SOBOLEV EMBEDDING THEOREM FOR IRREGULAR DOMAINS AND DISCONTINUITY OF $p \to p^*(p, n)$

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Abstract. For a domain $\Omega \subset \mathbb{R}^n$ we denote

$$q_\Omega(p) := \sup\big\{r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega))\big\}.$$

Let $p_0 \in [2,\infty)$. We construct a domain $\Omega \subset \mathbb{R}^2$ such that $q_\Omega(p)$ is discontinuous at p_0.

1. Introduction

We study the Sobolev embedding theorem on domains with non-Lipschitz boundary. The Sobolev embedding theorem on a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary claims

$$f \in W^{1,p}(\Omega),\ p \neq n \Rightarrow f \in L^{p^*(p)}(\Omega),\ \text{where}$$

$$p^*(p) = \begin{cases} \frac{np}{n-p}, \text{ for } 1 \leq p < n, \\ \infty, \text{ for } n < p < \infty. \end{cases} \tag{1.1}$$

Inspired by this theorem, we can define the function of the optimal embedding for a domain $\Omega \subset \mathbb{R}^n$ as

$$q_\Omega(p) := \sup\big\{r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega))\big\}. \tag{1.2}$$

There are a lot of results on the field of characterization of $q_\Omega(p)$ for classes of domains. For a Lipschitz domain Ω the function $p^*(p) = q_\Omega(p)$ is continuous and even smooth, (see (1.1)), this was proven by Sobolev in 1938 [12]. Later, the embedding was examined on some more problematic classes of domains by V. G. Maz'ya [9, 10], O. V. Besov and V. P. Il'in [3], T. Kilpeläinen and J. Malý [5], D. A. Labutin [6, 7], B. V. Trushin [13, 14] and others. For further results and motivation we recommend the introduction by O. V. Besov [2]. Even considering somehow irregular domains, examined classes of domains have always $q_\Omega(p)$ somehow nice and continuous. We construct a domain Ω such that the function of the optimal embedding $q_\Omega(p)$ is continuous up to some point, has a leap at this point and then it is continuous again. The point of discontinuity $p_0 \in [n,\infty)$ and the size of the leap can be chosen as desired.

Our work is inspired by the construction of a domain in [4], but our proof is completely different. The original article shows the construction of such a domain

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only in case $p_0 = n = 2$ and the proof is based on change of variables. We prove the same result by chaining Poincaré inequalities and we generalize the construction for the point of discontinuity anywhere in $[n, \infty)$. This result can be generalized to any dimension too, but for simplicity we show the calculations only in case $n = 2$.

It would be nice to see explicit example of domain with a point of discontinuity under the point of dimension, i.e. $p_0 \in (1, n)$.

First of all, we suggest the shape of a domain Ω in dependence on parameters such that $q_\Omega(p)$ is not continuous at point $p_0 \geq 2 = n$. We prove this statement by verifying the embedding $W^{1,p}(\Omega) \subset L^{q(p)}(\Omega)$ and by constructing the counterexamples that show the optimality of $q_\Omega(p)$.

1.1. **Construction of $\Omega_{\alpha,\beta}$ and embedding theorem.** Firstly, we construct a domain $\Omega_{\alpha,\beta} \subset \mathbb{R}^2$ for parameters $\alpha \geq 1, \beta > \alpha$. The point of discontinuity of $q_{\Omega_{\alpha,\beta}}(p)$ is $p_0 = \alpha + 1$, parameter β determinates the size of the leap $\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t)$.

Let us denote by T_i the family of domains in $\mathbb{R}^2$

$$\begin{aligned} T_i := \big\{[x_1, x_2] \in \mathbb{R}^2 :\ & x_1 \in \big(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1}\big), \\ & x_2 \in \big(2^{-i+1}, 2^{-i+1} + (x_1 + 2^{-i^2} i^{-1})^\alpha 2^{-i(\beta-\alpha)} i^{-1+\alpha}\big)\big\}. \end{aligned} \tag{1.3}$$

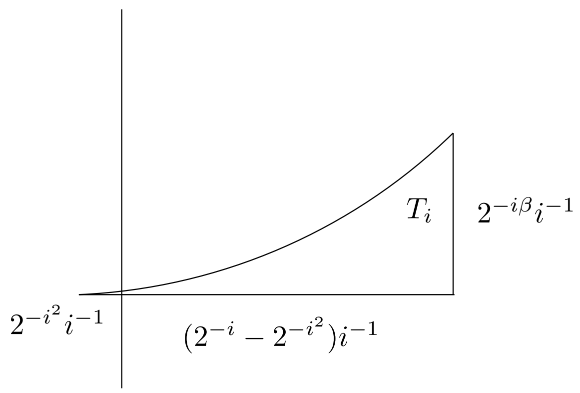

Figure 1. The domain T_i

The shape of T_i is the sub-graph of $y(x) = x^\alpha$ function on some right neighbourhood of 0. Let us denote open square $S \subset \mathbb{R}^2$ by $S := (-4, 0) \times (-2, 2)$. Now we define

