ERROR ESTIMATES FOR HIGHER-ORDER FINITE VOLUME SCHEMES FOR CONVECTION DIFFUSION PROBLEMS

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Abstract. It is still an open problem to prove a priori error estimates for finite volume schemes of higher order MUSCL type, including limiters, on unstructured meshes, which show some improvement compared to first order schemes. In this paper we use these higher order schemes for the discretization of convection dominated elliptic problems in a convex bounded domain Ω in $\mathbb{R}^2$ and we can prove such kind of an a priori error estimate. In the part of the estimate, which refers to the discretization of the convective term, we gain $h^{1/2}$. Although the original problem is linear, the numerical problem becomes nonlinear, due to MUSCL type reconstruction/limiter technique.

Key words. linear convection dominated diffusion equation in 2D, upwind finite volume scheme, first and higher order finite volume schemes, a priori error estimates, MUSCL type reconstruction/limiter.

AMS subject classifications. 65N15, 35J25, 76M25

1. Introduction. There are many Finite Volume and Discontinuous Galerkin schemes for solving elliptic convection dominated problems and nonlinear conservation laws on unstructured grids in multi dimensions, as

$$\begin{aligned} \partial_t v + \operatorname{div} f(v) &= 0 \qquad && \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) &= u_0(x) && \text{on } \mathbb{R}^n. \end{aligned} \tag{1.1}$$

While for strongly elliptic problems like

$$\begin{aligned} -\Delta v &= f \qquad && \text{in } \Omega \\ v(x) &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

with dominating diffusion, no stabilization is necessary for numerical schemes, we need some upwinding [21] or, for higher order schemes, a suitable stabilization [22], [32], [45] for convection dominated problems like

$$\begin{aligned} -\varepsilon\Delta v + \operatorname{div}(bv) + cv &= f \qquad && \text{in } \Omega \\ v(x) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

The same statement holds also for nonlinear conservation laws as in (1.1). In this case the stabilization is obtained by reconstruction technique with so called limiters. They make the scheme nonlinear, even in cases where the underlying partial differential equation (1.1) is linear. For finite volume schemes, the reconstruction with limiters can be realized in a very easy way even on unstructured grids, e.g. by MUSCL type discretizations. However, the theoretical background for these schemes, in particular when applied to conservation laws, is not yet satisfactorily developed. Concerning the convergence of both first and higher order schemes, there are results in the case of nonlinear scalar conservation laws [13], [17], [34], [33], [15], and in the case of weakly coupled systems of conservation laws [40]. For conservation laws as in (1.1), a priori error estimates of the form

$$||v - u_h||_{L^\infty(L^1)} \le ch^{\frac{1}{4}} + \text{approximation error of data} \tag{1.4}$$

are available [3], [12], [42], [13], [2], [7]. Here, v denotes the exact solution of the underlying partial differential equation and u_h the approximative numerical solution obtained by a first order finite volume scheme in multi dimensions on unstructured grids. From numerical experiments one would expect $h^{\frac{1}{2}}$ in (1.4), but the proof for this on unstructured grids is an open question.

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For smooth solutions of the linear transport equations

$$\partial_t v + (a \cdot \nabla) v = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+ \tag{1.5}$$

one gets [2]

$$||v - u_h||_p \leq ch. \tag{1.6}$$

There are also no error estimates of the type $||v - u_h|| \leq ch^\beta$ for higher order finite volume schemes for conservation laws in multi dimensions on unstructured grids including limiters, with $\beta > \frac{1}{4}$. To get results in this direction, concerning nonlinear hyperbolic conservation laws, seems to be very difficult. Theoretically justified error analysis for upwind finite volume schemes of higher order, which would also indicate the higher order convergence rate, remains an open problem. See for example [15], [42], [13], [12], [14].

Therefore in this paper we apply the higher order finite volume schemes with limiters to a linear convection dominated stationary diffusion equations like (1.3) in multi dimensions on partially unstructured, locally irregular grids (see Assumption 2.2 and Remark 2.3 (c), as well as left part of the Fig. 2.1). In this case the numerical scheme becomes nonlinear, because of the limiter. We can show that we gain $h^{\frac{1}{2}}$ in the error estimate for the term, which refers to the discretization of the convective term (i.e. $\frac{h^3}{\varepsilon}$), compared to first order schemes. For the higher order scheme we get (see Theorem 4.1)

$$\|z_h\|_\varepsilon^2 \leq c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 + c\frac{h^4}{\varepsilon}\sum_j R_j^2 |T_j| \tag{1.7}$$

and if $v \in W^{3,2}(\Omega)$, in particular for smooth solutions, we have

$$\|z_h\|_\varepsilon^2 \leq c\Big(\varepsilon h^3 + h^3 + \frac{h^3}{\varepsilon}\Big)\|v\|_{3,2}^2 + c\frac{h^4}{\varepsilon}\sum_j R_j^2 |T_j|\,.$$

