A LIPSCHITZ FUNCTION WHICH IS C^∞ ON A.E. LINE NEED NOT BE GENERICALLY DIFFERENTIABLE

LUDĚK ZAJÍČEK

Abstract. We construct a Lipschitz function f on $X = \mathbb{R}^2$ such that, for each $0 \neq v \in X$, the function f is C^∞ smooth on a.e. lines parallel to v and f is Gâteaux non-differentiable at all points of X except a first category set. Consequently, the same holds if X ($\dim X > 1$) is an arbitrary Banach space and "a.e." has any usual "measure sense". This example gives an answer to a natural question concerning a recent author's study of linearly essentially smooth functions (which generalize essentially smooth functions of Borwein and Moors).

1. Introduction

There exists a number of results which assert that some "partial or directional smoothness property" (e.g., smoothness on some lines or directional differentiability in some directions) of a function f on a Banach space X implies some "global smoothness property" (e.g. Gâteaux or Fréchet differentiability at many points). For results of this sort see e.g. [6], [13], [5], [12].

The present note is motivated by the special question whether a "smoothness on many lines" of a Lipschitz function f on X implies generic Fréchet differentiability of f (where "generic" has the usual meaning "at all points except a first category set").

A remarkable result in this direction ([13]) says that if an (a priori arbitrary) function f on $X = \mathbb{R}^n$ has all partial directional derivatives at all points (in other words, f is differentiable on each line parallel to a coordinate axis), then f is generically Fréchet differentiable. On the other hand, if $X = \ell_2$, then (see [9]) there exists a Lipschitz function on X which is everywhere Gâteaux differentiable (and so differentiable on all lines) which is generically Fréchet non-differentiable.

A contribution to this special question is given in the article [15] which was motivated by the papers [2], [3] of Borwein and Moors on "essentially smooth" functions.

For example [15, Theorem 5.2] reads as follows.

Theorem A. *Let X be an Asplund space and $f : X \to \mathbb{R}$ a Lipschitz function. Suppose that there exists a set D which is dense in the unit sphere S_X such that, for each $v \in D$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable.*

Here "f is essentially smooth on the line L" means "the restriction of f is a.e. strictly differentiable on L." So each function which is C^1 on a line L is essentially smooth on L. (Recall also that X is Asplund if and only if Y^* is separable for each separable subspace $Y \subset X$.)

In [15, Remark 1.4(iii)], it was announced that, in Theorem A, it is not possible to suppose only that f is essentially smooth on each line from a set of lines which is dense in the space of all lines parallel to $v \in D$. (So it is not sufficient to suppose that f is essentially smooth on each line from a set of lines which is dense in the space of all lines; cf. Remark 3.7).

The main aim of the present note is to construct the following much stronger example (Theorem 3.6 below), in which we obtain even generic *Gâteaux* non-differentiability.

1991 *Mathematics Subject Classification.* Primary 46G05; Secondary 26B05.

Key words and phrases. Gâteaux differentiability, essentially smooth functions, Lipschitz function, functions C^∞ on a.e. lines.

The research was supported by the grant GAČR P201/12/0436.

Let X be a Banach space, $\dim X > 1$. Then there exists a Lipschitz function f on X such that, for each $v \in S_X$, f is C^∞ on a.e. line parallel to v and f is generically Gâteaux non-differentiable.

Here "a.e. line parallel to v" is taken in a very strong sense (using "$*$-nullness", see Definition 3.5). Note that each $*$-null set is clearly Lebesgue null if $X = \mathbb{R}^n$ and is Gaussian (=Aronszajn) null and also Γ-null if X is separable.

Stress that our construction is "two-dimensional"; if we have an example in $\mathbb{R}^2$, then the construction in a general X is rather obvious. The notion of $*$-nullness is not of general interest, we introduce it only to be able shortly formulate our result in general X.

