to f and disjoint Cantor type sets $\mathcal{C}_j$ such that $\mathcal{L}_3(\mathcal{C}_j) > 0$ and $J_{F_j} = 0$ a.e. on $\mathcal{C}_j$ for $j \in \bigcup_{k\in\mathbb{N}}\{6k+1, 6k+2, 6k+3\}$ while $\mathcal{L}_3(F_j(\mathcal{C}_j)) > 0$ and $J_{F_j^{-1}} = 0$ a.e. on $F_j(\mathcal{C}_j)$ for $j \in \bigcup_{k\in\mathbb{N}}\{6k+4, 6k+5, 6k+6\}$. The mappings F_{6k+1} and F^{-1}_{6k+4} are squeezing the Cantor type set in the direction of x and y axes, F_{6k+2} and F^{-1}_{6k+5} are squeezing after rotation in the directions x and z and finally F_{6k+3} and F^{-1}_{6k+6} are squeezing after rotation in the directions y and z.

It would be nice to determine all possible values of p and q for which there is a bi-Sobolev mapping with $f \in W^{1,p}$, $f^{-1} \in W^{1,q}$ and $J_f = 0$ a.e. We have not pursued this direction. We will use the usual convention that C denotes a generic constant whose value may change at each occurrence.

2. Proof of Theorem 1.2

Suppose for contrary that there is a bi-Sobolev homeomorphism $f \in W^{1,N-1}$ such that $J_f = 0$ a.e. From [10] we know that each bi-Sobolev mapping is a mapping of finite inner distortion, i.e. for almost every x we have

$$J_f(x) = 0 \Rightarrow \operatorname{adj} Df(x) = 0 \qquad \text{a.e..}$$

Since $J_f(x) = 0$ a.e. we obtain that $\operatorname{adj} Df(x) = 0$ a.e.

From [4] (see also [7], [3] and [2]) we know that each $W^{1,N-1}$ homeomorphism of finite inner distortion satisfies $f^{-1}\in W^{1,1}$ and we have the following identity

$$\int_{(0,1)^N} |\operatorname{adj} Df(x)|\, dx = \int_{f((0,1)^N)} |Df^{-1}(y)|\, dy\ .$$

Since the left hand side equals to zero we obtain that $Df^{-1}(y)=0$ a.e. Using the absolute continuity of f^{-1} on almost all lines it is not difficult to deduce that f^{-1} maps everything to a point which clearly contradicts the fact that f is a homeomorphism.

Now we can proceed to the construction in Theorem 1.1. From some technical reasons we construct a mapping from some rhomboid onto the same rhomboid and not from the unit cube onto the unit cube. This difference is of course immaterial. We give the details of the construction $f=(f_1,f_2,f_3)$ in dimension $N=3$. In general dimension it is possible to use for example the mapping

$$f(x_1,x_2,\dots,x_N)=(f_1(x_1,x_2,x_3),f_2(x_1,x_2,x_3),f_3(x_1,x_2,x_3),x_4,\dots,x_N)$$

which is again a bi-Sobolev homeomorphism with zero Jacobian a.e.

3. Basic building block

We begin by defining "building blocks". For $0<w$ and $s\in(0,1)$, we denote the diamond of width w by

$$Q^z(w)=\{(x,y,z)\in\mathbb{R}^3 : |x|+|y|<w(1-|z|)\}.$$

We will often work with the inner smaller diamond and the outer annular diamond defined as

$$I^z(w,s)=Q^z(ws) \text{ and } O^z(w,s)=Q^z(w)\setminus Q^z(ws).$$

Given parameters $s\in[\frac12,1), s'\in[\frac14,1)$, we will repeatedly employ the mapping $\varphi^z_{w,s,s'}\colon Q^z(w)\to Q^z(w)$ defined by

$$\varphi^z_{w,s,s'}(x,y,z)=\begin{cases}\Big(\frac{1-s'}{1-s}x+(1-|z|)w\frac{s'-s}{1-s}\frac{x}{|x|+|y|},\ \frac{1-s'}{1-s}y+(1-|z|)w\frac{s'-s}{1-s}\frac{y}{|x|+|y|},z\Big) & (x,y,z)\in O^z(w,s),\\ \Big(\frac{s'}{s}x,\ \frac{s'}{s}y,z\Big) & (x,y,z)\in I^z(w,s).\end{cases}$$

If $s'<s$, then this homeomorphism horizontally compresses $I^z(w,s)$ onto $I^z(w,s')$, while stretching $O^z(w,s)$ onto $O^z(w,s')$. Note that $\varphi^z_{w,s,s'}$ is the identity on the boundary of $Q^z(w)$.

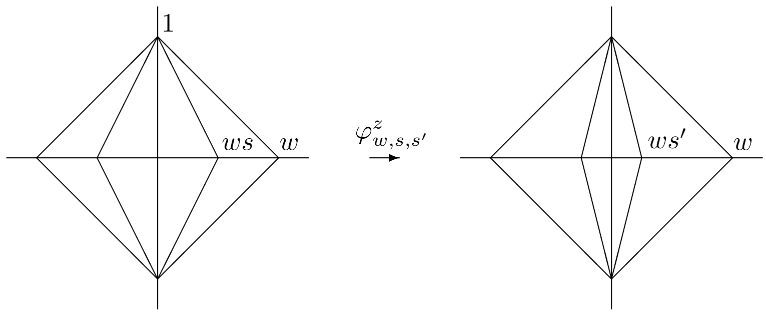

Fig. 1. The restriction of the mapping $\varphi^z_{w,s,s'}$ to the x,z-plane

If (x,y,z) is an interior point of $I^z(w,s)$, then

$$D\varphi^z_{w,s,s'}(x,y,z)=\begin{pmatrix}\frac{s'}{s}&0&0\\0&\frac{s'}{s}&0\\0&0&1\end{pmatrix} \tag{3.1}$$

and if (x,y,z) is an interior point of $O^z(w,s)$ and $z\neq 0$, then

(3.2)

$$D\varphi^z_{w,s,s'}(x,y,z)=$$
$$\begin{pmatrix} \frac{1-s'}{1-s}+(1-|z|)w\frac{s'-s}{1-s}\Bigl(\frac{1}{|x|+|y|}-\frac{\operatorname{sgn}(x)x}{(|x|+|y|)^2}\Bigr) & (1-|z|)w\frac{s'-s}{1-s}\Bigl(-\frac{\operatorname{sgn}(y)x}{(|x|+|y|)^2}\Bigr) & -w\frac{s'-s}{1-s}\frac{\operatorname{sgn}(z)x}{|x|+|y|}\\ (1-|z|)w\frac{s'-s}{1-s}\Bigl(-\frac{\operatorname{sgn}(x)y}{(|x|+|y|)^2}\Bigr) & \frac{1-s'}{1-s}+(1-|z|)w\frac{s'-s}{1-s}\Bigl(\frac{1}{|x|+|y|}-\frac{\operatorname{sgn}(y)y}{(|x|+|y|)^2}\Bigr) & -w\frac{s'-s}{1-s}\frac{\operatorname{sgn}(z)y}{|x|+|y|}\\ 0 & 0 & 1\end{pmatrix}.$$

Clearly $\frac{|x|}{|x|+|y|}\leq 1$ and since $s\geq\frac12$ we have $\frac{(1-|z|)w}{|x|+|y|}\leq 2$ for every $(x,y,z)\in O^z(w,s)$. We will have to work quite often with this matrix and therefore we will use a notation c to denote an expression which may depend on x,y,z but we know that $|c|\leq 1$. This expression may have a different value at each occurrence but it will not depend on various parameters w,s,k,l,t. Using this convention we may write

$$D\varphi^z_{w,s,s'}(x,y,z)=\begin{pmatrix} \frac{1-s'}{1-s}+4c\frac{s'-s}{1-s} & 2c\frac{s'-s}{1-s} & cw\frac{s'-s}{1-s}\\ 2c\frac{s'-s}{1-s} & \frac{1-s'}{1-s}+4c\frac{s'-s}{1-s} & cw\frac{s'-s}{1-s}\\ 0 & 0 & 1\end{pmatrix}. \tag{3.3}$$

Note that by choosing w sufficiently small we can make the first two terms in the last column arbitrarily small. Later we will rotate this matrix in the first two coordinates and we obtain almost upper triangular matrix.

We will need also to estimate the derivative of the inverse mapping

$$D(\varphi^z_{w,s,s'})^{-1}(\varphi^z_{w,s,s'}(x,y,z))=$$
$$\begin{pmatrix} \frac{1-s}{1-s'}+(1-|z|)w\frac{s-s'}{1-s'}(\frac{1}{|x|+|y|}-\frac{\operatorname{sgn}(x)x}{(|x|+|y|)^2}) & (1-|z|)w\frac{s-s'}{1-s'}(-\frac{\operatorname{sgn}(y)x}{(|x|+|y|)^2}) & w\frac{s-s'}{1-s'}(-\frac{\operatorname{sgn}(z)x}{(|x|+|y|)})\\ (1-|z|)w\frac{s-s'}{1-s'}(-\frac{\operatorname{sgn}(x)y}{(|x|+|y|)^2}) & \frac{1-s}{1-s'}+(1-|z|)w\frac{s-s'}{1-s'}(\frac{1}{|x|+|y|}-\frac{\operatorname{sgn}(y)y}{(|x|+|y|)^2}) & w\frac{s-s'}{1-s'}(-\frac{\operatorname{sgn}(z)y}{(|x|+|y|)})\\ 0 & 0 & 1\end{pmatrix}$$

and by $s'\geq\frac14$ we have $\frac{(1-|z|)w}{|x|+|y|}\leq 4$ and hence we may rewrite this as in (3.3)

$$D(\varphi^z_{w,s,s'})^{-1}(\varphi^z_{w,s,s'}(x,y,z))=\begin{pmatrix} \frac{1-s}{1-s'}+8c\frac{s-s'}{1-s'} & 4c\frac{s-s'}{1-s'} & cw\frac{s-s'}{1-s'}\\ 4c\frac{s-s'}{1-s'} & \frac{1-s}{1-s'}+8c\frac{s-s'}{1-s'} & cw\frac{s-s'}{1-s'}\\ 0 & 0 & 1\end{pmatrix}. \tag{3.4}$$

Suppose that Q^z is a scaled and translated version of $Q^z(w)$. We define $\varphi^{Q^z}_{w,s,s'}$ to be the corresponding scaled and translated version of $\varphi^z_{w,s,s'}$. By $I^s_{Q^z}$ and $O^s_{Q^z}$ we will denote the corresponding inner diamond and outer annular diamond.

Suppose that Q^y and Q^x are scaled and translated copy of rotated diamonds

$$Q^y(w)=\{(x,y,z)\in\mathbb{R}^3:|x|+|z|<w(1-|y|)\}\text{ and }Q^x(w)=\{(x,y,z)\in\mathbb{R}^3:|y|+|z|<w(1-|x|)\}$$

We define $\varphi^{Q^y}_{w,s,s'}$ and $\varphi^{Q^x}_{w,s,s'}$ to be the corresponding rotated, scaled and translated version of $\varphi^y_{w,s,s'}$ and $\varphi^x_{w,s,s'}$. That is $\varphi^{Q^y}_{w,s,s'}$ maps Q^y onto Q^y and it is the identity on the boundary; analogously for $\varphi^{Q^x}_{w,s,s'}$. We will also use a notation $I^s_{Q^y}$ and $O^s_{Q^y}$ for the corresponding inner diamond and outer annular diamond. It is also easy to see that each of these mappings is bi-Lipschitz. By the composition of finitely many of these mappings we always get a bi-Lipschitz mapping.

4. Choice of parameters and lemmata

Let C_1 and C_2 be absolute constants whose exact value we will specify later. We can clearly fix $t>1$ such that

$$C_1C_2\Bigl(\frac{\pi^2}{6}\Bigr)^6\frac{1}{t}<\frac{1}{2}\,. \tag{4.1}$$

For $k\in\mathbb{N}$, we set

$$w_k=\frac{k+1}{tk^2-1},\ s_k=1-\frac{1}{tk^2}\text{ and } s'_k=s_k\frac{k}{k+1}. \tag{4.2}$$

In this case,

$$\frac{1-s'_k}{1-s_k}=\frac{tk^2+k}{k+1}\text{ and }\frac{s_k-s'_k}{1-s_k}w_k=\frac{tk^2-1}{k+1}w_k=1. \tag{4.3}$$

It is also easy to check that $0<s_k<1$ and

$$\prod_{i=1}^{\infty}s_i>0.$$

We will need the following elementary consequence of area formula for mappings whose Jacobian is almost constant on some subset.

Lemma 4.1. *Let $\delta>0$, $0<s<1$ and let $A\subset P$ satisfy $|A|=s|P|$. Suppose that $F\in W^{1,1}((0,1)^N,\mathbb{R}^N)$ is a homeomorphism, $P\subset F((0,1)^N)$, F satisfies the Lusin (N) condition on $F^{-1}(P)$ and*

$$J_F(x)\le(1+\delta)J_F(y)\text{ for every } x,y\in F^{-1}(P)\ .$$

Then

$$\frac{1}{1+\delta}|F^{-1}(A)|\le|F^{-1}(P)|s\le(1+\delta)|F^{-1}(A)|\ .$$

Proof. Let us denote

$$m=\inf_{x\in F^{-1}(P)}|J_F(x)|\text{ and } M=\sup_{x\in F^{-1}(P)}|J_F(x)|\ .$$

The area formula is valid for Sobolev homeomorphisms that satisfy the Lusin (N) condition (see e.g. [5]) and hence

$$m|F^{-1}(A)|\le\int_{F^{-1}(A)}|J_F(x)|\,dx=|A|=s|P|=s\int_{F^{-1}(P)}|J_F(x)|\,dx\le sM|F^{-1}(P)|\ .$$

Since $M\le(1+\delta)m$ we obtain the first inequality and the second one can be shown analogously. □

Later we apply this lemma for parameters δ_k and we use that the product of $(1+\delta_k)$ is bounded

$$\delta_k=\frac{1}{k^2},\ \Delta_k=\Pi_{i=1}^{k}(1+\delta_i)\text{ and }\Delta:=\Pi_{i=1}^{\infty}(1+\delta_i)=\lim_{k\to\infty}\Delta_k<\infty\ . \tag{4.4}$$

Given a matrix B and a set $Q\subset R^N$ we use the notation $BQ=\{Bx:\ x\in Q\}$.