$$\Omega_{\alpha,\beta} := \bigcup_{i\in\mathbb{N}} T_i \cup S.$$

We define $q_{\Omega_{\alpha,\beta}}(p) : [1, \infty) \to [1, \infty)$ by

$$q_{\Omega_{\alpha,\beta}}(p) := \begin{cases} p & \text{for } 1 \leq p < \alpha + 1, \\ \frac{(\beta+1)p}{\beta+1-p} & \text{for } \alpha + 1 \leq p < \beta + 1. \end{cases}$$

The function $q_{\Omega_{\alpha,\beta}}(p)$ has a leap at $p_0 = \alpha + 1$ of size

$$\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t) = \frac{(\alpha+1)^2}{\beta - \alpha}.$$

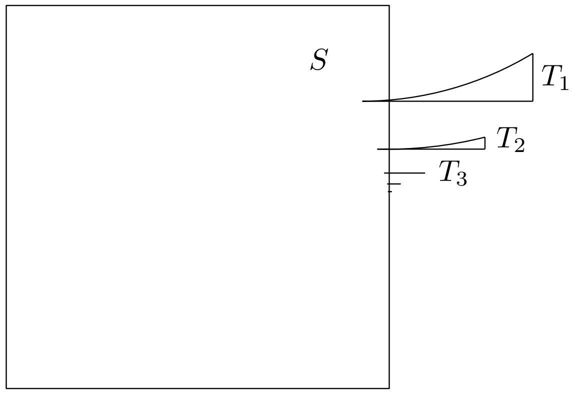

Figure 2. The domain $\Omega_{\alpha,\beta}$

From (1.1) it is easy to see that $q_{\Omega_{\alpha,\beta}}(p) \leq p^*(p)$ for $p \neq n, p \neq \alpha + 1$. This property holds for any $q_\Omega(p)$ from (1.2), the nicer domain Ω is, the lower function $q_\Omega(p)$ is. If the domain Ω has Lipschitz boundary then the function $q_\Omega(p) = p^*(p)$ is the lowest possible.

Theorem 1 (Optimal Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$)**.** *Let $\alpha > 0$, $\beta > \alpha$ and $1 \leq p < 1 + \beta$. Then*

$$W^{1,p}(\Omega_{\alpha,\beta}) \subset L^{q(p)}(\Omega_{\alpha,\beta}).$$

Moreover, for every $\varepsilon > 0$ there exists a function $g : \Omega_{\alpha,\beta} \to \mathbb{R}$ satisfying

$$g \in W^{1,p}(\Omega_{\alpha,\beta}) \text{ and } g \notin L^{q(p)+\varepsilon}(\Omega_{\alpha,\beta}).$$

We prove the first part of Theorem 1 in Section 3. The optimality part of Theorem 1 is proven in Section 4.

We would like to thank to author's supervisor Stanislav Hencl for introducing this field, pointing out this problem and supporting and useful advices during work itself.

2. Preliminaries

For simplicity we use notation $\Omega = \Omega_{\alpha,\beta}$ and $q(p) = q_{\Omega_{\alpha,\beta}}(p)$. By C we denote the generic positive constant whose exact value may change at each occurrence. We write for example $C(a, b, c)$ if C may depend on parameters a, b and c.

Definition 1. *We define Lebesgue norm $\|f\|_{L^p(\Omega)}$ for measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}, p \in [1, \infty]$ as*

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \text{ for } p \in [1, \infty), \\ \operatorname{esssup}_\Omega |f| \text{ for } p = \infty. \end{cases} \tag{2.1}$$

We define Lebesgue space $L^p(\Omega)$ as a set of all functions with finite norm $\|f\|_{L^p(\Omega)}$.

Definition 2. *Let $A \subset \mathbb{R}^n$ be an open set and $v \in L^1_{\text{loc}}(A)$ be a function. We call the function $u : A \to \mathbb{R}^n$ weak derivative of v, if for all $\phi \in C_0^\infty(A)$ we have*

$$\int_A \nabla \phi v dx = -\int_A u \phi dx.$$

We use the notation $u = Dv$, $u_i = D_i v$.