Further note that in the case $X = \mathbb{R}^n$ the function f from our example is C^∞ on a.e. line in X, which justifies the title of the note. It is immediately seen from the canonical definition of the measure on the set of all lines in $\mathbb{R}^n$ (see [8, p. 53]).

Note also that the main idea of the construction is similar to that of [11].

2. Preliminaries

In the following, if it is not said otherwise, X will be a real Banach space. We set $S_X := \{x \in X : \|x\| = 1\}$. If $a, b \in X$, then $\overline{a,b}$ denotes the closed segment. By $\operatorname{span} M$ we denote the linear span of $M \subset X$. The equality $X = X_1 \oplus \cdots \oplus X_n$ means that X is the direct sum of non-trivial closed linear subspaces $X_1, \dots, X_n$ and the corresponding projections $\pi_i : X \to X_i$ are continuous.

We say that a function $f : X \to \mathbb{R}$ is C^∞ on a line $L = a + \mathbb{R}v$ if the function $h(t) := f(a+tv)$ is C^∞ on $\mathbb{R}$. (Clearly, this definition does not depend on the choice of a and v.)

The symbol $B(x,r)$ will denote the open ball with center x and radius r. The word *"generically"* has the usual sense; it means "at all points except a first category set".

The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

We will need several times the following easy well-known fact.

Lemma 2.1. *Let X be a Banach space, $0 \neq u \in X$, and let $X = W \oplus \operatorname{span}\{u\}$. Then the mapping $w \in W \mapsto w + \mathbb{R}u \in X/\operatorname{span}\{u\}$ is a linear homeomorphism.*

In the following, f is a real function defined on an open subset G of X.

We say that f has a property generically on G, if f has this property at each point of G except a first category set.

We say that f is K-Lipschitz ($K \geq 0$), if f is Lipschitz with (not necessary least) constant K.

Recall the well-known easy fact that

(2.1) if f is Lipschitz and $\dim X < \infty$, then the Gâteaux and Fréchet derivatives coincide.

Recall (see [10]) that $x^* \in X^*$ is called a strict derivative of f at $a \in G$ if

$$\lim_{(x,y)\to(a,a),\ x\neq y} \frac{f(y) - f(x) - x^*(y-x)}{\|y-x\|} = 0.$$

It is well-known and easy to see that if $f'(a)$ is the strict derivative of f at $a \in X$ and $v \in X$, then

$$\lim_{n\to\infty} \frac{f(a_n + t_n v) - f(a_n)}{t_n} = f'(a)(v) \quad \text{whenever} \quad a_n \to a,\ t_n \to 0+\,. \tag{2.2}$$

Strict differentiability is a stronger condition than Fréchet differentiability, but (see e.g. [14, Theorem B, p. 476]), for an arbitrary f,

(2.3) generically Fréchet differentiability of f implies strict differentiability of f.

The directional and one-sided directional derivatives of f at x in the direction v are defined respectively by

$$f'(x,v) := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

We will need some well-known facts about mollification of functions. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be the function defined as $\eta(x) = 0$ for $\|x\| \geq 1$ and $\eta(x) = c\,\exp((\|x\|^2-1)^{-1})$ for $\|x\| < 1$, where c is such that $\int_{\mathbb{R}^n} \eta = 1$. For $\delta > 0$, we define (the standard mollifier, see [4])

$$\eta_\delta(x) = \frac{1}{\delta^n}\,\eta\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^n.$$

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$f^\delta(x) := \eta_\delta * f(x) = \int_{\mathbb{R}^n} \eta_\delta(x-y)\,f(y)\,dy = \int_{\mathbb{R}^n} \eta_\delta(y)\,f(x-y)\,dy, \quad x \in R^n.$$

We will need the following well-known facts.

Fact 2.2. Let f be a K-Lipschitz function on $\mathbb{R}^n$ and $\delta > 0$. Then

(i) $f^\delta \in C^\infty(\mathbb{R}^n)$.
(ii) $f^\delta \to f$ (when $\delta \to 0+$) uniformly on compact subsets of $\mathbb{R}^n$.
(iii) f^δ is K-Lipschitz.
(iv) If $x \in \mathbb{R}^n$, $\delta > 0$, and f equals to an affine function α on $B(x,\delta)$, then $f^\delta(x) = \alpha(x)$.