Lemma 4.2. *Let $k\in\mathbb{N}$, $L\ge1$ and $0<r_0<\frac{1}{4L}$. Suppose that for every $x\in[0,1]^3$ there is a matrix B_x with $\|B_x\|\le L$ and $\|B_x^{-1}\|\le L$. Then we can cover the whole set by scaled, translated and rotated copies of $Q^y_{w_k}$. In particular*

$$[0,1]^3=N\cup\bigcup_{j=1}^{\infty}r_jB_{x_j}(x_j+Q^y_{w_k})$$

where the sets are pairwise disjoint, $\mathcal{L}_3(N)=0$ and $r_j\le r_0$.

Proof. Set $\varepsilon_1=Lr_0$ and $r_1=\frac{\varepsilon_1}{L}=r_0$ and consider the ε_1-grid in $\mathbb{R}^3$

$$G_1:=\{x\in(\varepsilon_1\mathbb{N})^3:\ B(x,Lr_1)\subset(0,1)^3\}\ .$$

In the first step we choose diamonds

$$D_1=\bigcup_{x\in G_1}r_1B_x(x+Q^y_{w_k})$$

and we obtain finitely many diamonds that are pairwise disjoint. Since $|Q^y_{w_k}|\ge\frac{C}{k^2}$ it is not difficult to check that

$$\mathcal{L}_3(D_1)\ge\frac{C_0(L,t)}{k^2}\mathcal{L}_3((0,1)^3)\ .$$

Now we choose $\varepsilon_2 > 0$ so small that

$$\mathcal{L}_3\big(\{x \in (0,1)^3 \setminus D_1 :\ \operatorname{dist}(x, D_1) > \varepsilon_2\}\big) > \frac{1}{2}\mathcal{L}_3((0,1)^3 \setminus D_1)\ .$$

We set $r_2 = \frac{\varepsilon_2}{L}$ and we consider a grid

$$G_2 = \{x \in (\varepsilon_2\mathbb{N})^3 :\ B(x, Lr_2) \subset (0,1)^3 \setminus D_1\}\ .$$

We add disjoint diamonds

$$D_2 = \bigcup_{x\in G_2} r_2 B_x(x + Q^y_{w_k})$$

and it is easy to check that diamonds from D_1 and D_2 are pairwise disjoint. Again it is not difficult to check that

$$\mathcal{L}_3(D_1) \geq \frac{1}{2}\frac{C_0(L,t)}{k^2}\mathcal{L}_3((0,1)^3 \setminus D_1)\ .$$

We continue by induction. Now we choose $\varepsilon_j > 0$ so small that for the already covered set $D^j := \bigcup_{i=1}^{j-1} D_i$ we have

$$\mathcal{L}_3\big(\{x \in (0,1)^3 \setminus D^j :\ \operatorname{dist}(x, D_j) > \varepsilon_j\}\big) > \frac{1}{2}\mathcal{L}_3((0,1)^3 \setminus D^j)\ ..$$

We set $r_j = \frac{\varepsilon_j}{L}$ and we consider a grid

$$G_j = \{x \in (\varepsilon_j\mathbb{N})^3 :\ B(x, Lr_j) \subset (0,1)^3 \setminus D^j\}\ .$$

We add pairwise disjoint diamonds

$$D_j = \bigcup_{x\in G_j} r_j B_x(x + Q^y_{w_k})$$

and again it is not difficult to check that

$$\mathcal{L}_3(D_j) \geq \frac{1}{2}\frac{C_0(L,t)}{k^2}\mathcal{L}_3((0,1)^3 \setminus D^j)\ .$$

Since this inequality holds for all j it is easy to see that the measure of the set

$$N := [0,1]^3 \setminus \bigcup_{j=1}^{\infty} D_j \text{ satisfies } \mathcal{L}_3(N) = 0.$$

□

We will need to decompose some matrices to the product of rotation and upper triangular matrix (i.e. all terms below the diagonal are zero) with the help of the well-known QR decomposition theorem. Recall that the matrix Q is orthogonal if the columns are unitary vectors, $Q^{-1} = Q^T$ and $\|Q\| \leq 1$.

Theorem 4.3. *For every $N \times N$ matrix A we can find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.*

5. Construction and differentiability of F_1

5.1. **Construction of F_1.** Let us denote $Q_0 := Q^z(w_1)$. We will construct a sequence of bi-Lipschitz mappings $f_{k,1} : Q_0 \to Q_0$ and our mapping $F_1 \in W^{1,1}(Q_0, \mathbb{R}^3)$ will be later defined as $F_1(x) = \lim_{k\to\infty} f_{k,1}(x)$. We will also construct a Cantor-type set $\mathcal{C}_1$ of positive measure such that $J_{F_1} = 0$ almost everywhere on $\mathcal{C}_1$.

We define a sequence of families $\{\mathcal{Q}_{k,1}\}$ of building blocks, and a sequence of homeomorphisms $f_{k,1}\colon Q_0 \to Q_0$. Let $\mathcal{Q}_{1,1} = Q^z(w_1) = Q_0$, and define $f_{1,1}\colon Q_0 \to Q_0$ by

$$f_{1,1}(x, y, z) = \varphi^z_{w_1,s_1,s_1'}(x, y, z).$$

Clearly $f_{1,1}$ is a bi-Lipschitz homeomorphism. Now each $f_{k,1}$ will equal to $f_{1,1}$ on the set $G_{1,1} := O^z(w_1, s_1)$ and it remains to define it on $R_{1,1} := I^z(w_1, s_1)$. Clearly

$$\mathcal{L}_3(G_{1,1}) = (1 - s_1^2)\mathcal{L}_3(Q_0) \text{ and } \mathcal{L}_3(R_{1,1}) = s_1^2\mathcal{L}_3(Q_0).$$

Let $\mathcal{Q}_{2,1}$ be any collection of disjoint, scaled and translated copies of $Q^z(w_2)$ which covers $f_{1,1}(R_{1,1}) = I^z(w_1, s_1')$ up to a set of measure zero. That is any two elements of $\mathcal{Q}_{2,1}$ have disjoint interiors, and there is a set $E_{2,1} \subseteq I^z(w_1, s_1')$ of measure 0 such that

$$I^z(w_1, s_1') \setminus E_{2,1} \subseteq \bigcup_{Q^z \in \mathcal{Q}_{2,1}} Q^z \subseteq I^z(w_1, s_1').$$

Clearly such a collection exists. Note that if $Q^z \in \mathcal{Q}_{2,1}$, then the inverse image of Q^z under $f_{1,1}$ is a scaled and translated copy of $Q^z(\frac{s_1}{s_1'} w_2) = Q^z(2w_2)$ and

$$I^z(w_1, s_1) \setminus (f_{1,1})^{-1}(E_{2,1}) \subseteq \bigcup_{Q^z \in \mathcal{Q}_{2,1}} (f_{1,1})^{-1}(Q^z) \subseteq I^z(w_1, s_1).$$

Note that $J_{f_{1,1}} \neq 0$ a.e. and hence the inverse image of a null set $E_{2,1}$ has measure zero.

We define $f_{2,1} \colon Q_0 \to Q_0$ by

$$f_{2,1}(x,y,z) = \begin{cases} \varphi^{Q^z}_{w_2, s_2, s_2'} \circ f_{1,1}(x,y,z) & f_{1,1}(x,y,z) \in Q^z \in \mathcal{Q}_{2,1}, \\ f_{1,1}(x,y,z) & \text{otherwise.} \end{cases}$$

It is not difficult to check that $f_{2,1}$ is a bi-Lipschitz homeomorphism. From now on each $f_{k,1}$ will equal to $f_{2,1}$ on

$$G_{1,1} \cup G_{2,1} \cup (f_{1,1})^{-1}(E_{2,1}), \text{ where } G_{2,1} := f_{1,1}^{-1}\Big(\bigcup_{Q^z \in \mathcal{Q}_{2,1}} O^{s_2}_{Q^z} \Big)$$

and it remains to define it on

$$R_{2,1} := f_{1,1}^{-1}\Big(\bigcup_{Q \in \mathcal{Q}_{2,1}} I^{s_2}_{Q^z} \Big).$$

Since each $f_{1,1}^{-1}(Q^z)$ is a scaled and translated copy of our basic building block and the ratio s_2 is fixed, we obtain

$$\begin{aligned} \mathcal{L}_3(G_{2,1}) &= \sum_{Q^z \in \mathcal{Q}_{2,1}} \mathcal{L}_3\big(f_{1,1}^{-1}(O^{s_2}_{Q^z})\big) = \sum_{Q^z \in \mathcal{Q}_{2,1}} \mathcal{L}_3\big(O^{s_2}_{f_{1,1}^{-1}(Q^z)}\big) \\ &= \sum_{Q^z \in \mathcal{Q}_{2,1}} (1 - s_2^2) \mathcal{L}_3\big(f_{1,1}^{-1}(Q^z)\big) = (1 - s_2^2)\mathcal{L}_3(R_{1,1}). \end{aligned}$$

It is also easy to see that

$$\mathcal{L}_3(R_{2,1}) = s_2^2 \mathcal{L}_3(R_{1,1}).$$

We continue inductively. Assume that $\mathcal{Q}_{k,1}$, $f_{k,1}$, $G_{k,1}$ and $R_{k,1}$ have already been defined. We find a family of disjoint scaled and translated copies of $Q^z(w_{k+1})$ that cover $f_{k,1}(R_{k,1})$ up to a set of measure zero $E_{k+1,1}$. Define $\varphi_{k+1,1} \colon Q_0 \to Q_0$ by

$$\varphi_{k+1,1}(x,y,z) = \begin{cases} \varphi^{Q^z}_{w_{k+1}, s_{k+1}, s_{k+1}'}(x,y,z) & (x,y,z) \in Q^z \in \mathcal{Q}_{k+1,1}, \\ (x,y,z) & \text{otherwise.} \end{cases}$$

The mapping $f_{k+1,1} \colon Q_0 \to Q_0$ is now defined by $\varphi_{k+1,1} \circ f_{k,1}$. Clearly each mapping $f_{k+1,1}$ is a bi-Lipschitz homeomorphism. We further define the sets

$$G_{k+1,1} := f_{k,1}^{-1}\Big(\bigcup_{Q^z \in \mathcal{Q}_{k+1,1}} O^{s_{k+1}}_{Q^z} \Big) \text{ and } R_{k+1,1} := f_{k,1}^{-1}\Big(\bigcup_{Q^z \in \mathcal{Q}_{k+1,1}} I^{s_{k+1}}_{Q^z} \Big).$$

Again it is not difficult to check that

$$\mathcal{L}_3(G_{k+1,1}) = (1 - s_{k+1}^2)\mathcal{L}_3(R_{k,1}) \text{ and } \mathcal{L}_3(R_{k+1,1}) = s_{k+1}^2 \mathcal{L}_3(R_{k,1}).$$

Using $\mathcal{L}_3(G_{1,1}) = (1 - s_1^2)\mathcal{L}_3(Q_0)$ and $\mathcal{L}_3(R_{1,1}) = s_1^2 \mathcal{L}_3(Q_0)$ we easily obtain

$$\mathcal{L}_3(R_{k,1}) = s_1^2 s_2^2 \cdots s_k^2 \mathcal{L}_3(Q_0) \text{ and } \mathcal{L}_3(G_{k,1}) = s_1^2 s_2^2 \cdots s_{k-1}^2 (1 - s_k^2)\mathcal{L}_3(Q_0). \tag{5.1}$$

It follows that the resulting Cantor type set

$$\mathcal{C}_1 := \bigcap_{k=1}^{\infty} R_{k,1}$$

satisfies

$$\mathcal{L}_3(\mathcal{C}_1)=\mathcal{L}_3(Q_0)\prod_{i=1}^{\infty}s_i^2>0.$$

It is clear from the construction that $f_{k,1}$ converge uniformly and hence the limiting map $F_1(x):=\lim_{k\to\infty}f_{k,1}(x)$ exists and is continuous. It is not difficult to check that F_1 is a one-to-one mapping of Q_0 onto Q_0. Since Q_0 is compact and F_1 is continuous we obtain that F_1 is a homeomorphism. It remains to verify that $f_{k,1}$ and $f_{k,1}^{-1}$ form a Cauchy sequence in $W^{1,1}$ and thus F_1 is a bi-Sobolev mapping.

5.2. **Weak differentiability of F_1.** Let us estimate the derivative of our functions $f_{m,1}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^z\in\mathcal{Q}_{k,1}$ and $(x,y,z)\in\operatorname{int}(f_{k,1})^{-1}(I_{Q^z}^{s'_k})$, then we have squeezed our diamond k-times. Using (3.1), (4.2) and the chain rule we obtain

$$Df_{k,1}(x,y,z)=\prod_{i=1}^{k}\begin{pmatrix}\frac{i}{i+1}&0&0\\0&\frac{i}{i+1}&0\\0&0&1\end{pmatrix}=\begin{pmatrix}\frac{1}{k+1}&0&0\\0&\frac{1}{k+1}&0\\0&0&1\end{pmatrix}. \tag{5.2}$$

Moreover, if $(x,y,z)\in\operatorname{int}(f_{m,1})^{-1}(O_{Q^z}^{s'_k})$, then we have squeezed our diamond $k-1$ times and then we have stretched it once. It follows from (3.1), (4.2), (3.3), (4.3) and the chain rule that

$$\begin{aligned}Df_{m,1}(x,y,z)&=\begin{pmatrix}\frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1}&2c\frac{tk^2-1}{k+1}&c\\2c\frac{tk^2-1}{k+1}&\frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1}&c\\0&0&1\end{pmatrix}\left(\prod_{i=1}^{k-1}\begin{pmatrix}\frac{i}{i+1}&0&0\\0&\frac{i}{i+1}&0\\0&0&1\end{pmatrix}\right)\\&=\begin{pmatrix}\frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1}&\frac{2c}{k}\frac{tk^2-1}{k+1}&c\\\frac{2c}{k}\frac{tk^2-1}{k+1}&\frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1}&c\\0&0&1\end{pmatrix}=:A_k.\end{aligned}\tag{5.3}$$

It is easy to see that the norm of this matrix can be estimated by Ct.