Definition 3. *We define Sobolev norm $\|f\|_{W^{1,p}(\Omega)}$ for function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}, p \in [1, \infty]$ as*

$$\|f\|_{W^{1,p}(\Omega)} := \begin{cases} \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^n \|D_i f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \text{ for } p \in [1,\infty) \\ \max\{\|f\|_{L^p(\Omega)}, \|D_1 f\|_{L^p(\Omega)}, \dots \|D_n f\|_{L^p(\Omega)}\} \text{ for } p = \infty. \end{cases} \tag{2.2}$$

We define Sobolev space $W^{1,p}(\Omega)$ as a set of all functions with finite norm $\|f\|_{W^{1,p}(\Omega)}$.

We use notation $a_i \simeq b_i$, if there exists a constant $K > 0$ such that

$$\frac{1}{K} < \frac{a_i}{b_i} < K \text{ for every } i \in \mathbb{N}.$$

We denote the integral average by

$$f_A := \fint_A f = \frac{1}{|A|} \int_A f.$$

The following Poincaré-type inequality will be essential.

Lemma 1. *Let b be a bi-Lipschitz mapping $b : B(0,r) \subset \mathbb{R}^n \to \mathbb{R}^n$ with a bi-Lipschitz constant $L > 0$, and set $A := b(B(0,r))$. Let $1 \le p \le \infty$, $p \neq n$ and $1 \le m \le p^*(p)$. Then there exists a constant $C(n,p,m,L)$ such that for $f \in W^{1,p}(A)$ we have*

$$|A|^{-\frac{1}{m}} \|f - f_A\|_{L^m(A)} \le C(n,p,m,L) r |A|^{-\frac{1}{p}} \|Df\|_{L^p(A)}.$$

We use the convention $|A|^{-\frac{1}{\infty}} = 1$.

Let $p = n$ and $1 \le m < \infty$. Then there exists a constant $C(n,m,L)$, such that for $f \in W^{1,p}(A)$ it holds

$$|A|^{-\frac{1}{m}} \|f - f_A\|_{L^m(A)} \le C(n,m,L) r |A|^{-\frac{1}{n}} \|Df\|_{L^n(A)}.$$

Proof. In case b is identity and $p = q$ we get classical result. The more difficult case $1 \le q \le p^*(p)$ can be found in [8] as Theorem 12.23 and Exercise 12.24 and with the help of Hölder's inequality. The general case for b not being identity follows by a simple change of variables. □

3. The proof of Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$

In this section we prove Theorem 1 for the case $\alpha \ge 1$. We give the details for $\alpha > 1$ and the case $\alpha = 1$ is only sketched.

Let us suppose that $\alpha > 1$. Then for every $i \in \mathbb{N}$ we define the covering of $T_i \setminus S$ by domains bi-Lipschitz equivalent to balls. The proof of $W^{1,p} \subset L^{q(p)}$ for $p < \alpha + 1$ is elementary from the Definition 3, as every function in $W^{1,p}$ belongs to L^p. Further we suppose that $p > \alpha + 1$.

3.1. **Covering of T_i.** We define $k_\alpha = \frac{3}{2}\left(\frac{2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}$,

$$s_{i,j} := k_\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{1}{\alpha-1}},\qquad r_{i,j} := \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{\alpha}{\alpha-1}}. \tag{3.1}$$

For fixed $i \in \mathbb{N}$ we define the sequence of domains $Q_{i,j}$, $j \in \mathbb{N}$

$$Q_{i,j} := \left\{[x_1, x_2] \in T_i : \ x_1 \in \left(s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j}\right) \cap \left(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1}\right)\right\}. \tag{3.2}$$

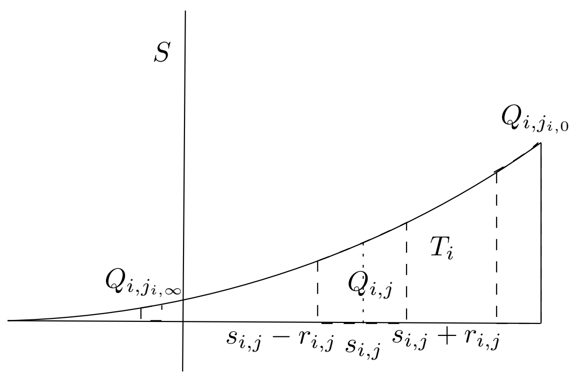

Figure 3. The covering of T_i

Lemma 2 (Covering lemma)**.** *Let $i \in \mathbb{N}$, T_i be given by* (1.3) *and the sequence of domains $Q_{i,j}$ by* (3.2)*. Then*

(i) $Q_{i,j}$ are bi-Lipschitz equivalent to balls with radius $r_{i,j}$ with the same bi-Lipschitz constant L independent on i and j.

(ii) For fixed j_0 there exists only a finite number of domains $Q_{i,j}$ with non-empty intersection with Q_{i,j_0}. This number is bounded by some constant $C(\alpha, \beta)$.