Later we will show that the term $\sum_j R_j^2 |T_j|$ (see (4.8)) is of the same order with respect to ε as $\|v\|_{2,2}^2$. In the case of the first order scheme we get (for comparison) the following result

$$\|z_h\|_\varepsilon^2 \leq c\Big(\varepsilon h^2 + h^3 + \frac{h^2}{\varepsilon}\Big)\|v\|_{2,2}^2\,. \tag{1.8}$$

Let us briefly mention some related results. In [28], [23] a convection diffusion equation like (1.3) with $\varepsilon = 1$ and general elliptic part is considered. They prove error estimates of the form

$$||v - u_h||_{L^2} \leq ch$$

for finite volume schemes of first order. Further results for elliptic and parabolic equations for finite volume schemes are obtained in [4], [24] and the results of an interesting benchmark problem are published in [25]. Lube considered in [36] discretizations of (1.3) but with $B(v)\nabla v$ instead of $\operatorname{div}(bv)$ and proved

$$||v - u_h|| \leq c_\varepsilon h^k (\varepsilon^{\frac{1}{2}} + h^{\frac{1}{2}})$$

for the streamline diffusion method. Here k is the degree of the local polynomials and

$$||v||^2 = \varepsilon |v|_{1,2}^2 + \sum_i \delta_i ||B(u_h)\nabla v||_{0,2,T_i}^2.$$

A-priori error estimates of the type (1.4), e.g. with the $\varepsilon||\cdot||_{H^{1,2}} + ||\cdot||_{L^2}$-norm , are also known for the streamline diffusion shock capturing method applied to the linear transport equation with $h^{\frac{3}{2}}$, see [31].

For dominating diffusion problems there are error estimates for first order schemes (cf. [28], [23] for stationary case), which show $||v - u_h||_{L^2} \le ch$. In [26], [37] convergence for a first order combined finite volume–finite element method in the non-stationary case was proved.

For second order TVD Rung-Kutta Discontinuous Galerkin methods with piecewise polynomials of order k in space a priori error estimates of the form

$$||u - u_h||_{L_2} \le ch^{k+\frac{1}{2}}$$

for smooth solutions u of (1.1) have been proved in [47]. More advanced results for (also hybridized) Discontinuous Galerkin methods can be found e.g. in [43], [16], [39]. For finite element approximations of convection diffusion problems we refer to [5], [41] and for systems to [29].

The following papers deal with finite volume schemes for elliptic problems which are not convection dominated. In [9], [10] and [8] general frameworks for the construction and analysis of higher-order finite volume methods are developed and optimal error estimates are derived. In particular this has also been done for higher order finite volume schemes on rectangular partitions in [6]. Multiscale discontinuous Galerkin schemes for second order elliptic equations with rough coefficients are considered in [44]. Detailed error estimates are derived and confirmed by numerical experiments. Limiters are not included.

Finite volume schemes for elliptic problems without any error estimate are considered in the following papers. Development of higher order schemes are studied in [30]. Oevermann and Klein consider in [38] finite volume schemes of second order for two-dimensional elliptic equations with variable, discontinuous coefficients. Different limiting procedure for diffusion problems are discussed in [35]. In [18] a review of some of the most successful higher-order numerical schemes for the compressible Navier-Stokes equations on unstructured grids are presented. They use Moving-Least-Squares approximations for the construction of higher order finite volume schemes on unstructured grids. They can be used for direct reconstruction of the fluxes at cell edges for hyperbolic and elliptic problems.

2. The problem.

Consider the following boundary value problem

$$Lv := -\varepsilon\Delta v + \operatorname{div}(bv) + cv = f \qquad \text{in } \Omega\,, \tag{2.1}$$

$$v = 0 \qquad \text{on } \partial\Omega \tag{2.2}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ and $b(x)$, $c(x)$, $f(x)$ are functions which are sufficiently smooth on $\overline{\Omega}$ and such that $0 < c_0 \le c(x) \le c_1$, $\operatorname{div} b = 0$ in $\overline{\Omega}$. Moreover we suppose that the diffusion parameter ε is a positive constant, $0 < \varepsilon \le 1$.

We consider that $\overline{\Omega} = \bigcup_j \overline{T}_j$, where $T_j \in \mathcal{T}_h$ are open triangles, $h := \sup_j \operatorname{diam}(T_j)$, $0 < h < h_0$. Furthemore, all boundary triangles are mirrored by the boundary of Ω to get a corresponding ghost triangles (see Fig. 2.1 on the right). The set of all ghost triangles will be denoted by $\mathcal{T}_G$, $\mathcal{T}_G \cap \mathcal{T}_h = \emptyset$.