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,1}=f_{m,1}$ outside of $R_{n,1}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f_{m,1}-f_{n,1})|&=\int_{R_{n,1}}|D(f_{m,1}-f_{n,1})|\\&\leq C\int_{R_{n,1}\setminus R_{m,1}}|Df_{n,1}|+C\int_{R_{m,1}}|Df_{m,1}-Df_{n,1}|+C\sum_{k=n+1}^{m}\int_{G_{k,1}}|Df_{m,1}|.\end{aligned}$$

From (5.2) and (5.1) we obtain

$$\int_{R_{n,1}\setminus R_{m,1}}|Df_{n,1}|\leq C\mathcal{L}_3(R_{n,1}\setminus R_{m,1})\overset{n\to\infty}{\to}0$$

and

$$\int_{R_{m,1}}|Df_{m,1}-Df_{n,1}|\leq C\Big(\frac{1}{n+1}-\frac{1}{m+1}\Big)\leq\frac{C}{n+1}\overset{n\to\infty}{\to}0.$$

From (5.3) and (5.1) we obtain

$$\begin{aligned}\sum_{k=n+1}^{m}\int_{G_{k,1}}|Df_{m,1}|&\leq C\sum_{k=n+1}^{m}\mathcal{L}_3(G_{k,1})t\\&\leq C\sum_{k=n+1}^{m}(1-s_k^2)t\\&\leq C\sum_{k=n+1}^{m}\frac{1}{tk^2}t\overset{n\to\infty}{\to}0.\end{aligned}$$

It follows that the sequence $Df_{k,1}$ is Cauchy in L^1 and thus we can easily obtain that $f_{k,1}$ is Cauchy in $W^{1,1}$. Since $f_{k,1}$ converge to F_1 uniformly we obtain that $F_1\in W^{1,1}$. Moreover, using (5.2) and (5.3) it is not difficult to see that F_1 is in fact Lipschitz mapping with Lipschitz constant Ct.

From (5.2) we obtain that the derivative of $f_{k,1}$ on $R_{k,1}$ and especially on $\mathcal{C}_1$ equals to

$$Df_{k,1}(x,y) = \begin{pmatrix} \frac{1}{k+1} & 0 & 0 \\ 0 & \frac{1}{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $Df_{k,1}$ converge to DF_1 in L^1 we obtain that for almost every $(x,y,z)\in\mathcal{C}_1$ we have

$$DF_1(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore $J_{F_1}(x,y,z)=0$. From now on each F_k will equal to F_1 on $\mathcal{C}_1$ and we need to define it only on $Q_0\setminus\mathcal{C}_1$. Moreover it is easy to see from the construction that $J_{F_1}\neq 0$ a.e. on $Q_0\setminus\mathcal{C}_1$. It follows that the preimage of each null set in $F_1(Q_0\setminus\mathcal{C}_1)$ has zero measure.

In the rest of the paper we will not explicitly mention all the exceptional null sets but we will keep in mind that they are not important for our considerations and estimates.

5.3. **Weak differentiability of F_1^{-1}.** Let us estimate the derivative of our functions $f_{m,1}^{-1}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^z\in\mathcal{Q}_{k,1}$ and $(x,y,z)\in\operatorname{int}(I_{Q^z}^{s'_k})$, then we have squeezed our diamond k-times by $f_{k,1}$ and the derivative of $f_{k,1}^{-1}$ can be computed as an inverse matrix to (5.2) and we get

$$Df_{k,1}^{-1}(x,y,z) = \begin{pmatrix} k+1 & 0 & 0 \\ 0 & k+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

Moreover, if $(x,y,z)\in\operatorname{int}(O_{Q^z}^{s'_k})$, then we have squeezed our diamond by $f_{m,1}$ $k-1$ times and then we have stretched it once. Hence we can compute its derivative as an inverse matrix to (5.3) and with the help of (3.4) we get

$$\begin{aligned} Df_{m,1}^{-1}(x,y,z) &= \begin{pmatrix} \frac{k+1}{tk^2+k}+8c\frac{tk^2-1}{k(tk+1)} & 4c\frac{tk^2-1}{k(tk+1)} & c\frac{k+1}{tk^2+k} \\ 4c\frac{tk^2-1}{k(tk+1)} & \frac{k+1}{tk^2+k}+8c\frac{tk^2-1}{k(tk+1)} & c\frac{k+1}{tk^2+k} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{k+1}{tk+1}+8c\frac{tk^2-1}{tk+1} & 4c\frac{tk^2-1}{tk+1} & c\frac{k+1}{tk^2+k} \\ 4c\frac{tk^2-1}{tk+1} & \frac{k+1}{tk+1}+8c\frac{tk^2-1}{tk+1} & c\frac{k+1}{tk^2+k} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{5.5}$$

and hence $\|Df_{m,1}^{-1}(x,y,z)\|\leq Ck$.

Analogously to the proof of (5.1) we may deduce from the construction that for every k we have

$$\mathcal{L}_3(f_{k+1,1}(R_{k+1,1})) = (s'_{k+1})^2\mathcal{L}_3(f_{k,1}(R_{k,1})) \text{ and hence } \mathcal{L}_3(f_{k,1}(R_{k,1}))\leq\frac{1}{k^2}\ . \tag{5.6}$$

Moreover, by (5.6) and (4.2) we can deduce that for every $m\geq k$

$$\mathcal{L}_3(f_{m,1}(G_{k,1})) = (1-(s'_k)^2)\mathcal{L}_3(f_{k-1,1}(R_{k-1,1})) = (s'_1)^2(s'_2)^2\cdots(s'_{k-1})^2(1-(s'_k)^2)\mathcal{L}_3(Q_0)\leq\frac{C}{k^3}\ . \tag{5.7}$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,1}=f_{m,1}$ outside of $R_{n,1}$ we obtain

$$\begin{aligned} \int_{Q_0}|D(f_{m,1}^{-1}-f_{n,1}^{-1})| &= \int_{f_{m,1}(R_{n,1})}|D(f_{m,1}^{-1}-f_{n,1}^{-1})| \\ &\leq \int_{f_{m,1}(R_{n,1})}|Df_{n,1}^{-1}| + \int_{f_{m,1}(R_{m,1})}|Df_{m,1}^{-1}| + \int_{f_{m,1}(\bigcup_{k=n+1}^m G_{k,1})}|Df_{m,1}^{-1}|. \end{aligned}$$

From (5.4) and (5.6) we obtain

$$\int_{f_{m,1}(R_{n,1})}|Df_{n,1}^{-1}| \leq \mathcal{L}_3(f_{m,1}(R_{n,1}))n = \mathcal{L}_3(f_{n,1}(R_{n,1}))n \leq \frac{1}{n^2}n \overset{n\to\infty}{\to} 0$$

and

$$\int_{f_{m,1}(R_{m,1})}|Df^{-1}_{m,1}|\leq \mathcal{L}_3(f_{m,1}(R_{m,1}))m\leq \frac{1}{m^2}m\overset{n\to\infty}{\to}0.$$

From (5.5) and (5.6) we obtain

$$\int_{f_{m,1}(\cup_{k=n+1}^m G_{k,1})}|Df^{-1}_{m,1}|\leq \sum_{k=n+1}^m \mathcal{L}_3(f_{m,1}(G_{k,1}))Ck\leq C\sum_{k=n+1}^m\frac{1}{k^3}Ck\overset{n\to\infty}{\to}0$$

It follows that the sequence $Df^{-1}_{k,1}$ is Cauchy in L^1 and thus we can easily obtain that $f^{-1}_{k,1}$ is Cauchy in $W^{1,1}$. Since $f^{-1}_{k,1}$ converge to F_1^{-1} uniformly we obtain that $F_1^{-1}\in W^{1,1}$ (see Lemma 3.1 [4]).

6. Construction and differentiability of F_2

6.1. **Construction of F_2.** We will construct a sequence of homeomorphisms $f_{k,2}:Q_0\to Q_0$ and our mapping $F_2\in W^{1,1}(Q_0,\mathbb{R}^3)$ will be later defined as $F_2(x)=\lim_{k\to\infty}f_{k,2}(x)$. We will also construct a Cantor-type set $\mathcal{C}_2\subset Q_0\setminus\mathcal{C}_1$ of positive measure such that $J_{F_2}=0$ almost everywhere on $\mathcal{C}_2$.

The set $\mathcal{C}_1$ is closed and thus we can find $\mathcal{Q}_{1,2}$, a collection of disjoint, scaled, translated and 'rotated'copies of $Q^y(w_1)$ which cover $F_1(Q_0\setminus\mathcal{C}_1)$ up to a set of measure zero $E_{1,2}$. In the later computations it will be essential for us to compute with almost upper diagonal but the matrix from (5.3) is not like that. Therefore we use the QR decomposition Theorem 4.3 and we cover the set $F_1(Q_0\setminus\mathcal{C}_1)$ using Lemma 4.2 by 'rotated' diamonds and then we apply similar procedure as in the construction of F_1. That is instead of a mapping $\varphi^{Q^y}_{w,s,s'}$ we work with the map $B^{-1}\circ\varphi^{Q^y}_{w,s,s'}\circ B$ for some properly chosen linear map B. We will use the symbol B to denote both the linear mapping and the corresponding matrix. By the chain we obtain that the derivative of this map is $B^{-1}D\varphi^{Q^y}_{w,s,s'}B$.

Recall that the constants c in (5.3) depend on x,y,z but in a locally continuous way. For each $(x,y,z)\in F_1(G_{k,1})$ we have a matrix

$$A_k(x,y,z)=\begin{pmatrix}\frac{tk+1}{k+1}+\frac{4c_1}{k}\frac{tk^2-1}{k+1} & \frac{2c_2}{k}\frac{tk^2-1}{k+1} & c\\ \frac{2c_3}{k}\frac{tk^2-1}{k+1} & \frac{tk+1}{k+1}+\frac{4c_4}{k}\frac{tk^2-1}{k+1} & c\\ 0 & 0 & 1\end{pmatrix}$$

where the constants c_1,c_2,c_3,c_4 are the evaluations of the constants from (5.3) for this particular point (x,y,z). By QR-decomposition Theorem 4.3 there exists a orthogonal matrix Q_k and an upper triangular matrix R_k such that $A_k(x,y,z)=Q_kR_k$. Hence, taking $B_1=Q_k^{-1}$ we know that $B_1A_k(x,y,z)$ is upper triangular matrix and $\|B_1A_k\|\leq\|A_k\|\leq Ct$. By the continuous dependence of constants c_1,c_2,c_3 and c_4 it is easy to see that there is $r(x,y,z)>0$ such that for every (x',y',z'), $|(x',y',z')-(x,y,z)|<r(x,y,z)$ we know that the matrix $B_1A_k(x',y',z')$ is almost upper triangular, i.e. the numbers below the diagonal are between -1 and 1. As t is chosen large enough these terms will not be important in the estimates of the norm of product of matrices in Section 7.

From the construction of F_1 we know that for every cube compactly inside each int $F_1(O_{Q^z})$ we may choose $r(x,y,z)\geq r_0>0$ by local continuity of constants c. Hence we can use Lemma 4.2 to cover this cube by scaled, translated and rotated copies of $Q^y(w_1)$. In this way we cover $F_1(Q_0\setminus\mathcal{C}_1)$ up to a set of measure zero. For simplicity of notation we denote the 'rotation' matrix by B_1 but we keep in mind that its entries are different for each rotated diamond from $\mathcal{Q}_{1,2}$.

We will moreover require two additional properties. We know that $Q_0\setminus\mathcal{C}_1$ is equal up to a set of measure zero to $\bigcup_{l=1}^\infty G_{l,1}$. Hence we will also require that

$$\text{for each } Q^y\in\mathcal{Q}_{1,2} \text{ there is } l\in\mathbb{N} \text{ such that } F_1^{-1}(B_1Q^y)\subset G_{l,1}. \tag{6.1}$$

Secondly, we know that J_{F_1} is continuous in each diamond from $G_{l,1}$ (see (3.2)) and thus we may assume that $F_1^{-1}(B_1Q^y)$ is a subset of one diamond and it is so small that

$$J_{F_1}(x_1,y_1,z_1)\leq(1+\delta_2)J_{F_1}(x_2,y_2,z_2) \text{ for every } (x_1,y_1,z_1),\ (x_2,y_2,z_2)\in F_1^{-1}(B_1Q^y). \tag{6.2}$$

This fact, $\mathcal{L}_3(B_1 I^{s_1}_{Q^y}) = s_1^2\mathcal{L}_3(B_1 Q^y)$, $\mathcal{L}_3(B_1 O^{s_1}_{Q^y}) = (1-s_1^2)\mathcal{L}_3(B_1 Q^y)$ and Lemma 4.1 imply that

$$\begin{aligned}&\frac{1}{1+\delta_2}\mathcal{L}_3\big(F_1^{-1}(B_1 I^{s_1}_{Q^y})\big) \le s_1^2\mathcal{L}_3\big(F_1^{-1}(B_1 Q^y)\big) \le (1+\delta_2)\mathcal{L}_3\big(F_1^{-1}(B_1 I^{s_1}_{Q^y})\big) \text{ and}\\ &\frac{1}{1+\delta_2}\mathcal{L}_3\big(F_1^{-1}(B_1 O^{s_1}_{Q^y})\big) \le (1-s_1^2)\mathcal{L}_3\big(F_1^{-1}(B_1 Q^y)\big) \le (1+\delta_2)\mathcal{L}_3\big(F_1^{-1}(B_1 O^{s_1}_{Q_y})\big).\end{aligned} \tag{6.3}$$

We define $f_{1,2}\colon Q_0 \to Q_0$ by

$$f_{1,2}(x,y,z) = \begin{cases} B_1^{-1}\circ \varphi^{Q^y}_{w_1,s_1,s_1'}\circ B_1 \circ F_1(x,y,z) & F_1(x,y,z) \in Q^y \in \mathcal{Q}_{1,2},\\ F_1(x,y,z) & \text{otherwise.}\end{cases}$$

It is not difficult to check that $f_{1,2}$ is a homeomorphism. Moreover it is a bi-Sobolev mapping since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. From now on each $f_{k,2}$ will equal to $f_{1,2}$ on

$$\mathcal{C}_1 \cup G_{1,2}, \text{ where } G_{1,2} := F_1^{-1}\Big(\bigcup_{Q^y\in\mathcal{Q}_{1,2}} B_1 O^{s_1}_{Q^y}\Big)$$

and it remains to define it on

$$R_{1,2} := F_1^{-1}\Big(\bigcup_{Q^y\in\mathcal{Q}_{1,2}} B_1 I^{s_1}_{Q^y}\Big).$$