For (i) and (ii) see [4, Theorem 1(i),(ii); p. 123]; (iii) and (iv) are also well-known and almost obvious. So I omit their proof, although I did not found an explicit reference.

3. Main result

Lemma 3.1. *Let $K \geq 4$ and let $f \in C^\infty(\mathbb{R}^2)$ be a K-Lipschitz function. Let $\emptyset \neq H \subset \mathbb{R}^2$ be an open set and $0 < \varepsilon < 1$. Then there exist $\tilde{f} \in C^\infty(\mathbb{R}^2)$, $c \in H$ and $t > 0$ with the following properties:*

(i) *$f(x) = \tilde{f}(x)$ for each $x \in \mathbb{R}^2 \setminus H$.*
(ii) *$|f(x) - \tilde{f}(x)| < \varepsilon$ for each $x \in \mathbb{R}^2$.*
(iii) *$\tilde{f}$ is a $(K+\varepsilon)$-Lipschitz function.*
(iv) *The points c, $c+te_1$ and $c-te_1$ (where $e_1 := (1,0)$) belong to H,*

$$\frac{\tilde{f}(c+te_1)-\tilde{f}(c)}{t} \geq 1 \quad \text{and} \quad \frac{\tilde{f}(c)-\tilde{f}(c-te_1)}{t} \leq -1. \tag{3.1}$$

Proof. Choose $c \in H$ and consider the affine function $\alpha(x) := f(c) + f'(c)(x-c)$, $x \in \mathbb{R}^2$. Since $f \in C^1(\mathbb{R}^2)$, we can clearly choose $r > 0$ such that

$$0 < r < 1, \quad B(c,r) \subset H \quad \text{and} \tag{3.2}$$

$$\text{the function } \ f-\alpha \ \text{ is } (\varepsilon/2)\text{-Lipschitz on } B(c,r). \tag{3.3}$$

Observe that $\|f'(c)\| \leq K$ and so α is a K-Lipschitz function. Set

$$\varphi(x) := \alpha(c) - \frac{\varepsilon^2 r}{8K^2} + (K+\varepsilon/2)\,\|x-c\|, \quad \text{and} \quad g(x) := \min(\varphi(x),\alpha(x)), \quad x \in \mathbb{R}^2.$$

We will need the following properties of the function g:

(P1) g is $(K+\varepsilon/2)$-Lipschitz.
(P2) $g(x) = \alpha(x)$ for each $x \in \mathbb{R}^2 \setminus B(c,r/4)$.
(P3) $|g(x)-\alpha(x)| < \varepsilon r/K$ for each $x \in \mathbb{R}^2$.
(P4) There exists $t > 0$ such that $c \pm te_1 \in B(c,r)$,

$$\frac{g(c+te_1)-g(c)}{t} = K+\varepsilon/2 \quad \text{and} \quad \frac{g(c)-g(c-te_1)}{t} = -(K+\varepsilon/2). \tag{3.4}$$

To prove these properties, first recall that α is K-Lipschitz and since φ is clearly $(K+\varepsilon/2)$-Lipschitz, we obtain (P1).

If $\|x-c\| \geq \varepsilon r/(4K^2)$, we obtain

$$\alpha(x) \leq \alpha(c) + K\|x-c\| = \varphi(x) + \frac{\varepsilon^2 r}{8K^2} - (\varepsilon/2)\|x-c\| \leq \varphi(x) + \frac{\varepsilon^2 r}{8K^2} - (\varepsilon/2)(\varepsilon r/(4K^2)) = \varphi(x)$$

and (P2) follows since $\varepsilon r/(4K^2) < r/4$.