Notation 2.1. *We denote by*

(i) $|T_j|$: *the volume of triangle* T_j; $T_j \in \mathcal{T}_h \cup \mathcal{T}_G$

(ii) x_j: *the center of gravity of* T_j *(i.e.,* x_j *is the center of the inscribed circle to the triangle* T_j*)*

(iii) $\overline{x_j}$: *intersection of the perpendicular bisectors of* T_j *(i.e.,* $\overline{x_j}$ *is the center of the circumscribed circle to the triangle* T_j*)*

(iv) $v_j := v(\overline{x_j})$

(v) N_j: *the set of the numbers of the neighboring triangles to* T_j, $T_j \in \mathcal{T}_h$

(vi) $T_{j\ell} := T_j \cup T_\ell$

(vii) $S_{j\ell}$, $\ell \in N_j$: *the joint edge of* T_j *and* T_ℓ *with length* $|S_{j\ell}|$, *where* $T_j \in \mathcal{T}_h$, $T_\ell \in \mathcal{T}_h \cup \mathcal{T}_G$

(viii) $x_{j\ell}$: *the midpoint of* $S_{j\ell}$

(ix) $d_{j\ell} := |x_\ell - x_j|$

(x) $\overline{d_{j\ell}} := |\overline{x_\ell} - \overline{x_j}|$

(xi) $\gamma_{j\ell} := \frac{|S_{j\ell}|}{d_{j\ell}}$; $\gamma := \min \gamma_{j\ell}$;

(xii) $n_{j\ell}$: *the outward unit normal to* $T_j \in \mathcal{T}_h$ *in the direction of* T_ℓ, $\ell \in N_j$.

We assume that there exists an $\eta > 0$ such that all angles of all triangles $T_j \in \mathcal{T}_h$ are less than $\frac{\pi}{2} - \eta$. Therefore, both x_j and $\overline{x_j}$ lie strictly inside of T_j for all j and there is a constant $c_\eta > 0$ independent of h such that $\gamma > c_\eta$.

Moreover we assume that $\mathcal{T}_h = \mathcal{T}_R \cup \mathcal{T}_S$, such that $\mathcal{T}_R \cap \mathcal{T}_S = \emptyset$, where

$$\mathcal{T}_R = \{T_j \in \mathcal{T}_h; T_j \text{ is equilateral } \textbf{and } T_\ell \text{ is equilateral } \forall \ell \in N_j\}\,, \tag{2.3}$$

and

$$\mathcal{T}_S = \{T_j \in \mathcal{T}_h; T_j \text{ is not equilateral } \textbf{or } \exists \ell \in N_j \text{ s.t. } T_\ell \text{ is not equilateral}\}\,. \tag{2.4}$$

The triangles in $\mathcal{T}_R$ and $\mathcal{T}_S$ are called regular and singular, respectively.

ASSUMPTION 2.2. *We assume that the triangulation is locally irregular in the sense of Heinrich (cf. [27, par. 2.2.2, p. 27]), i.e. that the set $\mathcal{T}_S$ consists of the finite number of strips of triangles, each being of the width of $O(h)$. See the left of Fig. 2.1 for the triangles of $\mathcal{T}_R$ (white color) and the triangles of $\mathcal{T}_G$ (darker color).*[1]

REMARK 2.3. (a) If we have $c = 0$ everywhere then we have to change the norm in (4.6): the c_0-term has to be cancelled. In the estimates from below and above in Sections 7 and 8 the term ψ_{N_j} is equal to zero and will not appear. Then all the arguments can be repeated. If $c = 0$ only in a subset of Ω then there will be some problems. We can change the definition of the norm in (4.6) by $\sum_{T_j} c_j |z_j|^2 |T_j|$ but then, e.g. the c_0-term in the last estimate in the proof of Lemma 7.1 (and correspondingly in the proof of Lemma 7.2) cannot be controlled by the left-hand side. Therefore degeneration of c in part of Ω is not allowed in this framework.

(b) The condition $\operatorname{div} b = 0$, mentioned in the beginning of Section 2, ensures that (3.10) and (3.13) are valid, and that we can obtain the estimate (8.16). This means that a local conservation property holds. This is essential to get the final estimates. Therefore, also a condition of the type $-\frac{1}{2} \operatorname{div} b + c \geq 0$ would not be sufficient.

(c) The analysis is performed essentially only on uniform grids consisting of identical equilateral triangles. The MUSCL procedure is used only on these elements and furthermore, the set of non-equilateral triangles is assumed to be very small (strips of width $O(h)$). Even then the non-equilateral triangles are assumed to have all angles less than $\frac{\pi}{2} - \eta$). These conditions are rather restrictive concerning domains which can be triangulated by the locally irregular triangulation. It can be shown, however, that every quadrilateral and pentagon can be triangulated by a reasonable finite amount of acute triangles (see [46]), so we can handle the acute triangulation of local quadrilaterals and pentagons occuring in the finite number of strips of irregularity.