Let us note that $J_{F_1}\neq 0$ on $Q_0\setminus\mathcal{C}_1$ and thus the preimage of the null set $E_{1,2}$ under F_1 is a null set. By summing up (6.3) we obtain

$$\begin{aligned}&\frac{1}{1+\delta_2}\mathcal{L}_3(R_{1,2}) \le s_1^2\mathcal{L}_3\big(Q_0\setminus\mathcal{C}_1\big) \le (1+\delta_2)\mathcal{L}_3(R_{1,2}) \text{ and}\\ &\frac{1}{1+\delta_2}\mathcal{L}_3(G_{1,2}) \le (1-s_1^2)\mathcal{L}_3\big(Q_0\setminus\mathcal{C}_1\big) \le (1+\delta_2)\mathcal{L}_3(G_{1,2}).\end{aligned}$$

We continue inductively. Assume that $\mathcal{Q}_{k,2}$, $f_{k,2}$, $G_{k,2}$ and $R_{k,2}$ have already been defined. We find a family of disjoint scaled, translated and rotated copies of $Q^y(w_{k+1})$ that cover $f_{k,2}(R_{k,2})$ up to a set of measure zero $E_{k+1,2}$. The rotation here is given by the same matrix B_1 as in the previous steps, i.e. in the rotated diamond we have smaller diamonds that are rotated in the same direction (but for each diamond in $\mathcal{Q}_{1,2}$ we have possibly different rotation B_1). Define $\varphi_{k+1,2}\colon Q_0\to Q_0$ by

$$\varphi_{k+1,2}(x,y,z) = \begin{cases} B_1^{-1}\circ\varphi^{Q^y}_{w_{k+1},s_{k+1},s_{k+1}'}\circ B_1(x,y,z) & (x,y,z)\in Q^y\in\mathcal{Q}_{k+1,2},\\ (x,y,z) & \text{otherwise.}\end{cases}$$

The mapping $f_{k+1,2}\colon Q_0\to Q_0$ is now defined by $\varphi_{k+1,2}\circ f_{k,2}$. Clearly each mapping $f_{k+1,2}$ is a homeomorphism. Moreover it is a bi-Sobolev mapping since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. We further define the sets

$$G_{k+1,2} := f_{k,2}^{-1}\Big(\bigcup_{Q^y\in\mathcal{Q}_{k+1,2}} O^{s_{k+1}}_{Q^y}\Big) \text{ and } R_{k+1,2} := f_{k,2}^{-1}\Big(\bigcup_{Q^y\in\mathcal{Q}_{k+1,2}} I^{s_{k+1}}_{Q^y}\Big).$$

The linear maps $\varphi_{j,2}$, $1\le j\le k$, on inner diamonds do not change the ratio of volumes of Q^y and $O^{s_{k+1}}_{Q^y}$, $Q^y\in\mathcal{Q}_{k+1,2}$. Therefore we obtain that

$$\mathcal{L}_3(F_1(G_{k+1,2})) = (1-s_{k+1}^2)\mathcal{L}_3(F_1(R_{k,2})) \text{ and } \mathcal{L}_3(F_1(R_{k+1,2})) = s_{k+1}^2\mathcal{L}_3(F_1(R_{k,2})).$$

Analogously as before we obtain

$$\mathcal{L}_3(F_1(R_{k,2})) = s_1^2 s_2^2\cdots s_k^2\mathcal{L}_3\big(F_1(Q_0\setminus\mathcal{C}_1)\big)$$

and

$$\mathcal{L}_3(F_1(G_{k,2})) = s_1^2 s_2^2\cdots s_{k-1}^2(1-s_k^2)\mathcal{L}_3\big(F_1(Q_0\setminus\mathcal{C}_1)\big).$$

Therefore using (6.2) and Lemma 4.1 we obtain that

$$\frac{1}{1+\delta_2}\mathcal{L}_3(R_{k,2}) \le s_1^2 s_2^2\cdots s_k^2\mathcal{L}_3\big(Q_0\setminus\mathcal{C}_1\big) \le (1+\delta_2)\mathcal{L}_3(R_{k,2}) \tag{6.4}$$

and

$$\frac{1}{1+\delta_2}\mathcal{L}_3(G_{k,2}) \le s_1^2 s_2^2\cdots s_{k-1}^2(1-s_k^2)\mathcal{L}_3\big(Q_0\setminus\mathcal{C}_1\big) \le (1+\delta_2)\mathcal{L}_3(G_{k,2}). \tag{6.5}$$

Since the sets Q^y, $Q^y \in \mathcal{Q}_{k,2}$, cover $F_1(G_{l,1})$ up to a null set (see (6.1)) we can moreover obtain the similar estimate on each $G_{l,1}$, $l \in \mathbb{N}$. Therefore

$$\frac{1}{1+\delta_2}\mathcal{L}_3(G_{k,2}\cap G_{l,1}) \leq s_1^2 s_2^2\cdots s_{k-1}^2(1-s_k^2)\mathcal{L}_3(G_{l,1}) \leq (1+\delta_2)\mathcal{L}_3(G_{k,2}\cap G_{l,1}). \tag{6.6}$$

It follows from (6.4) that the resulting Cantor type set

$$\mathcal{C}_2 := \bigcap_{k=1}^{\infty} R_{k,2}$$

satisfies

$$\mathcal{L}_3(\mathcal{C}_2) \geq \frac{1}{1+\delta_2}\mathcal{L}_3(Q_0\setminus\mathcal{C}_1)\prod_{i=1}^{\infty} s_i^2 > 0.$$

It is clear from the construction that $f_{k,2}$ converge uniformly and hence it is not difficult to check that the limiting map $F_2(x) := \lim_{k\to\infty} f_{k,2}(x)$ exists and is a homeomorphism. It remains to verify that $f_{k,2}$ and $f_{k,2}^{-1}$ form a Cauchy sequence in $W^{1,1}$ and thus F_2 is bi-Sobolev.

6.2. **Weak differentiability of F_2.** Let us estimate the derivative of our functions $f_{m,2}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^y\in\mathcal{Q}_{k,2}$ and $(x,y,z)\in \operatorname{int}(f_{k,2})^{-1}(I_{Q^y}^{s_k'})$, then after applying F_1 and B_1 we have squeezed our diamond k-times. Analogously to (5.2) we can use (3.1), (4.2), the chain rule and $B_1^{-1}B_1 = I$ to obtain

$$\begin{aligned} Df_{k,2}(x,y,z) &= \left[\prod_{i=1}^{k} B_1^{-1}\begin{pmatrix} \frac{i}{i+1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{i}{i+1}\end{pmatrix} B_1\right] DF_1(x,y,z)\\ &= B_1^{-1}\begin{pmatrix} \frac{1}{k+1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{k+1}\end{pmatrix} B_1 DF_1(x,y,z). \end{aligned} \tag{6.7}$$

Moreover, if $(x,y,z)\in\operatorname{int}(f_{m,2})^{-1}(O_{Q^y}^{s_k'})$, then after applying F_1 and B_1 we have squeezed our diamond $k-1$ times and then we have stretched it once. Analogously to (5.3) we can use (3.1), (4.2), (3.3), (4.3) and the chain rule to obtain that

$$\begin{aligned} Df_{m,2}(x,y,z) &= B_1^{-1}\begin{pmatrix} \frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1} & c & 2c\frac{tk^2-1}{k+1}\\ 0 & 1 & 0\\ 2c\frac{tk^2-1}{k+1} & c & \frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1}\end{pmatrix}\begin{pmatrix} \frac{1}{k} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{k}\end{pmatrix} B_1 DF_1(x,y,z)\\ &= B_1^{-1}\begin{pmatrix} \frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1} & c & \frac{2c}{k}\frac{tk^2-1}{k+1}\\ 0 & 1 & 0\\ \frac{2c}{k}\frac{tk^2-1}{k+1} & c & \frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1}\end{pmatrix} B_1 DF_1(x,y,z). \end{aligned} \tag{6.8}$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,2}=f_{m,2}$ outside of $R_{n,2}$ we obtain

$$\begin{aligned} \int_{Q_0} |D(f_{m,2}-f_{n,2})| &= \int_{R_{n,2}} |D(f_{m,2}-f_{n,2})|\\ &\leq C\int_{R_{n,2}\setminus R_{m,2}} |Df_{n,2}| + C\int_{R_{m,2}} |Df_{m,2}-Df_{n,2}| + C\sum_{k=n+1}^{m}\int_{G_{k,2}} |Df_{m,2}|. \end{aligned}$$

By (6.7), $\|B_1^{-1}\|\leq 1$ and $\|B_1\|\leq 1$ we get

$$\int_{R_{n,2}\setminus R_{m,2}} |Df_{n,2}| \leq C\int_{R_{n,2}\setminus R_{m,2}} |DF_1| \overset{n\to\infty}{\to} 0$$

since $|DF_1|\in L^1$ and $\mathcal{L}_3(R_{n,2}\setminus R_{m,2})\to 0$. Analogously we may use (6.7) to obtain

$$\int_{R_{m,2}} |Df_{m,2}-Df_{n,2}| \leq \Big(\frac{1}{n+1}-\frac{1}{m+1}\Big)\int_{R_{m,2}} |DF_1| \overset{n\to\infty}{\to} 0.$$

We need to estimate the norm of the matrix

(6.9)

$$A_{k_1,k_2} = B_1^{-1}A_{k_2}B_1A_{k_1} :=$$

$$B_1^{-1}\begin{pmatrix} \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1} \\ 0 & 1 & 0 \\ \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1} \end{pmatrix} B_1 \begin{pmatrix} \frac{tk_1+1}{k_1+1}+\frac{4c}{k_1}\frac{tk_1^2-1}{k_1+1} & \frac{2c}{k_1}\frac{tk_1^2-1}{k_1+1} & c \\ \frac{2c}{k_1}\frac{tk_1^2-1}{k_1+1} & \frac{tk_1+1}{k_1+1}+\frac{4c}{k_1}\frac{tk_1^2-1}{k_1+1} & c \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the norm of the second matrix can be estimated by Ct and the norm of the last also by Ct and hence $\|B_1^{-1}A_{k_2}B_1A_{k_1}\| \le Ct^2$ where C does not depend on k_1, k_2, x, y, z. Thus we may use (5.3), (6.8), (5.1) and (6.6) to obtain

$$\begin{aligned} \sum_{k_2=n+1}^{m}\int_{G_{k_2,2}}|Df_{m,2}| &\le \sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\int_{G_{k_2,2}\cap G_{k_1,1}}|Df_{m,2}| \\ &\le \sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\mathcal{L}_3(G_{k_2,2}\cap G_{k_1,1})\|A_{k_1,k_2}\| \\ &\le \sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}(1-s_{k_2}^2)(1-s_{k_1}^2)Ct^2 \\ &\le C\sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\frac{1}{tk_1^2tk_2^2}t^2 \le C\sum_{k_2=n+1}^{m}\frac{1}{k_2^2}\overset{n\to\infty}{\to} 0. \end{aligned} \tag{6.10}$$

It follows that the sequence $Df_{k,2}$ is Cauchy in L^1 and thus we can easily obtain that $f_{k,2}$ is Cauchy in $W^{1,1}$. Since $f_{k,2}$ converge to F_2 uniformly we obtain that $F_2 \in W^{1,1}$. Moreover, using (5.2), (6.7), (5.3) and (6.8) it is not difficult to see that F_2 is Lipschitz mapping with Lipschitz constant Ct^2.

From (6.7) we obtain that the derivative of $f_{k,2}$ on $R_{k,2}$ and especially on $\mathcal{C}_2$ equals to

$$Df_{k,2}(x,y,z) = B_1^{-1}\begin{pmatrix} \frac{1}{k+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k+1} \end{pmatrix} B_1 DF_1(x,y,z).$$

Since $Df_{k,2}$ converge to DF_2 in L^1 we obtain that for almost every $(x,y,z) \in \mathcal{C}_2$ we have

$$J_{F_2}(x,y,z) = \det\left(\lim_{k\to\infty} B_1^{-1}\begin{pmatrix} \frac{1}{k+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k+1} \end{pmatrix} B_1 DF_1(x,y,z)\right) = 0.$$

From now on each F_k will equal to F_2 on $\mathcal{C}_1 \cup \mathcal{C}_2$ and we need to define it only on $Q_0 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$. Analogously as before $J_{F_2} \neq 0$ a.e. on $Q_0 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$ and thus the preimages of the exceptional null sets will be null sets.

6.3. **Weak differentiability of F_2^{-1}.** Let us estimate the derivative of our functions $f_{m,2}^{-1}$. Let us fix $m, k \in \mathbb{N}$ such that $m \ge k$. If $Q^y \in \mathcal{Q}_{k,2}$ and $(x,y,z) \in \operatorname{int}(I_{Q^y}^{s_k'})$, then after applying F_1 we have squeezed our diamond k-times by $f_{k,2}$ and the derivative of $f_{k,2}^{-1}$ can be computed as an inverse matrix to (6.7) and we get

$$Df_{k,2}^{-1}(x,y,z) = \big(DF_1^{-1}(x',y',z')\big)B_1^{-1}\begin{pmatrix} k+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k+1 \end{pmatrix} B_1 \tag{6.11}$$

where $(x',y',z') = \varphi_{1,2}^{-1}\circ\varphi_{2,2}^{-1}\circ\cdots\circ\varphi_{k,2}^{-1}(x,y,z)$. Moreover, if $(x,y,z) \in \operatorname{int}(O_{Q^y}^{s_{k_2}'})$, then after applying F_1 and B_1 we have squeezed our diamond by $f_{m,2}$ $k_2 - 1$ times and then we have stretched it once.