(iii) For fixed j_0 let $A_{i,j_0} := Q_{i,j_0} \cap Q_{i,j_0+1}$. There exists some constant $C(\alpha, \beta) > 0$ such that $C(\alpha, \beta) < \frac{|A_{i,j_0}|}{|Q_{i,j_0}|} < \frac{|A_{i,j_0}|}{|Q_{i,j_0+1}|}$.

(iv) There exists $j_{i,\infty}$, the smallest index satisfying $Q_{i,j_{i,\infty}} \subset S$, and there exists $j_{i,0}$, the biggest index satisfying $s_{i,j_{i,0}} + r_{i,j_{i,0}} \geq (-2^{-i^2} + 2^{-i})i^{-1} =$ "height of T_i". Estimated values are

$$j_{i,\infty} \simeq 2^{i(\beta-\alpha)+i^2(\alpha-1)},\qquad j_{i,0} \simeq 2^{i(\beta-1)}. \tag{3.3}$$

The proof is rather technical but straightforward and can be done by basic calculus, therefore we only outline it.

Sketch of the proof of Lemma 2. We define two bi-Lipschitz mappings:

$$\begin{aligned} b_{1,i,j} :&B(0,r_{i,j}) \to (-r_{i,j}, r_{i,j}) \times (0, r_{i,j}), \\ b_{2,i,j} :&(-r_{i,j}, r_{i,j}) \times (0, r_{i,j}) \to Q_{i,j}, \\ b_{2,i,j}(x_1,x_2) :=& \big(x_1 + s_{i,j}, 2^{-i+1} + \frac{x_2}{r_{i,j}}(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha}\big). \end{aligned}$$

The mapping $b_{1,i,j}$ maps a ball to the half of a square and has bi-Lipschitz constant L_1 independent on i and j, its exact formula can be found easily. Let us consider Jacobi matrices of both $b_{2,i,j}$ and $b_{2,i,j}^{-1}$

$$D_{b_{2,i,j}}(x_1,x_2) = \begin{pmatrix} 1 & \frac{x_2}{r_{i,j}}\alpha(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha-1} 2^{-i(\beta-\alpha)} i^{-1+\alpha} \\ 0 & r_{i,j}^{-1}(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha} \end{pmatrix}$$

$$D_{b_{2,i,j}^{-1}}(b_{2,i,j}(x_1,x_2)) = \begin{pmatrix} 1 & -x_2\alpha(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{-1} \\ 0 & r_{i,j}(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{-\alpha} 2^{i(\beta-\alpha)} i^{1-\alpha} \end{pmatrix}.$$

By a direct computation it is not difficult to check that all partial derivatives are bounded by constant, i.e. the second mapping $b_{2,i,j}$ has bi-Lipschitz constant $L_{2,i,j}$ depended on i and j, and it can be estimated by L_2 common for all i and j. The key observation is, that $L_{2,i,j}$ is monotone sequence in both i and j. We found bi-Lipschitz mapping $b_{2,i,j} \circ b_{1,i,j} : B(0, r_{i,j}) \to Q_{i,j}$ with constant $L = L_1 L_2$ and the first part is proven.

Second part can be proven by verifying statement $\lim_{j\to\infty} s_{i,j} - s_{i,j+1} = r_{i,j}$ for every $i \in \mathbb{N}$.

To prove third part we define $P_{i,j} \subset A_{i,j}$, $P_{i,j} := (s_{i,j+1}, s_{i,j}) \times (2^{-i+1}, 2^{-i+1} + r_{i,j+1})$. We estimate $\frac{|P_{i,j}|}{|Q_{i,j}|}$ and we easily find $C(\alpha,\beta)$ such that $C(\alpha,\beta) < \frac{|P_{i,j}|}{|Q_{i,j}|}$.

The fourth part is important for further calculations. We estimate the indexes $j_{i,0}$ and $j_{i,\infty}$ by definition of $r_{i,j}$ (3.1). From $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq$ "width of $T_i \setminus S$ on left edge" and $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq r_{i,j_{i,\infty}}$ for $j_{i,\infty}$ and $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq$ "width of $T_i \setminus S$ on right edge" and $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq r_{i,j_{i,0}}$ for $j_{i,0}$ we get

$$\begin{aligned} 2^{-i\beta} i^{-1} &\simeq r_{i,j_{i,0}} \simeq \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j_{i,0}^{-\frac{\alpha}{\alpha-1}}, \\ (2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha} &\simeq r_{i,j_{i,\infty}} \simeq \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j_{i,\infty}^{-\frac{\alpha}{\alpha-1}} \end{aligned} \tag{3.4}$$

which implies (3.3). □

3.2. Proof of Theorem 1 for $p > \alpha + 1$, $\alpha \geq 1$.

Proof. We estimate the power of norm

$$\|f\|^q_{L^q(\Omega)} \leq \|f\|^q_{L^q(S)} + \|f\|^q_{L^q(\bigcup_{i\in\mathbb{N}} T_i \setminus S)} \leq \|f\|^q_{L^q(S)} + \sum_{i\in\mathbb{N}} \|f\|^q_{L^q(T_i\setminus S)}.$$

The part $\|f\|^q_{L^q(S)}$ is bounded for any $q \in [1,\infty)$ thanks to Sobolev embedding theorem for Lipschitz domains $W^{1,p}(S) \subset L^\infty(S), p > n = 2$. Therefore we have $\|f\|^q_{L^q(S)} \leq C$.