ASSUMPTION 2.4. *For the solution v of (2.1) and (2.2) we assume $v \in W^{2,2}(\Omega)$ and that v can be extended onto a small strip ω_d of the width of $O(h)$ outside of Ω such that we have $v(\overline{x_\ell}) = -v(\overline{x_j})$ if $T_\ell \subset \omega_d$ is the mirrored ghost triangle to T_j. For the continuation v_d of v we assume*

$$||v_d||_{W^{2,2}(\Omega_d)} \leq c||v||_{W^{2,2}(\Omega)},$$

where the constant c is independent of v and $\Omega_d := \Omega \cup \omega_d$.

In the context of the locally irregular grid we will also use the following result (cf. [27, p. 189] and the references there):

THEOREM 2.5. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\omega_h \subset \Omega$ be the strip of the width of $O(h)$, $0 < h < h_0$. Then there is a constant $c > 0$ independent of h such that for all $v \in W^{j+1,2}(\Omega)$, $j = 0, 1, 2$, we have*

$$\|v\|_{W^{j,2}(\omega_h)} \leq ch^{\frac{1}{2}} \|v\|_{W^{j+1,2}(\Omega)}\,, \qquad j = 0, 1, 2\,. \tag{2.5}$$

[1]For drawing the pictures we are grateful to Mr. Johannes Schäfer.

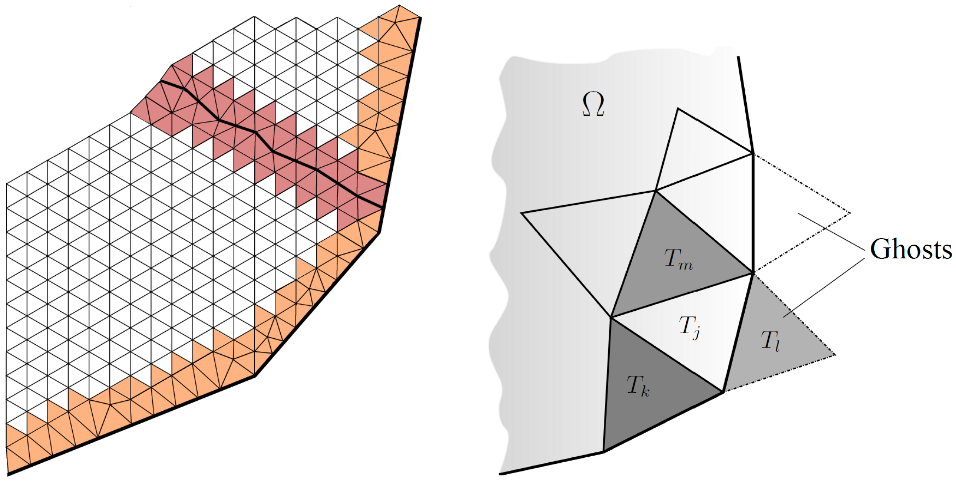

FIG. 2.1. *A locally irregular mesh and ghosts cells.*

We split the set $\mathcal{E}$ of the edges $S_{j\ell} = \overline{T_j} \cap \overline{T_\ell}$, with $T_j, T_\ell \in \mathcal{T}_h$, into three parts, $\mathcal{E} = \mathcal{E}_R \cup \mathcal{E}_S \cup \mathcal{E}_M$, where

$$\begin{aligned} \mathcal{E}_R &:= \{S_{j\ell};\, S_{j\ell} \not\subset \partial\Omega;\ \text{both } T_j \text{ and } T_\ell \text{ are regular}\}\,, \\ \mathcal{E}_S &:= \{S_{j\ell};\, S_{j\ell} \not\subset \partial\Omega;\ \text{both } T_j \text{ and } T_\ell \text{ are singular}\}\,, \\ \mathcal{E}_M &:= \{S_{j\ell};\, S_{j\ell} \not\subset \partial\Omega;\ T_j \text{ and } T_\ell \text{ are of different type (regular, singular)}\}\,. \end{aligned} \tag{2.6}$$

and call them regular, singular and mixed edges, respectively. We also denote by

$$\mathcal{E}_B := \{S_{j\ell};\, S_{j\ell} \subset \partial\Omega;\} \tag{2.7}$$

and call them the boundary edges.
Furthermore we denote

$$N_{jI} := \{\ell \,|\, T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \subset \Omega\}\,,$$

and

$$\begin{aligned} N_{jR} &:= \{\ell \,|\, T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_R\}\,, \\ N_{jS} &:= \{\ell \,|\, T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_S\}\,. \\ N_{jG} &:= \{\ell \,|\, T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_G\}\,, \end{aligned}$$

3. The scheme. Let $c_h(x) := c_j$, $f_h(x) := f_j$ for $x \in T_j \in \mathcal{T}_h$, be piecewise constant approximants of c, f, respectively, defined by

$$c_j := \frac{1}{|T_j|}\int_{T_j} c\,, \qquad f_j := \frac{1}{|T_j|}\int_{T_j} f\,. \tag{3.1}$$