If $\|x-c\| < \varepsilon r/(4K^2)$, then $|\alpha(x)-\alpha(c)| < K(\varepsilon r/(4K^2))$ and $|\varphi(x)-\varphi(c)| < (K+\varepsilon/2)(\varepsilon r/(4K^2))$. Consequently,

$$\begin{aligned} |g(x)-\alpha(x)| \leq |\varphi(x)-\alpha(x)| &\leq |\alpha(c)-\varphi(c)| + |\alpha(x)-\alpha(c)| + |\varphi(x)-\varphi(c)| \\ &\leq \frac{\varepsilon^2 r}{8K^2} + K(\varepsilon r/(4K^2)) + (K+\varepsilon/2)\varepsilon r/(4K^2)) < \varepsilon r/K \end{aligned}$$

which gives (P3), since we have proved that $g(x) = \alpha(x)$ if $\|x-c\| \geq \varepsilon r/(4K^2)$.

Since α and φ are continuous, we can clearly choose $t > 0$ so small that $c \pm te_1 \in B(c,r)$, $\varphi(c+te_1) < \alpha(c+te_1)$ and $\varphi(c-te_1) < \alpha(c-te_1)$. Then $g(c) = \varphi(c)$ and $g(c \pm te_1) = \varphi(c \pm te_1)$ and so, by the definition of φ, we clearly obtain (3.4). Thus we have proved (P4).

Now, for $\delta > 0$, consider the mollification g^δ of g. By Fact 2.2(i),(iii), we obtain that $g^\delta \in C^\infty(\mathbb{R}^2)$ and g^δ is $(K+\varepsilon/2)$-Lipschitz.

Using (P2) and Fact 2.2(iv) we obtain that, if $0 < \delta < r/4$, then

$$g^\delta(x) = g(x) = \alpha(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus B(c, r/2). \tag{3.5}$$

So, using Fact 2.2(ii) for the compact set $\overline{B(c,r)}$, we easily see that we can choose $\delta \in (0, r/4)$ so small that

$$|g^\delta(x) - g(x)| < \frac{\varepsilon r}{K} \text{ for each } x \in \mathbb{R}^2 \tag{3.6}$$

and, using (3.4), also

$$\frac{g^\delta(c+te_1) - g^\delta(c)}{t} \geq 2 \quad \text{and} \quad \frac{g^\delta(c) - g^\delta(c-te_1)}{t} \leq -2. \tag{3.7}$$

By (3.6) and (P3) we obtain that

$$|g^\delta(x) - \alpha(x)| < \frac{2\varepsilon r}{K} \quad \text{for each} \quad x \in \mathbb{R}^2. \tag{3.8}$$

Define $\tilde{f} := f + g^\delta - \alpha$. Clearly $\tilde{f} \in C^\infty(\mathbb{R}^2)$. We will show that $\tilde{f}$ has also properties (i) - (iv).

By (3.5) we have

$$\tilde{f}(x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus B(c, r/2), \tag{3.9}$$

which implies (i).

By (3.8) we obtain

$$|\tilde{f}(x) - f(x)| < \frac{2\varepsilon r}{K} < \varepsilon \quad \text{for each} \quad x \in \mathbb{R}^2, \tag{3.10}$$

so (ii) holds.

Since $\tilde{f} := (f-\alpha) + g^\delta$, g^δ is $(K+\varepsilon/2)$-Lipschitz and $f - \alpha$ is $(\varepsilon/2)$-Lipschitz on $B(c,r)$ (see (3.3)), we obtain that

$$\tilde{f} \text{ is a } (K+\varepsilon)\text{-Lipschitz function on } B(c,r). \tag{3.11}$$

Using (3.5) we obtain that

$$\tilde{f} = f + (g^\delta - \alpha) \text{ is } K\text{-Lipschitz on } \mathbb{R}^2 \setminus B(c, r/2). \tag{3.12}$$