Hence we can compute its derivative as an inverse matrix to (6.9) and we get

$$\text{(6.12)}\quad Df^{-1}_{m,2}(x,y,z)=\big(DF_1^{-1}(x',y',z')\big)B_1^{-1}\begin{pmatrix} \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1}\\ 0 & 1 & 0\\ \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1}\end{pmatrix}^{-1} B_1\ .$$

By using analogy of (5.5) and also the same estimate for DF_1^{-1} we obtain $\|Df^{-1}_{m,2}(x,y,z)\|\leq Ck_1k_2$ for every $(x,y,z)\in\operatorname{int}(O^{s'_{k_2}}_{Q^y})$ such that $f^{-1}_{m,2}(x,y,z)\in G_{k_1,1}$. Analogously to the proof of (5.1) we may deduce from the construction that for every k we have

$$\text{(6.13)}\qquad \mathcal{L}_3(f_{k+1,2}(R_{k+1,2}))=(s'_{k+1})^2\mathcal{L}_3(f_{k,2}(R_{k,2}))\text{ and hence }\mathcal{L}_3(f_{k,2}(R_{k,2}))\leq\frac{1}{k^2}\ .$$

By (5.7) we know that $\mathcal{L}_3(F_1(G_{k_1,1}))\leq\frac{C}{k_1^3}$ and analogously we can deduce that for every $m\geq k$

$$\mathcal{L}_3(f_{m,2}(G_{k,2}))=(s'_1)^2\cdots(s'_{k-1})^2(1-(s'_k)^2)\mathcal{L}_3(F_1(Q_0\setminus\mathcal{C}_1))\leq\tfrac{C}{k^3}\ .$$

Since $Q_0\setminus C_1=\bigcup_{k_1=1}^{\infty}G_{k_1,1}$ we can apply similar estimate on each $F_1(G_{k_1,1})$ and we obtain

$$\text{(6.14)}\qquad \mathcal{L}_3(f_{m,2}(G_{k_1,1}\cap G_{k_2,2}))\leq\frac{C}{k_1^3k_2^3}\text{ and }\mathcal{L}_3(f_{m,2}(G_{k_1,1}\cap R_{n,2}))\leq\frac{C}{k_1^3n^2}\ .$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,2}=f_{m,2}$ outside of $R_{n,2}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f^{-1}_{m,2}-f^{-1}_{n,2})|&=\int_{f_{m,2}(R_{n,2})}|D(f^{-1}_{m,2}-f^{-1}_{n,2})|\\ &\leq\int_{f_{m,2}(R_{n,2})}|Df^{-1}_{n,2}|+\int_{f_{m,2}(R_{m,2})}|Df^{-1}_{m,2}|+\int_{f_{m,2}(\cup_{k=n+1}^m G_{k,2})}|Df^{-1}_{m,2}|.\end{aligned}$$

As $Q_0\setminus\mathcal{C}_1=\bigcup_{k_1=1}^{\infty}G_{k_1,1}$, from (6.11) and (6.14) we obtain

$$\begin{aligned}\int_{f_{m,2}(R_{n,2})}|Df^{-1}_{n,2}|&\leq\sum_{k_1=1}^{\infty}\int_{f_{m,2}(G_{k_1,1}\cap R_{n,2})}|Df^{-1}_{n,2}|\\ &\leq\sum_{k_1=1}^{\infty}\mathcal{L}_3(f_{m,2}(G_{k_1,1}\cap R_{n,2}))Ck_1n\leq\sum_{k_1=1}^{\infty}\frac{1}{k_1^3n^2}Ck_1n\overset{n\to\infty}{\to}0\end{aligned}$$

and

$$\begin{aligned}\int_{f_{m,2}(R_{m,2})}|Df^{-1}_{m,2}|&\leq\sum_{k_1=1}^{\infty}\int_{f_{m,2}(G_{k_1,1}\cap R_{m,2})}|Df^{-1}_{m,2}|\\ &\leq\sum_{k_1=1}^{\infty}\mathcal{L}_3(f_{m,2}(G_{k_1,1}\cap R_{m,2}))Ck_1m\leq\sum_{k_1=1}^{\infty}\frac{1}{k_1^3m^2}Ck_1m\overset{n\to\infty}{\to}0\ .\end{aligned}$$

From (6.12), (5.5) and (6.14) we obtain

$$\begin{aligned}\int_{f_{m,2}(\cup_{k=n+1}^m G_{k,2})}|Df^{-1}_{m,2}|&\leq\sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\int_{f_{m,2}(G_{k_1,1}\cap G_{k_2,2})}|Df^{-1}_{m,2}|\\ &\leq\sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\mathcal{L}_3(f_{m,2}(G_{k_1,1}\cap G_{k_2,2}))Ck_1k_2\leq\sum_{k_2=n+1}^{m}\sum_{k_1=1}^{\infty}\frac{1}{k_1^3k_2^3}Ck_1k_2\overset{n\to\infty}{\to}0.\end{aligned}$$

It follows that the sequence $Df^{-1}_{k,2}$ is Cauchy in L^1 and thus we can easily obtain that $f^{-1}_{k,2}$ is Cauchy in $W^{1,1}$. Since $f^{-1}_{k,2}$ converge to F_2^{-1} uniformly we obtain that $F_2^{-1}\in W^{1,1}$.

The mapping F_3 is constructed in a similar way as mapping F_2 using translated and scaled copies of Q^x. Again we need to use 'rotations' B_2 in this step to adjust A_{k_2} to obtain almost upper triangular matrix. Derivatives of F_3 and F_3^{-1} can be estimated as in the general step below. Now we give details of the construction of F_4 which is different because we do similar construction in the target and not in the domain as for F_1, F_2 and F_3.

7. Construction and differentiability of F_4

7.1. **Key estimate.** Later we estimate the norm of the derivative by the chain rule as the norm of the product of corresponding matrices. The following estimate will be the key for the Sobolev regularity of function f. Let us estimate the derivative of the product

$$A_{k_1,k_2,k_3} = B_2^{-1}A_{k_3}B_2B_1^{-1}A_{k_2}B_1A_{k_1} := B_2^{-1}\begin{pmatrix} 1 & 0 & 0 \\ c & \frac{tk_3+1}{k_3+1}+\frac{4c}{k_3}\frac{tk_3^2-1}{k_3+1} & \frac{2c}{k_3}\frac{tk_3^2-1}{k_3+1} \\ c & \frac{2c}{k_3}\frac{tk_3^2-1}{k_3+1} & \frac{tk_3+1}{k_3+1}+\frac{4c}{k_3}\frac{tk_3^2-1}{k_3+1} \end{pmatrix}\cdot B_2B_1^{-1}\cdot$$

$$\cdot\begin{pmatrix} \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1} \\ 0 & 1 & 0 \\ \frac{2c}{k_2}\frac{tk_2^2-1}{k_2+1} & c & \frac{tk_2+1}{k_2+1}+\frac{4c}{k_2}\frac{tk_2^2-1}{k_2+1} \end{pmatrix}\cdot B_1\cdot\begin{pmatrix} \frac{tk_1+1}{k_1+1}+\frac{4c}{k_1}\frac{tk_1^2-1}{k_1+1} & \frac{2c}{k_1}\frac{tk_1^2-1}{k_1+1} & c \\ \frac{2c}{k_1}\frac{tk_1^2-1}{k_1+1} & \frac{tk_1+1}{k_1+1}+\frac{4c}{k_1}\frac{tk_1^2-1}{k_1+1} & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that we can obviously estimate the norm of each A_{k_i} by Ct and thus the norm of the product can be estimated by Ct^3. This would not be sufficient in the general step but in our construction we have chosen B_1 and then B_2 so that $B_1A_{k_1}$ and $B_2B_1^{-1}A_{k_2}$ are almost upper triangular matrices which leads to a better estimate. It is easy to see that in $B_1A_{k_1}$ we need to rotate only in the x,y-coordinates and in $B_2B_1^{-1}A_{k_2}$ we need to rotate only in x,z-coordinates. After these rotations we obtain a matrix (here Cct means a term that may depend on $(x,y,z),t,k$ but is bounded in absolute value by Ct)

(7.1)
$$\begin{aligned} A_{k_1,k_2,k_3} &= B_2^{-1}\begin{pmatrix} 1 & 0 & 0 \\ c & Cct & Cct \\ c & Cct & Cct \end{pmatrix}\cdot\begin{pmatrix} Cct & c & Cct \\ c & c & Cct \\ c & c & Cct \end{pmatrix}\cdot\begin{pmatrix} Cct & Cct & c \\ c & Cct & c \\ 0 & 0 & 1 \end{pmatrix} \\ &= B_2^{-1}\begin{pmatrix} 1 & 0 & 0 \\ c & Cct & Cct \\ c & Cct & Cct \end{pmatrix}\cdot\begin{pmatrix} Cct^2 & Cct^2 & Cct \\ Cct & Cct & Cct \\ Cct & Cct & Cct \end{pmatrix} = B_2^{-1}\cdot\begin{pmatrix} Cct^2 & Cct^2 & Cct \\ Cct^2 & Cct^2 & Cct^2 \\ Cct^2 & Cct^2 & Cct^2 \end{pmatrix}. \end{aligned}$$

and thus we may estimate $\|A_{k_1,k_2,k_3}\| \leq Ct^2$.

7.2. **Construction of F_4.** We will construct a sequence of homeomorphisms $f_{k,4}^{-1}: Q_0 \to Q_0$ and our mapping $F_4 \in W^{1,1}(Q_0,\mathbb{R}^3)$ will be later defined as $F_4(x) = \lim_{k\to\infty} f_{k,4}(x)$. So far we have constructed disjoint Cantor type sets such that $J_{F_1} = 0$ a.e. on $\mathcal{C}_1$, $J_{F_2} = 0$ a.e. on $\mathcal{C}_2$ and $J_{F_3} = 0$ a.e. on $\mathcal{C}_3$. Now we will construct a Cantor type set $\tilde{\mathcal{C}}_4$ of positive measure in the image so that $J_{F_4^{-1}} = 0$ a.e. on $\tilde{\mathcal{C}}_4$ and so that $\mathcal{L}_3(F_4^{-1}(\tilde{\mathcal{C}}_4)) = 0$.

The set $\mathcal{C}^4 = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is closed and thus we can find $\mathcal{Q}_{1,4}$, a collection of disjoint, scaled and translated copies of $Q^z(w_1)$ which cover $F_3(Q_0 \setminus \mathcal{C}^4)$ up to a set of measure zero. We will also require that

(7.2) for each $Q^z \in \mathcal{Q}_{1,4}$ there are $k_1,k_2,k_3 \in \mathbb{N}$ such that $F_3^{-1}(Q^z) \subset G_{k_1,1} \cap G_{k_2,2} \cap G_{k_3,3}$.

Secondly, we know that J_{F_3} is continuous in each diamond from $G_{k_1,1} \cap G_{k_2,2} \cap G_{k_3,3}$ and thus we may assume that $F_3^{-1}(Q^z)$ is a subset of one diamond and it is so small that

(7.3) $J_{F_3}(x_1,y_1,z_1) \leq (1+\delta_4)J_{F_3}(x_2,y_2,z_2)$ for every $(x_1,y_1,z_1),\ (x_2,y_2,z_2) \in F_3^{-1}(Q^z)$.

We define $f_{1,4}^{-1}\colon Q_0 \to Q_0$ by

$$f_{1,4}^{-1}(x,y,z) = \begin{cases} F_3^{-1}\circ\varphi^{Q^z}_{w_1,s_1,s_1'}(x,y,z) & (x,y,z) \in Q^z \in \mathcal{Q}_{1,4}, \\ F_3^{-1}(x,y,z) & \text{otherwise.} \end{cases}$$

It is not difficult to check that $f_{1,4}$ is a homeomorphism. Moreover it is a bi-Sobolev mapping since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. From now on each $f_{k,4}^{-1}$ will equal to $f_{1,4}^{-1}$ on

$$f_{1,4}(\mathcal{C}^4 \cup G_{1,4}), \text{ where } \tilde{G}_{1,4} = \bigcup_{Q^z\in\mathcal{Q}_{1,4}} O^{s_1}_{Q^z} \text{ and } G_{1,4} = f_{1,4}^{-1}(\tilde{G}_{1,4})$$

and it remains to define it on $\tilde{R}_{1,4} := f_{1,4}(R_{1,4})$, where

$$R_{1,4} := f_{1,4}^{-1}\Big(\bigcup_{Q^z\in\mathcal{Q}_{1,4}} I_{Q^z}^{s_1}\Big).$$

Analogously as before we obtain

$$\mathcal{L}_3(\tilde{R}_{1,4}) = s_1^2\mathcal{L}_3\big(F_3(Q_0\setminus\mathcal{C}^4)\big) \text{ and } \mathcal{L}_3(\tilde{G}_{1,4}) = (1-s_1^2)\mathcal{L}_3\big(F_3(Q_0\setminus\mathcal{C}^4)\big).$$

We continue inductively. Assume that $\mathcal{Q}_{k,4}$, $f_{k,4}$, $G_{k,4}$ and $R_{k,4}$ have already been defined. We find a family of disjoint scaled and translated copies of $Q^z(w_{k+1})$ that cover $\tilde{R}_{k,4}$ up to a set of measure zero. Define $\varphi_{k+1,4}\colon Q_0\to Q_0$ by

$$\varphi_{k+1,4}(x,y,z) = \begin{cases} \varphi^{Q^z}_{w_{k+1},s_{k+1},s'_{k+1}}(x,y,z) & (x,y,z)\in Q^z\in\mathcal{Q}_{k+1,4},\\ (x,y,z) & \text{otherwise.}\end{cases}$$

The mapping $f_{k+1,4}^{-1}\colon Q_0\to Q_0$ is now defined by $f_{k,4}^{-1}\circ\varphi_{k+1,4}$. Clearly each mapping $f_{k+1,4}$ is a homeomorphism. Moreover it is a bi-Sobolev mapping since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. We further define the sets

$$\begin{aligned} \tilde{G}_{k+1,4} &= \bigcup_{Q^z\in\mathcal{Q}_{k+1,4}} O_{Q^z}^{s_{k+1}}, & G_{k+1,4} &= f_{k+1,4}^{-1}(\tilde{G}_{k+1,4}),\\ \tilde{R}_{k+1,4} &= \bigcup_{Q^z\in\mathcal{Q}_{k+1,4}} I_{Q^z}^{s_{k+1}} \text{ and} & R_{k+1,4} &= f_{k+1,4}^{-1}(\tilde{R}_{k+1,4}).\end{aligned}$$

The linear maps $\varphi_{j,4}$, $1\le j\le k$, on inner diamonds do not change the ratio of volumes of Q^z and $O_{Q^z}^{s_{k+1}}$, $Q^z\in\mathcal{Q}_{k+1,4}$. Therefore we obtain that

$$\mathcal{L}_3(\tilde{G}_{k+1,4}) = (1-s_{k+1}^2)\mathcal{L}_3(\tilde{R}_{k,4}) \text{ and } \mathcal{L}_3(\tilde{R}_{k+1,4}) = s_{k+1}^2\mathcal{L}_3(\tilde{R}_{k,4}). \tag{7.4}$$

Analogously as before we obtain

$$\mathcal{L}_3(\tilde{R}_{k,4}) = s_1^2 s_2^2\cdots s_k^2\mathcal{L}_3\big(F_3(Q_0\setminus\mathcal{C}^4)\big) \tag{7.5}$$

and

$$\mathcal{L}_3(\tilde{G}_{k,4}) = s_1^2 s_2^2\cdots s_{k-1}^2(1-s_k^2)\mathcal{L}_3\big(F_3(Q_0\setminus\mathcal{C}^4)\big).$$