Let $u_h(x) = u_j$ for $x \in T_j \in \mathcal{T}_h \cup T_G$, be a piecewise constant solution of the discrete problem

$$(L_h u_h)_j = f_j\,, \qquad \text{if } T_j \in \mathcal{T}_h, \tag{3.2}$$

$$u_\ell = -u_j\,, \qquad \text{if } S_{j\ell} \subset \partial\Omega, \text{ and } T_\ell \in \mathcal{T}_G \text{ is the ghost triangle to } T_j \subset \Omega, \tag{3.3}$$

where the discrete operator is given by

$$(L_h u_h)_j := -\frac{\varepsilon}{|T_j|}\sum_{\ell \in N_j}(u_\ell - u_j)\gamma_{j\ell} + \frac{1}{|T_j|}\sum_{\ell \in N_j} g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) + c_j u_j\,. \tag{3.4}$$

The first term in (3.4) approximates the value of the diffusion term $-\varepsilon\Delta v$ in $\overline{x_j}$, while $\sum_{\ell\in N_j} g_{j\ell}(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j})$ approximates the values of the convective term $\operatorname{div}(bv)$ along $S_{j\ell}$. Here, $g_{j\ell}$ stands for an upwind finite volume flux, and,

$$\mathcal{U}_{j\ell} = \mathcal{U}_{j\ell}(u_j, u_\ell), \qquad \mathcal{U}_{\ell j} = \mathcal{U}_{\ell j}(u_\ell, u_j) \tag{3.5}$$

will be defined more precisely later. A particular scheme of the type (3.2)–(3.5) is then chosen by the particular choice of functions $\mathcal{U}_{j\ell}$ and $g_{j\ell}$.

Example 3.1. (**General numerical flux**) In general, we suppose that the upwind finite volume flux $g_{j\ell}(u,v)$ is a *Lipschitz continuous* function, i.e., we suppose that there is a constant $c > 0$ such that

$$|g_{j\ell}(u,v) - g_{j\ell}(u',v')| \le c\,h\ (|u-u'| + |v-v'|)\ . \tag{3.6}$$

Furthermore we suppose that $g_{j\ell}$ satisfies the following three basic properties:

$$g_{j\ell}(u,u) = u \int_{S_{j\ell}} bn_{j\ell}\,ds\,, \tag{3.7}$$

$$g_{j\ell}(u,v) = -g_{\ell j}(v,u)\,, \tag{3.8}$$

$$\frac{\partial}{\partial u} g_{j\ell}(u,v) \ge 0 \ge \frac{\partial}{\partial v} g_{j\ell}(u,v)\,, \tag{3.9}$$

which are referred to as *consistency*, *conservativity*, and *monotonicity* of the numerical flux $g_{j\ell}$, respectively. (See [33] or [34] for more discussion on general upwind finite volume numerical fluxes.) Moreover, due to (3.7) and $\operatorname{div} b(x) = 0$, we have that (cf. (3.13)):

$$\sum_{\ell\in N_j} g_{j\ell}(u,u) = 0 \qquad \text{for all } j\,. \tag{3.10}$$

Example 3.2. (**First order Engquist-Osher scheme**) As a particular example of the numerical flux we choose the Engquist-Osher type upwind finite volume flux $g_{j\ell}$ defined by

$$g_{j\ell}(u,v) := b^+_{j\ell}\,u + b^-_{j\ell}\,v\,, \qquad b^\pm_{j\ell} := \int_{S_{j\ell}} (bn_{j\ell})^\pm\,ds\,. \tag{3.11}$$

It can be easily shown that this particular numerical flux satisfies (3.6)–(3.10). The easiest choice of $\mathcal{U}_{j\ell}$ in (3.5), namely

$$\mathcal{U}_{j\ell} := u_j\,, \qquad \mathcal{U}_{\ell j} := u_\ell\,, \tag{3.12}$$

used together with (3.11) in (3.4) defines a *first order* numerical scheme.

Remark 3.3. Due to the properties of b we have for all $T_j \in \mathcal{T}_h$

$$\sum_{\ell\in N_j} (b^+_{j\ell} + b^-_{j\ell}) = \sum_{\ell\in N_j} b_{j\ell} = \sum_{\ell\in N_j} \int_{S_{j\ell}} bn_{j\ell}\,ds = \int_{\partial T_j} bn_{j\ell}\,ds = \int_{T_j} \operatorname{div} b\,dx = 0\,, \tag{3.13}$$

and, for all $S_{j\ell} \in \mathcal{E}$,

$$b_{\ell j} = -b_{j\ell}\,, \quad b^+_{\ell j} = -b^-_{j\ell}\,, \quad b^-_{\ell j} = -b^+_{j\ell}\,. \tag{3.14}$$