Further consider arbitrary $x_1, x_2 \in \mathbb{R}^2$ such that $x_1 \in B(c, r/2)$ and $x_2 \notin B(c,r)$. Then, using (3.9) and (3.10), we obtain

$$\begin{aligned}|\tilde{f}(x_2) - \tilde{f}(x_1)| &= |f(x_2) - \tilde{f}(x_1)| \le |f(x_2) - f(x_1)| + |f(x_1) - \tilde{f}(x_1)| \\ &\le K|x_2 - x_1| + \frac{2\varepsilon r}{K} \le K|x_2 - x_1| + (4\varepsilon/K)|x_2 - x_1| \le (K+\varepsilon)|x_2 - x_1|.\end{aligned}$$

This inequality together with (3.11) and (3.12) clearly imply (iii).

Finally, since $\tilde{f} := (f - \alpha) + g^\delta$, (3.3), (3.7) and the fact that the points c, $c + te_1$, $c - te_1$ belong to $B(c,r)$ easily imply (iv). □

Lemma 3.2. *Let $M_n \subset \mathbb{R}^2$, $n \in \mathbb{N}$, be nowhere dense sets. Then there exists a Lipschitz function f on $\mathbb{R}^2$ such that*

(a) *f is C^∞ on each line which is contained in a set M_n, $n \in \mathbb{N}$, and*
(b) *f is generically Gâteaux non-differentiable.*

Proof. We can clearly choose a set $D = \{d_n : n \in \mathbb{N}\}$ which is dense in $\mathbb{R}^2$ and $D \cap \bigcup_{k\in\mathbb{N}} \overline{M_k} = \emptyset$. For each $n \in \mathbb{N}$, choose $0 < r_n < 1/n$ such that $B(d_n, r_n) \cap \bigcup_{k=1}^n M_k = \emptyset$ and denote $B_n := B(d_n, r_n)$. Set $\varepsilon_n := 2^{-n}$ and $e_1 := (1,0)$.

Now we will inductively construct sequences $(c_n)_{n=1}^\infty$ of points in $\mathbb{R}^2$, $(f_n)_{n=0}^\infty$ of C^∞ functions on $\mathbb{R}^2$ and $(t_n)_{n=1}^\infty$ of positive reals such that $f_0(x) = 0$, $x \in X$, and for each $n \in \mathbb{N}$ the following hold:

(i) $\{c_n, c_n + t_n e_1, c_n - t_n e_1\} \subset B_n$.
(ii) $\frac{f_n(c_n + t_n e_1) - f_n(c_n)}{t_n} \ge 1$ and $\frac{f_n(c_n) - f_n(c_n - t_n e_1)}{t_n} \le -1$.
(iii) $f_n(x) = f_{n-1}(x)$ whenever $x \in (\mathbb{R}^2 \setminus B_n) \cup \bigcup_{k=1}^{n-1} \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$.
(iv) $|f_n(x) - f_{n-1}(x)| < \varepsilon_n$ for each $x \in \mathbb{R}^2$.
(v) f_n is a $(4 + \sum_{k=1}^n \varepsilon_k)$-Lipschitz function.

Of course, we put $\bigcup_{k=1}^0 \{c_k, c_k + t_k e_1, c_k - t_k e_1\} := \emptyset$ (and also $\sum_{k=1}^0 \varepsilon_k := 0$ below).

We set $f_0(x) := 0$, $x \in X$. Further suppose that $m \in \mathbb{N}$ is given, c_n, f_n, t_n are defined for $1 \le n < m$, and the conditions (i)-(v) hold whenever $1 \le n < m$.

Applying Lemma 3.1 to $K := 4 + \sum_{k=1}^{m-1} \varepsilon_k$, $f := f_{m-1}$, $H := B_m \setminus \bigcup_{k=1}^{m-1} \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$ and $\varepsilon := \varepsilon_m$, we obtain a function $\tilde{f} =: f_m$, $c =: c_m \in H$ and $t =: t_m > 0$ such that the conditions (i)-(v) clearly hold for $n = m$.