For every $x \in T_i \setminus S$ we find $j_{i,x}$ such that $x \in Q_{i,j_{i,x}}$. We estimate

$$
\begin{aligned}
\|f\|^q_{L^q(\Omega)} \leq & \|f\|^q_{L^q(S)} + C\sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(|f(x)-f_{Q_{i,j_{i,x}}}| + \sum_{j=j_{i,x}}^{j_{i,\infty}} |f_{Q_{i,j}} - f_{Q_{i,j-1}}|\Big)^q dx \\
\leq & C + C\sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(\sum_{j=j_{i,x}}^{j_{i,\infty}} |f_{Q_{i,j}} - f_{Q_{i,j-1}}|\Big)^q dx \\
& + C\sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(|f(x)-f_{Q_{i,j_{i,x}}}|\Big)^q dx.
\end{aligned} \tag{3.5}
$$

By (1.3), Lemma 1 for $m=\infty$, Lemma 2 (ii) and $r_{i,j} \leq 1$ we have

$$
\begin{aligned}
\sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(|f(x)-f_{Q_{i,j_{i,x}}}|\Big)^q & \leq \sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(\|f(x)-f_{Q_{i,j_{i,x}}}\|_{L^\infty(Q_{i,j_{i,x}})}\Big)^q \\
& \leq \sum_{i=1}^{\infty}\int_{T_i\setminus S}\Big(Cr_{i,j_{i,x}}^{\frac{p-2}{p}}\|Df\|_{W^{1,p}(Q_{i,j_{i,x}})}\Big)^q \\
& \leq C\sum_{i=1}^{\infty}|T_i|\|Df\|^q_{W^{1,p}(\Omega)} \leq C
\end{aligned} \tag{3.6}
$$

By Lemma 1 and Lemma 2 (iii) we have an estimate

$$
\begin{aligned}
|f_{Q_{i,j}} - f_{Q_{i,j-1}}| & \leq \Big(\fint_{A_{i,j-1}}|f_{Q_{i,j}} - f(y)|dy + \fint_{A_{i,j-1}}|f_{Q_{i,j-1}} - f(y)|dy\Big) \\
& \leq C\Big(\fint_{Q_{i,j}}|f_{Q_{i,j}} - f(y)|dy + \fint_{Q_{i,j-1}}|f_{Q_{i,j-1}} - f(y)|dy\Big) \\
& \leq C\Big(r_{i,j}^{\frac{p-2}{p}}\Big(\int_{Q_{i,j}}|Df(y)|^p dy\Big)^{\frac{1}{p}} + r_{i,j-1}^{\frac{p-2}{p}}\Big(\int_{Q_{i,j-1}}|Df(y)|^p dy\Big)^{\frac{1}{p}}\Big).
\end{aligned} \tag{3.7}
$$

By this estimate and Hölder inequality for sums and Lemma 2 (ii) we get

$$
\begin{aligned}
\int_{T_i\setminus O}\Big(\sum_{j=j_{i,x}}^{j_{i,\infty}}|f_{Q_{i,j}} - f_{Q_{i,j-1}}|\Big)^q & \leq \int_{T_i} C\Big(\sum_{j=j_{i,x}}^{j_{i,\infty}} r_{i,j}^{\frac{p-2}{p}}\Big(\int_{Q_{i,j}}|Df(y)|^p dy\Big)^{\frac{1}{p}}\Big)^q dx \\
& \leq C\int_{T_i}\Big(\sum_{j=j_{i,x}}^{j_{i,\infty}} r_{i,j}^{\frac{p-2}{p}(\frac{p}{p-1})}\Big)^{\frac{q(p-1)}{p}}\Big(\sum_{j=j_{i,x}}^{j_{i,\infty}}\Big(\int_{Q_{i,j}}|Df(y)|^p dy\Big)^{p\frac{1}{p}}\Big)^{\frac{q}{p}} dx \\
& \leq C|T_i|\Big(\sum_{j=j_{i,0}}^{j_{i,\infty}} r_{i,j}^{\frac{p-2}{p-1}}\Big)^{\frac{q(p-1)}{p}}.
\end{aligned} \tag{3.8}
$$