Example 3.4. (**Higher order scheme using MUSCL type reconstruction**) Let T_k, T_ℓ, T_m be all neighboring triangles to T_j with centers of gravity x_k, x_ℓ, x_m, x_j, respectively. Let $w \in L^\infty(\Omega)$ with $w|_{T_j} \in C^0(T_j)$ and $w_i := w(x_i)$ for $i = k, \ell, m, j$, respectively. Let (cf. [19])

$$\begin{aligned}
&R^w_k \ \text{ be a plane passing through } (x_\ell,w_\ell),(x_m,w_m),(x_j,w_j)\,,\\
&R^w_\ell \ \text{ be a plane passing through } (x_k,w_k),(x_m,w_m),(x_j,w_j)\,,\\
&R^w_m \ \text{ be a plane passing through } (x_k,w_k),(x_\ell,w_\ell),(x_j,w_j)\,.
\end{aligned}$$

Define an index i by

$$|\nabla R_i^w| = \min\left\{|\nabla R_k^w|, |\nabla R_\ell^w|, |\nabla R_m^w|\right\} \tag{3.15}$$

and put

$$G_j^w := \nabla R_i^w . \tag{3.16}$$

If $w_j \geq \max\{w_k, w_\ell, w_m\}$ or $w_j \leq \min\{w_k, w_\ell, w_m\}$, we say that w_j is a local extremum. Let the coefficients $\alpha_j = \alpha_j^w \in \{0, 1\}$ be such that

$$\alpha_j^w = \begin{cases} 0 & \text{if } w_j \text{ is the local extremum,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.17}$$

Then define

$$L_j^w(x) := w_j + \alpha_j^w G_j^w (x - x_j) . \tag{3.18}$$

Finally, the higher order MUSCL type Engquist-Osher scheme is defined by (3.4) with the numerical flux (3.11) and

$$\mathcal{U}_{j\ell} := L_j^u(x_{j\ell}) , \qquad \mathcal{U}_{\ell j} := L_\ell^u(x_{j\ell}) . \tag{3.19}$$

It can be shown that, on the regular grid, the reconstruction operator L_j^u defined by (3.18) has the following properties.

LEMMA 3.5. *For all $T_j \in \mathcal{T}_R$ we have*

$$(a) \qquad |L_j^u(x_\ell) - u_j| \leq |u_j - u_\ell| \qquad \text{for all} \quad \ell \in N_j, \tag{3.20}$$

$$(b) \qquad |L_j^u(x_{j\ell}) - u_j| \leq \frac{1}{2}|u_j - u_\ell| \qquad \text{for all} \quad \ell \in N_j, \tag{3.21}$$

$$(c) \qquad (u_\ell - L_j^u(x_\ell))(u_j - u_\ell) \leq 0 \qquad \text{for all} \quad \ell \in N_j. \tag{3.22}$$

We will give the proof of this lemma in the following section.

4. Main result. We will use the scheme (3.2)–(3.4) with the following definition of the numerical flux:

- If $S_{j\ell} \in \mathcal{E}_R$ we use the higher order flux using MUSCL type reconstruction, i.e. we set (cf. (3.11), (3.18)–(3.19))

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ L_j^u(x_{j\ell}) + b_{j\ell}^- L_\ell^u(x_{j\ell}) . \tag{4.1}$$

- If $S_{j\ell} \in \mathcal{E}_M$ or $S_{j\ell} \in \mathcal{E}_S$ we use the first order flux, i.e. we set (cf. (3.11), (3.12))

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ u_j + b_{j\ell}^- u_\ell . \tag{4.2}$$

- If $S_{j\ell} \in \mathcal{E}_B$ we use

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ u_j + b_{j\ell}^- u_\ell \tag{4.3}$$

  where in this case u_ℓ is the value in the ghost cell of the cell T_j satisfying $u_\ell = -u_j$ (cf. (3.3)).

The main result of this paper is formulated in the following theorem.

THEOREM 4.1. *Let $u_h(x) = u_j$ for $x \in T_j \in \mathcal{T}_h$, be a piecewise constant numerical solution of the discrete problem* (3.2)–(3.4) *with a numerical flux satisfying* (3.11), (3.18)–(3.19), (4.1)–(4.3) *and let the Assumption 2.4 hold. We define*

$$z_h := I_h v - u_h \tag{4.4}$$

where

$$I_h v(x) := v(\overline{x_j}) = v_j \quad \text{if } x \in T_j \in \mathcal{T}_h\,. \tag{4.5}$$

Then, defining

$$\|z_h\|_\varepsilon^2 := \varepsilon\gamma \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 + c_0 \sum_{T_j\in\mathcal{T}_h} z_j^2 |T_j|\,, \tag{4.6}$$

we have the following error estimate for any $\delta > 0$:

$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 + c\frac{h^4}{\varepsilon}\sum_{T_j\in\mathcal{T}_R} R_j^2|T_j|\,. \tag{4.7}$$

If, moreover, $v \in W^{3,2}(\Omega)$, we have

$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^3 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{3,2}^2 + c\frac{h^4}{\varepsilon}\sum_{T_j\in\mathcal{T}_R} R_j^2|T_j|\,. \tag{4.8}$$

Here, $R_j := \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}$.

REMARK 4.2.