Condition (iv) gives that the series

$$f_1 + (f_2 - f_1) + (f_3 - f_2) + \dots$$

(uniformly) converges on $\mathbb{R}^2$ and consequently the sequence (f_n) converges to a function f. Since all f_n are 5-Lipschitz by (v), we obtain that f is a 5-Lipschitz function too.

To prove (a), suppose that L is a line in $\mathbb{R}^2$, $k \in \mathbb{N}$ and $L \subset M_k$. Since $M_k \subset \mathbb{R}^2 \setminus B_n$ for each $n \ge k$, we obtain by (iii) that $f_n(x) = f_{n-1}(x)$ for each $x \in L$ and $n \ge k$, and consequently $f(x) = f_k(x)$, $x \in L$. Since f_k is C^∞ on $\mathbb{R}^2$, we obtain that f is C^∞ on L.

To prove (b), first observe that, by (iii), for each $n > k$ and $x \in \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$ we have $f_n(x) = f_{n-1}(x)$, and so $f(x) = f_k(x)$. Thus (ii) implies that, for each $k \in \mathbb{N}$,

$$\text{(3.13)} \qquad \frac{f(c_k + t_k e_1) - f(c_k)}{t_k} \ge 1 \quad \text{and} \quad \frac{f(c_k) - f(c_k - t_k e_1)}{t_k} \le -1.$$

This easily implies that

(3.14) f is not strictly differentiable at each point of $\mathbb{R}^2$.

Indeed, suppose to the contrary that f is strictly differentiable at a point $x \in \mathbb{R}^2$. Using (i), we can clearly find a subsequence (c_{n_i}) of (c_n) with $c_{n_i} \to x$. Then clearly $t_{n_i} \to 0$ and so by (2.2) and (3.13)

$$\lim_{i\to\infty} \frac{f(c_{n_i} + t_{n_i} e_1) - f(c_{n_i})}{t_{n_i}} = f'(x)(e_1) \geq 1 \quad \text{and}$$

$$\lim_{i\to\infty} \frac{f(c_{n_i}) - f(c_{n_i} - t_{n_i} e_1)}{t_{n_i}} = f'(x)(e_1) \leq -1,$$

which is a contradiction. By (3.14), (2.3) and (2.1) we obtain (b). □

Proposition 3.3. *There exists a Lipschitz function f on $\mathbb{R}^2$ such that*

(a) *for each $0 \neq v \in \mathbb{R}^2$, f is C^∞ on a.e. line parallel to v, and*
(b) *f is generically Gâteaux non-differentiable.*

Proof. Choose a set $\{d_k : k \in \mathbb{N}\}$ dense in $\mathbb{R}^2$. For each $n, k \in \mathbb{N}$, set

$$B_{n,k} := B(d_k, (2^k n)^{-1}) \quad \text{and} \quad G_n := \bigcup_{k=1}^{\infty} B_{n,k}. \tag{3.15}$$

Then each G_n is clearly open dense, and consequently $M_n := \mathbb{R}^2 \setminus G_n$ is nowhere dense. Applying Lemma 3.2, we obtain a Lipschitz function f on $\mathbb{R}^2$ such that f is generically Gâteaux non-differentiable and

(3.16) f is C^∞ on each line which is contained in a set M_n, $n \in \mathbb{N}$.

Fix an arbitrary $0 \neq v \in \mathbb{R}^2$. Let W be the orthogonal complement of span$\{v\}$ and let π be the orthogonal projection on W. Then $\pi(G_n) = \bigcup_{k=1}^{\infty} \pi(B_{n,k})$ and so

$$\mathcal{H}^1(\pi(G_n)) \leq \sum_{k=1}^{\infty} \mathcal{H}^1(\pi(B_{n,k})) = \sum_{k=1}^{\infty} 2(2^k n)^{-1} = \frac{2}{n}.$$

Consequently

$$\mathcal{H}^1\left(\bigcap_{n=1}^{\infty} \pi(G_n)\right) = 0. \tag{3.17}$$

Let now $w \in W \setminus \bigcap_{n=1}^{\infty} \pi(G_n)$. Then there exists n with $w \notin \pi(G_n)$ and so the line which contains w and is parallel to v is contained in M_n. So, by (3.16) and (3.17), f is C^∞ on a.e. line parallel to v. □