From (3.1), (3.5), (3.6), (3.8) and (1.3) we have

$$\|f\|^q_{L^q(\Omega)} \le C + C\sum_{i=1}^{\infty} i^{-2}2^{-i(\beta+1)}\Big(\sum_{j=j_{i,0}}^{j_{i,\infty}} r_{i,j}^{\frac{p-2}{p-1}}\Big)^{\frac{q(p-1)}{p}}$$
$$\le C + C\sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}}2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p}-(\beta+1)\right)}\Big(\sum_{j=j_{i,0}}^{j_{i,\infty}} j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}}\Big)^{\frac{q(p-1)}{p}}.$$

We estimate the sum over j as an integral and we get

$$\sum_{j=j_{i,0}}^{j_{i,\infty}} j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}} \le C\big[x^{\frac{\alpha+1-p}{(\alpha-1)(p-1)}}\big]_{j_{i,0}}^{j_{i,\infty}} \le C2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}}, \tag{3.9}$$

where the integral can be estimated by smaller index (that is $j_{i,0}$ by (3.3)) since $p > \alpha + 1$. Finally we put the estimates together and we get

$$\|f\|^q_{L^q(\Omega)} \le C + C\sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}}2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p}-(\beta+1)\right)}\Big(2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}}\Big)^{\frac{q(p-1)}{p}}$$
$$\le C + C\sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}}2^{i\left(-(\beta+1)+q\frac{\beta+1-p}{p}\right)}.$$

The proof is done, because the sum is finite if $q \le \frac{(\beta+1)p}{\beta+1-p}$.

Let us consider the case $\alpha = 1$. We have to change the definition (3.1) of $s_{i,j}$ and $r_{i,j}$ and the definition (3.2) of $Q_{i,j}$ as

$$r_{i,j} := r_{i,0}(1 + 2^{-i(\beta-1)-1})^j, \text{ for } r_{i,0} = 2^{-i^2-i(\beta-1)-1}i^{-1} \text{ and } s_{i,j} := \sum_{k=0}^{j-1} r_{i,j}. \tag{3.10}$$

We define $Q_{i,j}$ as trapezoids with average of basis equal to height and half of this height we denote by $r_{i,j}$, that is

$$Q_{i,j} = T_i \cap \{x \in \mathbb{R}^2 : x_2 \in (s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j})\}.$$

Let us denote, that the sequences $r_{i,j}$ and $s_{i,j}$ are strictly decreasing with respect to

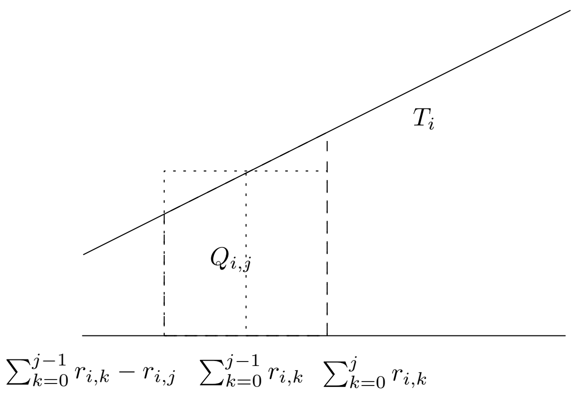

Figure 4. The domain $Q_{i,j}$

index j in case $\alpha > 1$, but these sequences are strictly increasing in case $\alpha = 1$.

The Lemma 2 holds and it is proven in the same way as for $\alpha > 1$, only the indexes of border $Q_{i,j}$ are

$$j_{i,\infty} = -1$$

and by analogy of (3.4)

$$2^{-i\beta} i^{-1} \simeq r_{i,j_{i,0}} = (1 + 2^{-i(\beta-1)-1})^{j_{i,0}} 2^{-i^2-i(\beta-1)-1} i^{-1}$$

we get

$$j_{i,0} \simeq \frac{\ln(2)(i^2 - i)}{\ln(1 + 2^{-i(\beta-1)-1})}.$$

The idea of chaining Poinceré inequality is analogous, and after easy modification we get our result. We can copy all arguments and calculations from (3.5), (3.6), (3.8), then we use (1.3) for $\alpha = 1$, new definition of $r_{i,j}$ and estimates for $j_{i,0}, j_{i,\infty}$ and we get

$$\begin{aligned}
\|f\|^q_{L^q(\Omega)} \leq & C + C \sum_{i=1}^{\infty} i^{-2} 2^{-i(\beta+1)} \Big(\sum_{j=j_{i,\infty}}^{j_{i,0}} r_{i,j}^{\frac{p-2}{p-1}} \Big)^{\frac{q(p-1)}{p}} \\
\leq & C + C \sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{-i(1+\beta)+(-i^2-i(\beta-1))\frac{q(p-2)}{p}} \\
& \Big(\frac{(1 + 2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}(\frac{\ln(2)(i^2-i)}{\ln(1+2^{-i(\beta-1)-1})}+2)} - 1}{(1 + 2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}} - 1} \Big)^{\frac{q(p-1)}{p}},
\end{aligned}$$

where the final term comes from the sum of geometric series. The right hand side can be estimated and after easy calculation we have

$$\|f\|^q_{L^q(\Omega)} \leq C + C \sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{i\frac{q(\beta+1-p)-(\beta+1)p}{p}}.$$