- Note that if T_ℓ is the mirrored ghost triangle to T_j, we have $v(\overline{x_\ell}) = -v(\overline{x_j})$ (see Assumption 2.4), and also $u_\ell = -u_j$ (see (3.3)). Therefore,
$$z_\ell = v_\ell - u_\ell = -v_j + u_j = -z_j\,, \tag{4.9}$$
if T_ℓ is the mirrored ghost triangle to T_j.
- In the case when the first order scheme is used in the whole domain we get (for comparison) the following result:
$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^2}{\varepsilon}\Big)\|v\|_{2,2}^2\,. \tag{4.10}$$
For the higher order MUSCL type scheme we thus gain $h^{1/2}$ inside the estimate of the norm of $\|z_h\|_\varepsilon$ for the term corresponding (as it shows) to the convective part of the equation, compared to the first order scheme: compare $\frac{h^3}{\varepsilon}$ to $\frac{h^2}{\varepsilon}$ in the estimate of the norm of $\|z_h\|_\varepsilon^2$.
- For the context of the definition of R_j see (8.6). It follows from Lemma 8.4 that the sum $\sum_{T_j\in\mathcal{T}_R} R_j^2|T_j|$ is of the same order in ε as $\|v\|_{2,2}^2(\Omega)$, cf. (8.14) and (8.20), namely we have both
$$\sum_{\mathcal{T}_R} R_j^2|T_j| \;\le\; \frac{c}{\varepsilon^3}\,\|f\|_{L^2(\Omega)}^2 \qquad \text{and} \qquad \|v\|_{W^{2,2}(\Omega)}^2 \le \frac{c}{\varepsilon^3}\|f\|_{L^2(\Omega)}^2\,,$$
the second estimate being sharp due to Remark 8.5. In such a way, the estimate (4.7) can be put in form of
$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 + c\frac{h^4}{\varepsilon^4}\|f\|_{L^2(\Omega)}^2$$
or
$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\frac{1}{\varepsilon^3}\|f\|_{L^2(\Omega)}^2 + c\frac{h^4}{\varepsilon^4}\|f\|_{L^2(\Omega)}^2$$

while in the case of the first order scheme we would get analogously instead of (4.10) the following estimate:

$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^2}{\varepsilon}\Big)\frac{1}{\varepsilon^3}\|f\|^2_{L^2(\Omega)}\,.$$

- For the particular numerical calculation for which $\varepsilon \approx h$, we get, using (4.10) and (4.7), the error estimates of the order $O(\sqrt{h})$ and $O(h)$ in the cases of first order and higher order scheme, respectively. If $\varepsilon \approx \sqrt{h}$, we get in the corresponding cases the error estimates of the order $O(h^{3/4})$ and $O(h^{5/4})$, respectively.
- In the estimate (4.7), the norm $||v||^2_{2,2}$ still depends singularly on ε. Due to Lemma 8.4 it behaves like $\frac{1}{\varepsilon^3}$. This means altogether the term $\frac{h^3}{\varepsilon}||v||^2_{2,2}$ behaves like $\frac{h^3}{\varepsilon^4}$. These types of estimate are usual also for stabilized finite elements but for numerical computations on a non adaptive grid they are of limited importance. Nevertheless numerical experiments indicate that adaptive grids in the neighbourhood of the boundary layer allow acceptable resolutions.

In the remaining part of the paper we will give the proof of Theorem 4.1.

5. The basic estimates.

First we give a proof of Lema 3.5.

Proof of Lemma 3.5 from page 7: Of course, for $T_j \in \mathcal{T}_R$, (b) follows immediately from (a). To prove (a), let us consider the following geometrical situation. Without loss of generality we assume that $x_j = (0,0)$, $x_\ell = (1,0)$, $x_m = (\cos\alpha, \sin\alpha)$, $x_k = (\cos\alpha, -\sin\alpha)$, $\alpha = \frac{2}{3}\pi$. Therefore,

$$\sin\alpha = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\alpha = -\frac{1}{2}\,. \tag{5.1}$$

Let u_j, u_ℓ, u_m, u_k be arbitrary. Denote $P_i = (x_i, u_i)$ for $i = j, \ell, m, k$. The planes p_k, p_ℓ, p_m through the set of points (P_j, P_ℓ, P_m), (P_j, P_m, P_k), (P_j, P_ℓ, P_k), respectively, are given by

$$p_k(y) = u_j + (u_\ell - u_j)y_1 + \frac{(u_m - u_j) - \cos\alpha(u_\ell - u_j)}{\sin\alpha}y_2\,, \tag{5.2}$$

$$p_\ell(y) = u_j + \frac{(u_m - u_j) + (u_k - u_j)}{2\cos\alpha}y_1 + \frac{(u_m - u_j) - (u_k - u_j)}{2\sin\alpha}y_2\,, \tag{5.3}$$

$$p_m(y) = u_j + (u_\ell - u_j)y_1 - \frac{(u_k - u_j) - \cos\alpha(u_\ell - u_j)}{\sin\alpha}y_2\,, \tag{5.4}$$