Remark 3.4. The assertion of Proposition 3.3 can be easily strengthened; namely we can consider "a.e." with respect to any generalized Hausdorff measure Λ_h given by a non-decreasing $h : [0, \infty) \to [0, \infty)$, see [8, p. 60]. Indeed, it is easy to slightly refine the proof of Proposition 3.3. Namely, it is sufficient to make two changes:

(a) to set $B_{n,k} := B(d_k, r_{n,k})$, where $r_{n,k} > 0$ are so small that $\sum_{k=1}^{\infty} h(2r_{n,k}) < 1/n$;
(b) in the proof of $\Lambda_h(\bigcap_{n=1}^{\infty} \pi(G_n)) = 0$ to use the definition of Λ_h (instead of the subadditivity of $\mathcal{H}^1$).

To apply Proposition 3.3 in infinite dimensional spaces, we found useful to introduce the following terminology.

Definition 3.5. Let X be a Banach space with $\dim X > 1$. We say that $M \subset X$ is $*$-*null* if there exists $0 \neq x^* \in X^*$ such that $x^*(M) \subset \mathbb{R}$ is Lebesgue null.

Obviously, if $X = \mathbb{R}^n$, then each $*$-null set in X is Lebesgue null. If X is an infinite dimensional separable space, then each $*$-null set M in X is contained in an Aronszajn null (= Gauss null) and is also Γ-null. It can be proved directly from definitions, but we can use also the following standard quicker argument:

Let x^* be as in Definition 3.5 and let h be a Lipschitz function on $\mathbb{R}$ which is differentiable at no point of $x^*(M)$ (see [1, p. 165]). Then $f := h \circ x^*$ is clearly a Lipschitz function on X

which is Gâteaux differentiable at no point of M. So our assertions follows from [1, Theorem 6.42] and [7, Theorem 5.2.3].

Note also that if X is non-separable then it is easy to see that each $*$-null set $M \subset X$ is Haar null. Moreover, using [7, Corollary 5.6.2], it is not difficult to prove that M is Γ-null.

Theorem 3.6. *Let X be a Banach space and $\dim X \geq 2$. Then there exists a Lipschitz function f on X such that*

(i) *for each $0 \neq v \in X$, the function f is C^∞ on $*$-a.e. lines parallel to v and*
(ii) *f is generically Gâteaux non-differentiable.*

(Of course, condition (i) is a short expression of the statement that there exists a $*$-null set N in $X/\operatorname{span}\{v\}$ such that f is C^∞ on each line $L \in X/\operatorname{span}\{v\} \setminus N$.)

Proof. If $\dim X = 2$, then the assertion clearly follows from Proposition 3.3.

So suppose $\dim X \geq 3$. Write $X = P \oplus Y$ with $\dim P = 2$. By Proposition 3.3 choose a Lipschitz function g on P and a first category set $A \subset P$ such that g is Gâteaux non-differentiable at all points of $P \setminus A$ and, for each $0 \neq u \in P$, the function g is C^∞ on a.e. line parallel to u. Let $\pi : X \to P$ be the linear projection of X on P in the direction of V. Set $f := g \circ \pi$.

It is easy to see that f is a Lipschitz function which is Gâteaux non-differentiable at all points outside the (first category) set $\pi^{-1}(A)$. So (ii) holds.

To prove (i), consider an arbitrary $0 \neq v \in X$. If $u := \pi(v) = 0$, then f is clearly constant on each line parallel to v. So suppose $u \neq 0$. Then we can write $P = \operatorname{span}\{u\} \oplus Z$ with $\dim Z = 1$. Let $\varphi : Z \to \mathbb{R}$ be a linear homeomorphism. By the choice of g and Lemma 2.1 there exists $N \subset Z$ such that $\varphi(N) \subset \mathbb{R}$ is Lebesgue null and the function $h(t) := g(d + tu)$, $t \in \mathbb{R}$, is C^∞ for each $d \in Z \setminus N$.