The right hand side is finite if $q \leq \frac{p(\beta+1)}{p-(\beta+1)}$ and the proof is done.

The complete proof for $\alpha = 1$ with all details can be found in [11]. $\square$

4. Optimality of $q(p)$ for $\Omega_{\alpha,\beta}$

Proof of the optimality. We construct the function g by the choice of the proper functions $g_i : T_i \to \mathbb{R}$ and the sequence d_i of positive numbers. We denote $q := q(p) + \varepsilon$. We define

$$g(x_1, x_2) = \begin{cases} 0 & \text{for } (x_1, x_2) \in S, \\ d_i g_i(x_1, x_2) & \text{for } (x_1, x_2) \in T_i \setminus S, \forall i \in \mathbb{N}. \end{cases}$$

Clearly

$$\|g\|^p_{W^{1,p}(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{W^{1,p}(T_i)} \text{ and } \|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)}. \tag{4.1}$$

The choice of g_i and d_i depends on p and $\alpha + 1$, so we split the proof into two parts.

4.1. **The case $p < \alpha + 1$.** Let us consider $p \in [1, \alpha + 1)$. We define

$$\begin{aligned} g_i(x_1, x_2) &:= \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha} \text{ for } (x_1, x_2) \in T_i \setminus S, \\ d_i &:= (2^{-i^2} i^{-1})^{\alpha} 2^{\frac{i(\beta+1)}{q}} i^{\frac{2}{q}}. \end{aligned} \tag{4.2}$$

For fixed $i \in \mathbb{N}$ we estimate the norm in space $L^q(T_i)$. By (1.3) the width of T_i for $x_1 \in \big(- i^{-1} 2^{-i^2}, i^{-1}(2^{-i} - 2^{-i^2})\big)$ is

$$l(x_1) = \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1} \tag{4.3}$$

and we get

$$\begin{aligned} \|g_i\|^q_{L^q(T_i)} &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2} i^{-1}\big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha}\Big|^q l(x_1) dx_1 \\ &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2} i^{-1}\big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha}\Big|^q \big(x_1 + 2^{-i^2} i^{-1}\big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1} dx_1. \end{aligned} \tag{4.4}$$

We can see that the important part is only some left neighbourhood of $\frac{2^{-i}-2^{-i^2}}{i}$, which determine the size of integral. We estimate

$$\begin{aligned} \|g_i\|^q_{L^q(T_i)} &\simeq \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} (2^{-i^2} i^{-1})^{-\alpha q} \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1} dx_1 \\ &\simeq i^{\alpha q - 2} 2^{i^2 \alpha q - i - i\beta}. \end{aligned} \tag{4.5}$$

By (4.2) we get

$$\|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)} \geq C \sum_{i=1}^{\infty} i^0 2^0 = \infty.$$

We need to prove the convergence of $\|g\|^p_{W^{1,p}(\Omega_\alpha)}$. First of all we estimate

$$\|g_i\|^p_{W^{1,p}(T_i)} \leq 2 \max\{\|g_i\|^p_{L^p(T_i)}, \|Dg_i\|^p_{L^p(T_i)}\}.$$

The estimate of the norm of g_i in $L^p(T_i)$ is analogical to (4.5), by interchanging the role of p and q we get

$$\|g_i\|^p_{L^p(T_i)} \simeq i^{\alpha p - 2} 2^{i^2 \alpha p - i - i\beta}.$$

We use $q = q(p) + \varepsilon = p + \varepsilon$ and estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \leq C \sum_{i=1}^{\infty} i^{-2} (2^{-i - i\beta})^{\frac{q-p}{q}} < \infty.$$

Let us express the norm of g_i by derivative

$$\|Dg_i(x_1, x_2)\|^p_{L^p(T_i)} = \int_{T_i} \left|\frac{\partial g_i(x_1, x_2)}{\partial x_1}\right|^p.$$