Therefore using (7.4) and Lemma 4.1 with $A=\tilde{G}_{k_4,4}$ and $P=F_3(Q_0\setminus\mathcal{C}^4)$, we obtain that

$$\frac{1}{1+\delta_4}\mathcal{L}_3(G_{k_4,4}) \le s_1^2 s_2^2\cdots s_{k_4-1}^2(1-s_{k_4}^2)\mathcal{L}_3\big(Q_0\setminus\mathcal{C}^4\big) \le (1+\delta_4)\mathcal{L}_3(G_{k_4,4}).$$

Since the sets Q^z, $Q^z\in\mathcal{Q}_{k+1,4}$ are uniformly places among $F_3(G_{l,i})$, $i\in\{1,2,3\}$, up to a null set (see (7.2) and (4.4)) we can moreover obtain

$$\frac{1}{\Delta_4}\mathcal{L}_3\Big(\bigcap_{i=1}^4 G_{k_i,i}\Big) \le s_1^2 s_2^2\dots s_{k_4-1}^2(1-s_{k_4}^2)\mathcal{L}_3\Big(\bigcap_{i=1}^3 G_{k_i,i}\Big) \le \Delta_4\mathcal{L}_3\Big(\bigcap_{i=1}^4 G_{k_i,i}\Big). \tag{7.6}$$

It follows from (7.5) that the resulting Cantor type set

$$\tilde{\mathcal{C}}_4 := \bigcap_{k=1}^\infty \tilde{R}_{k,4}$$

satisfies

$$\mathcal{L}_3(\tilde{\mathcal{C}}_4) \ge \frac{1}{\Delta_4}\mathcal{L}_3(Q_0\setminus\mathcal{C}^4)\prod_{i=1}^\infty s_i^2 > 0.$$

It is clear from the construction that $f_{k,4}^{-1}$ converge uniformly and hence it is not difficult to check that the limiting map $F_4(x) := \lim_{k\to\infty} f_{k,4}(x)$ exists and it is a homeomorphism. It remains to verify that $f_{k,4}^{-1}$ and $f_{k,4}$ form a Cauchy sequence in $W^{1,1}$ and thus F_4 is bi-Sobolev.

7.3. **Weak differentiability of F_4^{-1}.** Let us estimate the derivative of our functions $f_{m,4}^{-1}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^z\in\mathcal{Q}_{k,4}$ and $(x,y,z)\in\operatorname{int}(I_{Q^z}^{s'_k})$, then we have squeezed our diamond k-times and then we apply F_3^{-1}. Analogously to (5.2) we can use (3.1), (4.2) and the chain rule to obtain

$$Df_{k,4}^{-1}(x,y,z)=DF_3^{-1}(x',y',z')\begin{pmatrix}\frac{1}{k+1} & 0 & 0\\ 0 & \frac{1}{k+1} & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{7.7}$$

where $(x',y',z')=\varphi_{1,4}^{-1}\circ\ldots\circ\varphi_{k,4}^{-1}(x,y,z)$. Moreover, if $(x,y,z)\in\operatorname{int}(O_{Q^z}^{s'_k})$, then we stretch our diamond once, then we squeeze it $k-1$ times and then we apply F_3^{-1}. Analogously to (5.3) we can use (3.1), (4.2), (3.3), (4.3) and the chain rule to obtain that

$$Df_{k,4}^{-1}(x,y,z)=DF_3^{-1}(x',y',z')\begin{pmatrix}\frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1} & \frac{2c}{k}\frac{tk^2-1}{k+1} & c\\ \frac{2c}{k}\frac{tk^2-1}{k+1} & \frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1} & c\\ 0 & 0 & 1\end{pmatrix}. \tag{7.8}$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,4}^{-1}=f_{m,4}^{-1}$ outside of $\tilde{R}_{n,4}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f_{m,4}^{-1}-f_{n,4}^{-1})|&=\int_{\tilde{R}_{n,4}}|D(f_{m,4}^{-1}-f_{n,4}^{-1})|\\ &\leq\int_{\tilde{R}_{n,4}\setminus\tilde{R}_{m,4}}(|Df_{m,4}^{-1}|+|Df_{n,4}^{-1}|)+\int_{\tilde{R}_{m,4}}|D(f_{m,4}^{-1}-f_{n,4}^{-1})|+\sum_{k_4=n+1}^{m}\int_{\tilde{G}_{k_4,4}}|Df_{m,4}^{-1}|.\end{aligned}$$

By (7.7), (7.8), $DF_3^{-1}\in L^1$ and $\mathcal{L}_3(\tilde{R}_{n,4}\setminus\tilde{R}_{m,4})\to 0$ we obtain

$$\int_{\tilde{R}_{n,4}\setminus\tilde{R}_{m,4}}(|Df_{m,4}^{-1}|+|Df_{n,4}^{-1}|)\leq C\int_{\tilde{R}_{n,4}\setminus\tilde{R}_{m,4}}|DF_3^{-1}|\overset{n\to\infty}{\to}0\,.$$

By (7.7) and $DF_3^{-1}\in L^1$ we obtain

$$\int_{\tilde{R}_{m,4}}|D(f_{m,4}^{-1}-f_{n,4}^{-1})|\leq C\Big(\frac{1}{n+1}-\frac{1}{m+1}\Big)\int_{\tilde{R}_{m,4}}|DF_3^{-1}|\overset{n\to\infty}{\to}0\,.$$

The last term can be estimated with the help of (7.6) by

$$\begin{aligned}\sum_{k_4=n+1}^{m}\int_{\tilde{G}_{k_4,4}}|Df_{m,4}^{-1}|&\leq\sum_{k_1,k_2,k_3=1}^{\infty}\sum_{k_4=n+1}^{m}\mathcal{L}_3(f_{n,4}(G_{k_1,1}\cap G_{k_2,2}\cap G_{k_3,3})\cap\tilde{G}_{k_4,4})Ck_1k_2k_3t\\ &\leq\sum_{k_1,k_2,k_3=1}^{\infty}\sum_{k_4=n+1}^{m}\frac{C}{k_1^3k_2^3k_3^3tk_4^2}Ck_1k_2k_3t\overset{n\to\infty}{\to}0\,.\end{aligned}$$

It follows that the sequence $Df_{k,4}^{-1}$ is Cauchy in L^1 and thus we can easily obtain that $f_{k,4}^{-1}$ is Cauchy in $W^{1,1}$. Since $f_{k,4}^{-1}$ converge to F_4^{-1} uniformly we obtain that $F_4^{-1}\in W^{1,1}$.

From (7.7) we obtain that the derivative of $f_{k,4}^{-1}$ on $\tilde{R}_{k,4}$ and especially on $\widetilde{\mathcal{C}}_4$ equals to

$$Df_{k,4}^{-1}(x,y,z)=DF_3^{-1}(x',y',z')\begin{pmatrix}\frac{1}{k+1} & 0 & 0\\ 0 & \frac{1}{k+1} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

In the limit we obtain that $J_{F_4^{-1}}=0$ a.e. on $\widetilde{\mathcal{C}}_4$.

7.4. **Weak differentiability of F_4.** Let us estimate the derivative of our functions $f_{m,4}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^z\in\mathcal{Q}_{k,4}$ and $(x,y,z)\in\operatorname{int}f_{m,4}^{-1}(I_{Q^z}^{s'_k})$, then after applying F_3 we have squeezed our diamond k-times and the derivative of $f_{k,4}$ can be computed as an inverse matrix to (7.7) and we get

$$Df_{k,4}(x,y,z) = \begin{pmatrix} k+1 & 0 & 0 \\ 0 & k+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} DF_3(x,y,z). \tag{7.9}$$

Moreover, if $(x,y,z) \in \operatorname{int} f_{m,4}^{-1}(O_{Q^z}^{s'_{k_4}})$, then after applying F_3 we have stretched our diamond $k_4 - 1$ times and then we have squeezes it once. Hence we can compute its derivative as an inverse matrix to (7.8) and we get

$$Df_{m,4}(x,y,z) = \begin{pmatrix} \frac{tk_4+1}{k_4+1} + \frac{4c}{k_4}\frac{tk_4^2-1}{k_4+1} & \frac{2c}{k_4}\frac{tk_4^2-1}{k_4+1} & c \\ \frac{2c}{k_4}\frac{tk_4^2-1}{k_4+1} & \frac{tk_4+1}{k_4+1} + \frac{4c}{k_4}\frac{tk_4^2-1}{k_4+1} & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} DF_3(x,y,z) \ . \tag{7.10}$$

By using analogy of (5.5) we obtain $\|Df_{m,4}(x,y,z)\| \leq Ck_4\|DF_3\|$ for every $(x,y,z) \in \operatorname{int} f_{m,4}^{-1}(O_{Q^z}^{s'_{k_4}})$. Analogously to the proof of (5.1) we may deduce from the construction that for every k we have

$$\mathcal{L}_3(R_{k+1,4}) = (s'_{k+1})^2 \mathcal{L}_3(R_{k,4}) \text{ and hence } \mathcal{L}_3(R_{k,4}) \leq \frac{1}{k^2}. \tag{7.11}$$

and

$$\mathcal{L}_3(G_{k+1,4}) = (1 - (s'_{k+1})^2)\mathcal{L}_3(R_{k,4}).$$

By (5.1) and (6.5), we can deduce that:

$$\mathcal{L}_3(\bigcap_{i=1}^{4} G_{k_i,i}) \leq \Delta_3 \frac{C}{tk_1^2 tk_2^2 tk_3^2 k_4^3} \text{ and } \mathcal{L}_3(\bigcap_{i=1}^{3} G_{k_i,i} \cap R_{n,4}) \leq \Delta_3 \frac{C}{tk_1^2 tk_2^2 tk_3^2 n^2} \ .$$

Now let us fix $m, n \in \mathbb{N}$, $m > n$. Since $f_{n,4} = f_{m,4}$ outside of $R_{n,4}$ we obtain

$$\begin{aligned} \int_{Q_0} |D(f_{m,4} - f_{n,4})| &= \int_{R_{n,4}} |D(f_{m,4} - f_{n,4})| \\ &\leq \int_{R_{n,4}} |Df_{n,4}| + \int_{R_{m,4}} |Df_{m,4}| + \int_{\bigcup_{k_4=n+1}^{m} G_{k_4,4}} |Df_{m,4}|. \end{aligned}$$

From (7.9), (7.10) and (7.11) we obtain

$$\begin{aligned} \int_{R_{n,4}} |Df_{n,4}| &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \mathcal{L}_3(\bigcap_{i=1}^{3} G_{k_i,i} \cap R_{n,4}) Ct^3 n \\ &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \frac{C}{tk_1^2 tk_2^2 tk_3^2 n^2} Ct^3 n \overset{n\to\infty}{\to} 0 \end{aligned}$$

and

$$\begin{aligned} \int_{R_{m,4}} |Df_{m,4}| &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \mathcal{L}_3(\bigcap_{i=1}^{3} G_{k_i,i} \cap R_{m,4}) Ct^3 m \\ &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \frac{C}{tk_1^2 tk_2^2 tk_3^2 m^2} Ct^3 m \overset{n\to\infty}{\to} 0 \ . \end{aligned}$$

From (7.10) and (7.11) we obtain

$$\begin{aligned} \int_{\bigcup_{k_4=n+1}^{m} G_{k_4,4}} |Df_{m,4}| &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \sum_{k_4=n+1}^{m} \mathcal{L}_3(\bigcap_{i=1}^{4} G_{k_i,i}) Ct^3 k_4 \\ &\leq \sum_{k_1,k_2,k_3=1}^{\infty} \sum_{k_4=n+1}^{m} \frac{C}{tk_1^2 tk_2^2 tk_3^2 k_4^3} Ct^3 k_4 \overset{n\to\infty}{\to} 0 \ . \end{aligned}$$

It follows that the sequence $Df_{k,4}$ is Cauchy in L^1 and thus we can easily obtain that $f_{k,4}$ is Cauchy in $W^{1,1}$. Since $f_{k,4}$ converge to F_4 uniformly we obtain that $F_4 \in W^{1,1}$.

Set $\mathcal{C}_4 = F_4^{-1}(\tilde{\mathcal{C}}_4)$. From now on each F_k will equal to F_4 on $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ and we need to define it only on the complement of this compact set. It is not difficult to check that $J_{F_4} \neq 0$ a.e. on $Q_0 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$ and thus the preimages of the exceptional null sets will be null sets.

8. Construction and differentiability of general F_j

8.1. **Construction of F_j.** Assume that the mapping F_{j-1} and the Cantor type set $\mathcal{C}_{j-1}$ have already been defined. We will construct a sequence of homeomorphisms $f_{k,j} : Q_0 \to Q_0$ and our mapping $F_j \in W^{1,1}(Q_0, \mathbb{R}^3)$ will be later defined as $F_j(x) = \lim_{k\to\infty} f_{k,j}(x)$. We will also construct a Cantor-type set $\mathcal{C}_j \subset Q_0 \setminus (\cup_{i=1}^{j-1}\mathcal{C}_i)$ such that $\mathcal{L}_3(\mathcal{C}_j) > 0$ and $J_{F_j} = 0$ a.e. on $\mathcal{C}_j$ for $j \in \bigcup_{l\in\mathbb{N}}\{6l+1, 6l+2, 6l+3\}$ while $\mathcal{L}_3(F_j(\mathcal{C}_j)) > 0$ and $J_{F_j^{-1}} = 0$ a.e. on $F_j(\mathcal{C}_j)$ for $j \in \bigcup_{l\in\mathbb{N}}\{6l+4, 6l+5, 6l+6\}$. The mappings F_{6l+1} and F^{-1}_{6l+4} are constructed using diamonds Q^z, F_{6l+2} and F^{-1}_{6l+5} are constructed using 'rotated' diamonds Q^y and finally F_{6l+3} and F^{-1}_{6l+6} are constructed using 'rotated' diamonds Q^x. For simplicity we give the details of the construction only for $j = 6l+3$ as the estimates in other cases are similar. The construction in the case $j \in \bigcup_{l\in\mathbb{N}}\{6l+4, 6l+5, 6l+6\}$ is similar to the construction of F_4, i.e. we are constructing in the target and not in the domain.