Observe that $N+Y$ is $*$-null in $Z+Y$. Indeed, for $\psi := \varphi \circ (\pi \restriction_{Z+Y})$ we have $0 \neq \psi \in (Z+Y)^*$ and $\psi(N+Y) = \varphi(N)$ is Lebesgue null. Now let $p \in (Z+Y) \setminus (N+Y)$. Then we can write $p = d+y$, where $d \in Z \setminus N$ and $y \in Y$. Observing that $f(p+tv) = f(d+y+tv) = g(d+tu) = h(t)$ and using Lemma 2.1, we easily obtain (i). □

Remark 3.7. Each set containing $*$-a.e. line parallel to v is clearly dense in the space $X/\operatorname{span}\{v\}$.

Consequently the function f from Theorem 3.6 is C^∞ on a dense set of lines in the space $\mathcal{L}$ of all lines in X. Here we consider the topology on $\mathcal{L}$ in which, for a line $L = a_0 + \mathbb{R}v_0$,

$$\mathcal{B}_L := \{\{a + \mathbb{R}v : \ \|a - a_0\| < \varepsilon, \|v - v_0\| < \varepsilon\} : \ \varepsilon > 0\}$$

is a basis of the filter of all neigbourhoods of L.

References

[1] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Vol. 1,* Colloquium Publications **48**, American Mathematical Society, Providence, 2000.
[2] J.M. Borwein, W.B. Moors, Essentially smooth Lipschitz functions, J. Funct. Anal. 149 (1997) 305–351.
[3] J. Borwein, W.B. Moors, Null sets and essentially smooth Lipschitz functions, SIAM J. Optim. 8 (1998) 309–323.
[4] L. Evans, R. Gariepy, *Measure theory and fine properties of functions,* Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.
[5] D.D. Ilmuradov, On differential properties of real functions, Ukrainian Math. J. 46 (1994), 922-928 (translated from Ukrain. Mat. Zh. 46 (1994), 842–848).
[6] M. Jarnicki, P. Pflug, Directional regularity vs. joint regularity, Notices Amer. Math. Soc. 58 (2011), 896-904.
[7] J. Lindenstrauss, D. Preiss, J. Tišer, *Fréchet differentiability of Lipschitz maps and porous sets in Banach spaces*, Princeton University Press, Princeton 2012.
[8] P. Mattila, *Geometry of sets and measures in Euclidean spaces, Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.
[9] D. Preiss, Gâteaux differentiable Lipschitz functions need not be Fréchet differentiable on a residual subset, Proceedings of the 10th Winter School on Abstract Analysis (Srní, 1982), Rend. Circ. Mat. Palermo (2) (1982) 217–222.

[10] B.S. Mordukhovich, *Variational analysis and generalized differentiation I, Basic theory*, Grundlehren der Mathematischen Wissenschaften 330, Springer-Verlag, Berlin, 2006.
[11] D. Pokorný, The approximate and the Clarke subdifferentials can be different everywhere, J. Math. Anal. Appl. 347 (2008), 652–658.
[12] D. Preiss, L. Zajíček, Directional derivatives of Lipschitz functions, Israel J. Math. 125 (2001), 1-27.
[13] J. Saint-Raymond, Sur les fonctions munies de dérivées partielles, Bull. Sci. Math. (2) 103 (1979), 375-378.
[14] L. Zajíček, Fréchet differentiability, strict differentiability and subdifferentiability, Czechoslovak Math. J. 41(116) (1991) 471–489.
[15] L. Zajíček, Generic Fréchet differentiability on Asplund spaces via a.e. strict differentiability on many lines, J. Convex Anal. 19 (2012), 23–48.

E-mail address: `zajicek@karlin.mff.cuni.cz`

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8-Karlín, Czech Republic