The estimate is similar to (4.4). The proof splits in two cases, firstly, we consider $p > 1$ and we get

$$\begin{aligned}\int_{T_i}\left|\frac{\partial g_i}{\partial x_1}\right|^p \le& C\int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1 + 2^{-i^2}i^{-1}\right)^{\alpha+p(-\alpha-1)}2^{i(-\beta+\alpha)}i^{\alpha-1}dx_1 \\ \le& C2^{i(-\beta+\alpha)}i^{\alpha-1}\left[\left(x_1+2^{-i^2}i^{-1}\right)^{(1-p)(\alpha+1)}\right]_0^{i^{-1}(2^{-i}-2^{-i^2})}.\end{aligned} \tag{4.6}$$

We can see analogously to (4.5), that the important part is only some right neighbourhood of 0, so we estimate

$$\|Dg_i\|^p_{L^p(T_i)} \le Ci^{p(\alpha+1)}2^{i(-\beta+\alpha)-i^2(1-p)(\alpha+1)}.$$

It follows that

$$\|Dg\|^p_{L^p(\Omega)} = \sum_{i=1}^{\infty} d_i^p\|Dg_i\|^p_{L^p(T_i)} \le \sum_{i=1}^{\infty} Ci^{2\frac{p-q}{q}+p(\alpha+1)}2^{i(-\beta+\alpha+\frac{(\beta+1)p}{q})}2^{i^2(p-\alpha-1)} < \infty.$$

The proof of finiteness of the norm in case $p = 1$ is similar, except the estimate in (4.6) involves $\int(x_1 + C)^{-1} = \log|x_1 + C|$. It is easy to finish the proof in this case too.

4.2. **The case $p > \alpha + 1$.** We define

$$\begin{aligned} g_i(x_1,x_2) :=& \left(x_1 + 2^{-i^2}i^{-1}\right)^{\alpha} - (2^{-i^2}i^{-1})^{\alpha} \text{ for } (x_1,x_2)\in T_i\setminus S,\\ d_i :=& 2^{i\left(\frac{\beta+1}{q}+\alpha\right)}i^{\alpha+\frac{2}{q}}.\end{aligned} \tag{4.7}$$

We use (4.1), (4.3) and we estimate the norms of g_i as in previous case. Analogously to (4.4) and (4.5) we have

$$\begin{aligned}\|g_i\|^q_{L^q(T_i)} =& \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left|\left(x_1+2^{-i^2}i^{-1}\right)^{\alpha} - (2^{-i^2}i^{-1})^{\alpha}\right|^q l(x_1)dx_1 \\ \simeq& \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1+2^{-i^2}i^{-1}\right)^{(q+1)\alpha}2^{i\alpha-i\beta}i^{\alpha-1}dx_1 \\ \simeq& \, i^{-\alpha q-2}2^{-i(q\alpha+1+\beta)}.\end{aligned} \tag{4.8}$$

We estimate $\|g\|^q_{L^q(\Omega)}$ by

$$\|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q\|g_i\|^q_{L^q(T_i)} \ge C\sum_{i=1}^{\infty} i^0 2^0 = \infty.$$

Now we need to prove the convergence of norm of g and Dg in $L^p(\Omega)$. Analogously to (4.8), by interchanging the role of p and q we get

$$\|g_i\|^p_{L^p(T_i)} \simeq i^{-\alpha p-2}2^{-i(p\alpha+1+\beta)}.$$

We use $q = q(p) + \varepsilon > p + \varepsilon$ and we estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \leq C \sum_{i=1}^{\infty} (i^{-2} 2^{-i-i\beta})^{\frac{q-p}{q}} < \infty.$$

Let us express the norm of g_i by derivative and we estimate

$$\begin{aligned}\int_{T_i} \Big|\frac{\partial g_i}{\partial x_1}\Big|^p \leq & C \int_0^{\frac{2^{-i} - 2^{-i^2}}{i}} \big(x_1 + 2^{-i^2} i^{-1}\big)^{(\alpha-1)p+\alpha} 2^{i(-\beta+\alpha)} i^{\alpha-1} dx_1 \\ \leq & C 2^{i(-\beta+\alpha)} i^{\alpha-1} \Big[\big(x_1 + 2^{-i^2} i^{-1}\big)^{\alpha p - p + \alpha + 1}\Big]_0^{\frac{2^{-i} - 2^{-i^2}}{i}} \\ \leq & C i^{p(-\alpha+1)-2} 2^{i(-\alpha p + p - \beta - 1)}.\end{aligned}$$

It follows that

$$\|Dg\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|Dg_i\|^p_{L^p(T_i)} \leq \sum_{i=1}^{\infty} C i^{p-2\frac{q-p}{q}} 2^{i\left(p+(\beta+1)\frac{p-q}{q}\right)} < \infty,$$

where the finiteness follows from $q = \frac{(\beta+1)p}{\beta+1-p} + \varepsilon$. $\square$

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