The set $C^j := \cup_{i=1}^{j-1}\mathcal{C}_i$ is closed and thus we can find $\mathcal{Q}_{1,j}$, a collection of disjoint, scaled, translated and rotated copies of $Q^x(w_1)$ (recall that $j = 6l+3$) which cover $F_{j-1}(Q_0 \setminus \mathcal{C}^j)$ up to a set of measure zero. We will moreover require that

$$\text{for each } Q^x \in \mathcal{Q}_{1,j} \text{ there are } k_1, \dots, k_{j-1} \in \mathbb{N} \text{ such that } F^{-1}_{j-1}(Q^x) \subset \bigcap_{i=1}^{j-1} G_{k_i,i}\,. \tag{8.1}$$

Secondly, we know that $J_{F_{j-1}}$ is continuous in each diamond from $G_{l,j}$ (see (3.2)) and thus we may assume that $F^{-1}_{j-1}(Q^x)$ is a subset of one diamond from the previous construction and it is so small that

$$J_{F_{j-1}}(x_1, y_1, z_1) \leq (1 + \delta_j) J_{F_{j-1}}(x_2, y_2, z_2) \text{ for every } (x_1, y_1, z_1),\ (x_2, y_2, z_2) \in F^{-1}_{j-1}(Q^x). \tag{8.2}$$

We define $f_{1,j} \colon Q_0 \to Q_0$ by

$$f_{1,j}(x, y, z) = \begin{cases} B^{-1}_{j-1} \circ \varphi^{Q^x}_{w_1,s_1,s'_1} \circ B_{j-1} \circ F_{j-1}(x, y, z) & F_{j-1}(x, y, z) \in Q^x \in \mathcal{Q}_{1,j}, \\ F_{j-1}(x, y, z) & \text{otherwise.} \end{cases}$$

It is not difficult to check that $f_{1,j}$ is a bi-Sobolev homeomorphism since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. From now on each $f_{k,j}$ will equal to $f_{1,j}$ on

$$\mathcal{C}^j \cup G_{1,j}, \text{ where } G_{1,j} := F^{-1}_{j-1}\Big(\bigcup_{Q^x\in\mathcal{Q}_{1,j}} O^{s_1}_{Q^x}\Big)$$

and it remains to define it on

$$R_{1,j} := F^{-1}_{j-1}\Big(\bigcup_{Q^x\in\mathcal{Q}_{1,j}} I^{s_1}_{Q^x}\Big).$$

Clearly

$$\mathcal{L}_3(F_{j-1}(R_{1,j})) = s_1^2 \mathcal{L}_3\big(F_{j-1}(Q_0 \setminus \mathcal{C}^j)\big) \text{ and}$$
$$\mathcal{L}_3(F_{j-1}(G_{1,j})) = (1 - s_1^2) \mathcal{L}_3\big(F_{j-1}(Q_0 \setminus \mathcal{C}^j)\big).$$

We continue inductively. Assume that $\mathcal{Q}_{k,j}$, $f_{k,j}$, $G_{k,j}$ and $R_{k,j}$ have already been defined. We find a family of disjoint scaled, translated and rotated copies of $Q^x(w_{k+1})$ that cover $f_{k,j}(R_{k,j})$ up to a set of measure zero $E_{k+1,j}$. Define $\varphi_{k+1,j} \colon Q_0 \to Q_0$ by

$$\varphi_{k+1,j}(x, y, z) = \begin{cases} B^{-1}_{j-1} \circ \varphi^{Q^x}_{w_{k+1},s_{k+1},s'_{k+1}} \circ B_{j-1}(x, y, z) & (x, y, z) \in Q^x \in \mathcal{Q}_{k+1,j}, \\ (x, y, z) & \text{otherwise.} \end{cases}$$

The matrix B_{j-1} is chosen so that $B_{j-1}B^{-1}_{j-2}A_{k_{j-1}}$ is almost upper triangular. The mapping $f_{k+1,j} \colon Q_0 \to Q_0$ is now defined by $\varphi_{k+1,j} \circ f_{k,j}$. Clearly each mapping $f_{k+1,j}$ is a bi-Sobolev homeomorphism since it is a composition of a bi-Sobolev and bi-Lipschitz mapping. We further define the sets

$$G_{k+1,j} := f_{k,j}^{-1}\Big(\bigcup_{Q^x\in\mathcal{Q}_{k+1,j}} O_{Q^x}^{s_{k+1}}\Big) \text{ and } R_{k+1,j} := f_{k,j}^{-1}\Big(\bigcup_{Q^x\in\mathcal{Q}_{k+1,j}} I_{Q^x}^{s_{k+1}}\Big).$$

The maps $\varphi_{i,j}$, $1\le i\le k$, on inner diamonds do not change the ratio of volumes of Q^x and $O_{Q^x}^{s_{k+1}}$. Therefore we obtain that

$$\mathcal{L}_3(F_{j-1}(G_{k+1,j})) = (1-s_{k+1}^2)\mathcal{L}_3(F_{j-1}(R_{k,j})) \text{ and } \mathcal{L}_3(F_{j-1}(R_{k+1,j})) = s_{k+1}^2\mathcal{L}_3(F_{j-1}(R_{k,j})).$$

Analogously as before we obtain using (8.2) and Lemma 4.1 that

$$\frac{1}{1+\delta_j}\mathcal{L}_3(R_{k,j}) \le s_1^2 s_2^2\cdots s_k^2\mathcal{L}_3\big(Q_0\setminus\mathcal{C}^j\big) \le (1+\delta_j)\mathcal{L}_3(R_{k,j})$$

and

$$\frac{1}{1+\delta_j}\mathcal{L}_3(G_{k,j}) \le s_1^2 s_2^2\cdots s_{k-1}^2(1-s_k^2)\mathcal{L}_3\big(Q_0\setminus\mathcal{C}^j\big) \le (1+\delta_j)\mathcal{L}_3(G_{k,j}).$$

Since the sets Q^x are uniformly placed among all $F_{j-1}(G_{l,i})$ for $i=1,\dots,j-1$ (see (8.1)) we moreover obtain using (4.4) that

$$\frac{1}{\Delta_j}\mathcal{L}_3\Big(\bigcap_{i=1}^{j} G_{k_i,i}\Big) \le s_1^2 s_2^2\cdots s_{k_j-1}^2(1-s_{k_j}^2)\mathcal{L}_3\Big(\bigcap_{i=1}^{j-1} G_{k_i,i}\Big) \le \Delta_j\mathcal{L}_3\Big(\bigcap_{i=1}^{j} G_{k_i,i}\Big). \tag{8.3}$$

It follows that the resulting Cantor type set

$$\mathcal{C}_j := \bigcap_{k=1}^{\infty} R_{k,j}$$

satisfies

$$\mathcal{L}_3(\mathcal{C}_j) \ge \frac{1}{\Delta_j}\mathcal{L}_3(Q_0\setminus\mathcal{C}^j)\prod_{i=1}^{\infty} s_i^2 > 0. \tag{8.4}$$

It is clear from the construction that $f_{k,j}$ converge uniformly and hence it is not difficult to check that the limiting map $F_j(x) := \lim_{k\to\infty} f_{k,j}(x)$ exists and is a homeomorphism. It remains to verify that $f_{k,j}$ and $f_{k,j}^{-1}$ form a Cauchy sequence in $W^{1,1}$ and thus F_j is a bi-Sobolev mapping.

8.2. **Weak differentiability of F_j.** Let us estimate the derivative of our functions $f_{m,j}$. Let us fix $m,k\in\mathbb{N}$ such that $m\ge k$. If $Q^x\in\mathcal{Q}_{k,j}$ and $(x,y,z)\in\operatorname{int}(f_{k,j})^{-1}(I_{Q^x}^{s_k'})$, then after applying F_{j-1} we have squeezed our diamond k-times. Analogously to (5.2) we can use (3.1), (4.2) and the chain rule to obtain

$$Df_{k,j}(x,y,z) = B_{j-1}^{-1}\begin{pmatrix} 1 & 0 & 0\\ 0 & \frac{1}{k+1} & 0\\ 0 & 0 & \frac{1}{k+1}\end{pmatrix} B_{j-1}DF_{j-1}(x,y,z). \tag{8.5}$$

Moreover, if $(x,y,z)\in\operatorname{int}(f_{m,j})^{-1}(O_{Q^x}^{s_k'})$, then after applying F_{j-1} we have squeezed our diamond $k-1$ times and then we have stretched it once. Analogously to (5.3) we can use (3.1), (4.2), (3.3), (4.3) and the chain rule to obtain that

(8.6)
$$\begin{aligned} Df_{m,j}(x,y,z) &= B_{j-1}^{-1}\begin{pmatrix} 1 & 0 & 0\\ c & \frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1} & 2c\frac{tk^2-1}{k+1}\\ c & 2c\frac{tk^2-1}{k+1} & \frac{tk^2+k}{k+1}+4c\frac{tk^2-1}{k+1}\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & \frac{1}{k} & 0\\ 0 & 0 & \frac{1}{k}\end{pmatrix} B_{j-1}DF_{j-1}(x,y,z)\\ &= B_{j-1}^{-1}\begin{pmatrix} 1 & 0 & 0\\ c & \frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1} & \frac{2c}{k}\frac{tk^2-1}{k+1}\\ c & \frac{2c}{k}\frac{tk^2-1}{k+1} & \frac{tk+1}{k+1}+\frac{4c}{k}\frac{tk^2-1}{k+1}\end{pmatrix} B_{j-1}DF_{j-1}(x,y,z). \end{aligned}$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,j}=f_{m,j}$ outside of $R_{n,j}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f_{m,j}-f_{n,j})| &= \int_{R_{n,j}}|D(f_{m,j}-f_{n,j})|\\ &\leq C\int_{R_{n,j}\setminus R_{m,j}}|Df_{n,j}|+C\int_{R_{m,j}}|Df_{m,j}-Df_{n,j}|+C\sum_{k_j=n+1}^{m}\int_{G_{k_j,j}}|Df_{m,j}|.\end{aligned}$$

By (8.5) we get

$$\int_{R_{n,j}\setminus R_{m,j}}|Df_{n,j}|\leq\int_{R_{n,j}\setminus R_{m,j}}|DF_{j-1}|\overset{n\to\infty}{\to}0$$

since $DF_{j-1}\in L^1$ and $\mathcal{L}_3(R_{n,j}\setminus R_{m,j})\to 0$. From (8.5) we obtain

$$\int_{R_{m,j}}|Df_{m,j}-Df_{n,j}|\leq\frac{1}{n+1}\int_{R_{m,j}}|DF_{j-1}|\overset{n\to\infty}{\to}0.$$

In the estimate of the norm of the derivative in the remaining term we use the chain rule and then we multiply triples of adjacent matrices and then we use our key estimate (7.1). Now we use (8.6), (8.3), $\Delta_j\leq\Delta$, $\sum\frac{1}{k^2}=\frac{\pi^2}{6}$ and we proceed similarly to (6.10)

$$\begin{aligned}\sum_{k_j=n+1}^{m}\int_{G_{k_j,j}}|Df_{m,j}| &\leq\sum_{k_j=n+1}^{m}\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\int_{\bigcap_{i=1}^{j}G_{k_i,i}}|Df_{m,j}|\\ &\leq C\sum_{k_j=n+1}^{m}\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\mathcal{L}_3\big(\bigcap_{i=1}^{j}G_{k_i,i}\big)\|A_{k_1,k_2,k_3}\|\cdot\|A_{k_4,k_5,k_6}\|\cdots\|A_{k_{j-2},k_{j-1},k_j}\|\\ &\leq C\Delta\Big(\sum_{k_1,k_2,k_3,k_4,k_5,k_6=1}^{\infty}\frac{C_1t^2}{tk_1^2tk_2^2tk_3^2}\frac{C_2k_4k_5k_6}{k_4^3k_5^3k_6^3}\Big)\cdots\Big(\sum_{k_j=n+1}^{m}\sum_{k_{j-2},k_{j-1}=1}^{\infty}\frac{C_1t^2}{tk_{j-2}^2tk_{j-1}^2tk_j^2}\Big)\\ &\leq C\Big(C_1C_2\big(\frac{\pi^2}{6}\big)^6t^{-1}\Big)^{\frac{j-3}{6}}\cdot\Big(C_1\frac{\pi^4}{6^2}t^{-1}\sum_{k_j=n+1}^{m}\frac{1}{k_j^2}\Big)\overset{n\to\infty}{\to}0.\end{aligned}\tag{8.7}$$

As before this implies that $F_j\in W^{1,1}$ and similarly we also obtain that $J_{F_j}=0$ almost everywhere on $\mathcal{C}_j$ and that $J_{F_j}\neq 0$ almost everywhere on $Q_0\setminus\mathcal{C}^j$.

8.3. **Weak differentiability of F_j^{-1}.** Let us estimate the derivative of our functions $f_{m,j}^{-1}$. Let us fix $m,k\in\mathbb{N}$ such that $m\geq k$. If $Q^x\in\mathcal{Q}_{k,j}$ and $(x,y,z)\in\operatorname{int}(I_{Q^x}^{s'_k})$, then after applying F_{j-1} we have squeezed our diamond k-times by $f_{k,j}$ and the derivative of $f_{k,j}^{-1}$ can be computed as an inverse matrix to (8.5) and we get

$$Df_{k,j}^{-1}(x,y,z)=\big(DF_{j-1}(x',y',z')\big)^{-1}B_{j-1}^{-1}\begin{pmatrix}1&0&0\\0&k+1&0\\0&0&k+1\end{pmatrix}B_{j-1}.\tag{8.8}$$

where $(x',y',z')=\varphi_{1,j}^{-1}\circ\varphi_{2,j}^{-1}\circ\cdots\varphi_{k,j}^{-1}$. Moreover, if $(x,y,z)\in\operatorname{int}(O_{Q^x}^{s'_{k_j}})$, then after applying F_{j-1} we have squeezed our diamond by $f_{m,j}$ k_j-1 times and then we have stretched it once. Hence we can compute its derivative as an inverse matrix to (8.6) and we get

(8.9)

$$Df_{m,j}^{-1}(x,y,z)=\big(DF_{j-1}(x',y',z')\big)^{-1}B_{j-1}^{-1}\begin{pmatrix}1&0&0\\c&\frac{tk_j+1}{k_j+1}+\frac{4c}{k_j}\frac{tk_j^2-1}{k_j+1}&\frac{2c}{k_j}\frac{tk_j^2-1}{k_j+1}\\c&\frac{2c}{k_j}\frac{tk_j^2-1}{k_j+1}&\frac{tk_j+1}{k_j+1}+\frac{4c}{k_j}\frac{tk_j^2-1}{k_j+1}\end{pmatrix}^{-1}B_{j-1}\ .$$

By using analogy of (5.5) we obtain $\|Df_{m,j}^{-1}(x,y,z)\|\leq Ck_j\|DF_{j-1}^{-1}\|$ for every $(x,y,z)\in\operatorname{int}(O_{Q^x}^{s'_{k_j}})$. Analogously to the proof of (5.6) we may deduce from the construction that for every k we have

$$\mathcal{L}_3(f_{m,j}(R_{k+1,j}))=(s'_{k+1})^2\mathcal{L}_3(f_{k,j}(R_{k,j}))\text{ and hence }\mathcal{L}_3(f_{m,j}(R_{k,j}))\leq\frac{1}{k^2}\ .\tag{8.10}$$

Moreover, we can deduce that

$$\mathcal{L}_3(f_{m,j}(\bigcap_{i=1}^{j} G_{k_i,i})) \leq \frac{1}{k_1^3k_2^3k_3^3}\frac{1}{tk_4^2tk_5^2tk_6^2}\cdots\frac{1}{k_{j-2}^3k_{j-1}^3k_j^3} \text{ and}$$

$$\mathcal{L}_3(f_{m,j}(\bigcap_{i=1}^{j-1} G_{k_i,i}\cap R_{n,j})) \leq \frac{1}{k_1^3k_2^3k_3^3}\frac{1}{tk_4^2tk_5^2tk_6^2}\cdots\frac{1}{k_{j-2}^3k_{j-1}^3n^2}\ .$$

Now let us fix $m,n\in\mathbb{N}$, $m>n$. Since $f_{n,j}=f_{m,j}$ outside of $R_{n,j}$ we obtain

$$\begin{aligned}\int_{Q_0}|D(f_{m,j}^{-1}-f_{n,j}^{-1})| &= \int_{f_{m,j}(R_{n,j})}|D(f_{m,j}^{-1}-f_{n,j}^{-1})|\\ &\leq \int_{f_{m,j}(R_{n,j})}|Df_{n,j}^{-1}|+\int_{f_{m,j}(R_{m,j})}|Df_{m,j}^{-1}|+\int_{f_{m,j}(\bigcup_{k_j=n+1}^m G_{k,j})}|Df_{m,j}^{-1}|.\end{aligned}$$

From (8.8) and (8.10) we obtain

$$\begin{aligned}\int_{f_{m,j}(R_{n,j})}|Df_{n,j}^{-1}| &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\mathcal{L}_3(f_{m,j}(\bigcap_{i=1}^{j-1}G_{k_i,i}\cap R_{n,j}))C_2k_1k_2k_3C_1t^2\cdots Ck_{j-2}k_{j-1}n\\ &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\frac{C_2k_1k_2k_3}{k_1^3k_2^3k_3^3}\frac{C_1t^2}{tk_4^2tk_5^2tk_6^2}\cdots\frac{Ck_{j-1}k_{j-2}n}{k_{j-2}^3k_{j-1}^3n^2}\overset{n\to\infty}{\to}0\end{aligned}$$

and

$$\begin{aligned}\int_{f_{m,j}(R_{m,j})}|Df_{m,j}^{-1}| &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\mathcal{L}_3(f_{m,j}(\bigcap_{i=1}^{j-1}G_{k_i,i}\cap R_{m,j}))C_2k_1k_2k_3C_1t^2\cdots Ck_{j-2}k_{j-1}m\\ &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\frac{C_2k_1k_2k_3}{k_1^3k_2^3k_3^3}\frac{C_1t^2}{tk_4^2tk_5^2tk_6^2}\cdots\frac{Ck_{j-1}k_{j-2}m}{k_{j-2}^3k_{j-1}^3m^2}\overset{n\to\infty}{\to}0\ .\end{aligned}$$

From (8.9) and (8.10) we obtain

(8.11)

$$\begin{aligned}\int_{f_{m,j}(\cup_{k_j=n+1}^m G_{k_j,j})}|Df_{m,j}^{-1}| &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\sum_{k_j=n+1}^{m}\mathcal{L}_3(f_{m,j}(\bigcap_{i=1}^{j}G_{k_i,i}))C_2k_1k_2k_3C_1t^2\cdots Ck_{j-2}k_{j-1}k_j\\ &\leq \sum_{k_1,\dots,k_{j-1}=1}^{\infty}\sum_{k_j=n+1}^{m}\frac{C_2k_1k_2k_3}{k_1^3k_2^3k_3^3}\frac{C_1t^2}{tk_4^2tk_5^2tk_6^2}\cdots\frac{C_2k_{j-1}k_{j-2}k_j}{k_{j-2}^3k_{j-1}^3k_j^3}\overset{n\to\infty}{\to}0\ .\end{aligned}$$

It follows that the sequence $Df_{k,j}^{-1}$ is Cauchy in L^1 and thus we can easily obtain that $f_{k,j}^{-1}$ is Cauchy in $W^{1,1}$. Since $f_{k,j}^{-1}$ converge to F_j^{-1} uniformly we obtain that $F_j^{-1}\in W^{1,1}$.

9. Properties of f

Now we define $f(x)=\lim_{j\to\infty}F_j(x)$. Since F_j converge uniformly it is easy to see that f is a homeomorphism. It remains to show that DF_j and DF_j^{-1} is Cauchy in L^1 and thus f is bi-Sobolev.

Since $F_j=F_{j-1}$ on $\bigcup_{i=1}^{j-1}\mathcal{C}_i$ we obtain

$$\int_{Q_0}|D(F_j-F_{j-1})|\leq\int_{C_j}\big(|DF_j|+|DF_{j-1}|\big)+\sum_{k_j=1}^{\infty}\int_{G_{k_j,j}}\big(|DF_j|+|DF_{j-1}|\big).$$

We will proceed analogously to (8.7) but we will estimate the multiplicative constant more carefully. Again we will suppose that $j=6l+3$ but everything works for other j analogously. Analogously to

(8.7) we can use (4.1) to obtain

$$\begin{aligned}\sum_{k_j=1}^{\infty}\int_{G_{k_j,j}}\big(|DF_j|+|DF_{j-1}|\big)&\leq\sum_{k_1,\dots,k_j=1}^{\infty}\int_{\bigcap_{i=1}^{j}G_{k_i,i}}\big(|DF_j|+|DF_{j-1}|\big)\\ &\leq C\sum_{k_1,\dots,k_j=1}^{\infty}\mathcal{L}_3\big(\bigcap_{i=1}^{j}G_{k_i,i}\big)\|A_{k_1,k_2,k_3}\|\cdot\|A_{k_4,k_5,k_6}\|\cdots\|A_{k_{j-2},k_{j-1},k_j}\| \qquad (9.1)\\ &\leq C\Big(C_1C_2\big(\frac{\pi^2}{6}\big)^6t^{-1}\Big)^{\frac{j-3}{6}}\frac{C}{t}\leq C\Big(\frac{1}{2}\Big)^{\frac{j-3}{6}}.\end{aligned}$$

From (8.5) we know that

$$Df_{k,j}(x,y,z)=B_{j-1}^{-1}\begin{pmatrix}1&0&0\\0&\frac{1}{k+1}&0\\0&0&\frac{1}{k+1}\end{pmatrix}B_{j-1}DF_{j-1}(x,y,z)$$

on $\mathcal{C}_j$. Since the limit as $k\to\infty$ exists it is easy to see that $|DF_j|\leq|DF_{j-1}|$ there. Hence

$$\begin{aligned}\int_{\mathcal{C}_j}\big(|DF_j|+|DF_{j-1}|\big)&\leq C\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\int_{\mathcal{C}_j\cap\bigcap_{i=1}^{j-1}G_{k_i,i}}|DF_{j-1}|\\ &\leq\sum_{k_1,\dots,k_{j-1}=1}^{\infty}\mathcal{L}_3\big(\bigcap_{i=1}^{j-1}G_{k_i,i}\big)\|A_{k_1,k_2,k_3}\|\cdots\|A_{k_{j-2},k_{j-1}}\|\leq C\Big(\frac{1}{2}\Big)^{\frac{j-3}{6}}.\end{aligned}$$

It follows that

$$\sum_{j=1}^{\infty}\int_{Q_0}|D(F_j-F_{j-1})|<\infty$$

and thus DF_j forms a Cauchy sequence in L^1 and $f\in W^{1,1}$.

From (8.4) we know that

$$\mathcal{L}_3(\mathcal{C}_j)\geq\frac{1}{\Delta}\mathcal{L}_3\Big(Q_0\setminus\bigcup_{i=1}^{j-1}\mathcal{C}_i\Big)\prod_{i=1}^{\infty}s_i^2$$

for each $j\in\bigcup_{l\in\mathbb{N}}\{6l+1,6l+2,6l+3\}$. For $j\in\bigcup_{l\in\mathbb{N}}\{6l+4,6l+5,6l+6\}$ we can easily deduce from area formula for F_j^{-1} that $\mathcal{L}_3(\mathcal{C}_j)=0$ since $J_{F_j^{-1}}=0$ a.e. on $F_j(\mathcal{C}_j)$. Since $\prod_{i=1}^{\infty}s_i^2>0$ we easily obtain

$$\mathcal{L}_3\Big(\bigcup_{j=1}^{\infty}\mathcal{C}_j\Big)=\mathcal{L}_3(Q_0).$$

Together with $J_{F_j}=0$ on C_j for each $j\in\bigcup_{l\in\mathbb{N}}\{6l+1,6l+2,6l+3\}$ and $F_k=F_j$ on $\mathcal{C}_j$ for each $k>j$ this implies that $J_f=0$ almost everywhere on Q_0. Analogously we will deduce that $J_{f^{-1}}=0$ a.e. on Q_0.

It remains to show that DF_j^{-1} is Cauchy in L^1. For simplicity we again assume that $j=6l+3$. Since $F_j=F_{j-1}$ on $\bigcup_{i=1}^{j-1}\mathcal{C}_i$ and $\mathcal{L}_3(F_j(\mathcal{C}_j))=0$ we obtain

$$\int_{Q_0}|D(F_j^{-1}-F_{j-1}^{-1})|\leq\sum_{k_j=1}^{\infty}\int_{F_j(G_{k_j,j})}\big(|DF_j^{-1}|+|DF_{j-1}^{-1}|\big)\ .$$

Analogously to (8.11) and (9.1) we may estimate

$$\begin{aligned}\sum_{k_j=1}^{\infty}\int_{F_j(G_{k_j,j})}\big(|DF_j^{-1}|+|DF_{j-1}^{-1}|\big)&\leq\sum_{k_1,\dots,k_j=1}^{\infty}\int_{F_j(\bigcap_{i=1}^{j}G_{k_i,i})}\big(|DF_j^{-1}|+|DF_{j-1}^{-1}|\big)\\ &\leq C\sum_{k_1,\dots,k_j=1}^{\infty}\frac{C_1k_1k_2k_3}{k_1^3k_2^3k_3^3}\frac{C_1t^2}{tk_4^2tk_5^2tk_6^2}\cdots\frac{Ck_{j-1}k_{j-2}k_j}{k_{j-2}^3k_{j-1}^3k_j^2}\\ &\leq C\Big(C_1C_2\big(\frac{\pi^2}{6}\big)^6t^{-1}\Big)^{\frac{j-3}{6}}C\leq C\Big(\frac{1}{2}\Big)^{\frac{j-3}{6}}.\end{aligned}$$

It follows that
$$\sum_{j=1}^{\infty}\int_{Q_0}|D(F_j^{-1}-F_{j-1}^{-1})|<\infty$$
and thus DF_j^{-1} forms a Cauchy sequence in L^1 and $f^{-1}\in W^{1,1}$.

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References

[1] R. Černý, Homeomorphism with zero Jacobian: Sharp integrability of the derivative, *J. Math. Anal. Appl* **373** (2011), 161-174.
[2] L. D'Onofrio and R. Schiattarella, *On the total variation for the inverse of a BV-homeomorphism*, to appear in Adv. Calc. Var.
[3] P. di Gironimo, L. D'Onofrio, C. Sbordone and R. Schiattarella, Anisotropic Sobolev homeomorhisms, *Ann. Acad. Sci. Fenn. Math.* **36** (2011), 593-602.
[4] N. Fusco,G. Moscariello and C. Sbordone, The limit of $W^{1,1}$ homeomorphisms with finite distortion, *Calc. Var.* **33** (2008), 377-390.
[5] P. Hajl'asz, Change of variables formula under minimal assumptions, *Colloq. Math.* **64** (1993), 93–101.
[6] S. Hencl, Sobolev homeomorphism with zero Jacobian almost everywhere, *J. Math. Pures Appl.* **95** (2011), 444-458.
[7] S. Hencl, Sharpness of the assumptions for the regularity of a homeomorphism, *Michigan Math. J.* **59** (2010), 667-678.
[8] S. Hencl and P. Koskela, *Lecture notes on Mappings of finite distortion*, in preparation.
[9] S. Hencl and J. Malý, Jacobians of Sobolev homeomorphisms, *Calc. Var. Partial Differential Equations* **38** (2010), 233-242.
[10] S. Hencl, G. Moscariello, A. Passarelli di Napoli and C. Sbordone, Bi-Sobolev mappings and elliptic equations in the plane, *J. Math. Anal. Appl.* **355** (2009), 22-32.
[11] T. Iwaniec and G. Martin, *Geometric function theory and nonlinear analysis*, Oxford Mathematical Monographs, Clarendon Press, Oxford 2001.
[12] J. Kauhanen, An example concerning the zero set of the Jacobian, *J. Math. Anal. Appl.* **315** (2006), 656-665.
[13] J. Kauhanen, P. Koskela and J. Malý, Mappings of finite distortion: Condition N, *Michigan Math. J.* **49** (2001), 169–181.
[14] Kleprlík L., The zero set of the Jacobian and composition of mappings, *J. Math. Anal. Appl.* **386** (2012), 870–881.
[15] P. Koskela and J. Malý, Mappings of finite distortion: the zero set of the Jacobian, *J. Eur. Math. Soc.* **5** (2003), 95–105.
[16] J. Malý and O. Martio, Lusin's condition (N) and mappings of the class $W^{1,n}$, *J. Reine Angew. Math.* **458** (1995), 19–36.
[17] S. Ponomarev, Examples of homeomorphisms in the class $ACTL^p$ which do not satisfy the absolute continuity condition of Banach (Russian), *Dokl. Akad. Nauk USSR* **201** (1971), 1053-1054.

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