where $y = (y_1, y_2)$. If there is a local extremum in x_j we have (see (3.18)) $L^u_j(x_i) = u_j$ for $i = k, \ell, m$ and (3.20) holds. Therefore we can assume that there is no local extremum in x_j, i.e., $\alpha^u_j = 1$. Furthermore we assume without loss of generality that (see (3.18))

$$\nabla L^u_j = \nabla p_m,$$

which means that

$$|\nabla p_m| \le |\nabla p_\ell|, \tag{5.5}$$

$$|\nabla p_m| \le |\nabla p_k|. \tag{5.6}$$

Considering $L^u_j(x_i) - u_j$ for $i = k, \ell, m$, we will discuss two cases.
First case: if $i = \ell$, or $i = k$, we obtain

$$L^u_j(x_\ell) - u_j = \nabla p_m \cdot (1,0) = u_\ell - u_j\,,$$
$$L^u_j(x_k) - u_j = \nabla p_m \cdot (\cos\alpha, -\sin\alpha) = u_k - u_j\,,$$

and (3.20) follows.

Second case: if $i = m$, we have

$$L_j^u(x_m) - u_j = \nabla p_m \cdot (\cos\alpha, \sin\alpha) = (u_j - u_k) + (u_j - u_\ell)\,,$$

using (5.1). Therefore, in order to prove (3.20) we have to show that

$$|(u_j - u_k) + (u_j - u_\ell)| \le |u_j - u_m|\,. \tag{5.7}$$

Using (5.1) we obtain for the gradients of p_k, p_ℓ, p_m,

$$\begin{aligned} |\nabla p_k|^2 &= \frac{4}{3}\left[(u_\ell - u_j)^2 + (u_m - u_j)^2 + (u_\ell - u_j)(u_m - u_j)\right] \\ |\nabla p_\ell|^2 &= \frac{4}{3}\left[(u_m - u_j)^2 + (u_k - u_j)^2 + (u_m - u_j)(u_k - u_j)\right] \\ |\nabla p_m|^2 &= \frac{4}{3}\left[(u_k - u_j)^2 + (u_\ell - u_j)^2 + (u_k - u_j)(u_\ell - u_j)\right]. \end{aligned} \tag{5.8}$$

It follows that (5.5), (5.6) are equivalent to

$$(u_m - u_\ell)(u_m + u_k + u_\ell - 3u_j) \ge 0\,, \tag{5.9}$$

$$(u_m - u_k)(u_m + u_k + u_\ell - 3u_j) \ge 0\,, \tag{5.10}$$

respectively.

We will further distinguish three cases:

(i) $u_m + u_k + u_\ell - 3u_j = 0$. This implies $(u_j - u_k) + (u_j - u_\ell) = u_m - u_j$ and (5.7) follows.

(ii) $u_m + u_k + u_\ell - 3u_j > 0$. Then (5.10) and (5.9) imply $u_m \ge u_k$ and $u_m \ge u_\ell$, respectively. Since u_j is not a local maximum, we also have $u_m \ge u_j$. Then

$$\begin{aligned} u_j - u_k + u_j - u_\ell &= 3u_j - u_k - u_\ell - u_m + u_m - u_j \\ &< u_m - u_j = |u_m - u_j|\,. \end{aligned}$$

On the other hand, $u_j \ge \min(u_k, u_\ell)$, since u_j is not a local minimum. Therefore, $u_j + u_m \ge u_k + u_\ell$ and

$$u_j - u_k + u_j - u_\ell \ge u_j - u_m = -|u_m - u_j|\,.$$

Altogether, (5.7) follows.

(iii) $u_m + u_k + u_\ell - 3u_j < 0$. In this case we proceed quite analogously as in the case (ii) which finishes the proof of Lemma 3.5, part (a).

Finally let us prove (c). We have to show

$$\Big(u_i - L_j^u(x_i)\Big)(u_j - u_i) \le 0\,, \qquad i = k, \ell, m\,. \tag{5.11}$$

Using the same notation as in the part (a), we see that $L_j^u(x_i) = u_i$ for $i = k$ or $i = l$, and (5.11) follows. For $i = m$ we have

$$\Big(u_m - L_j^u(x_m)\Big)(u_j - u_m) = (u_m + u_k + u_\ell - 3u_j)(u_j - u_m)\,. \tag{5.12}$$

Discussing the cases (i), (ii), (iii) as in the part (a), we see that (5.11) follows. The Lemma 3.5 is proven. □

We continue with a rather technical result. Sometimes we have to change the sum over all triangles into a sum over all edges of the triangulation. The general rules for this are given by the following lemma.

LEMMA 5.1. *Let $A_{j\ell} \in \mathbb{R}$ for all $j, T_j \in \mathcal{T}_h$. Then we have*

$$
\begin{aligned}
\sum_{T_j\in\mathcal{T}_h}\sum_{\ell\in N_j} A_{j\ell} &= \sum_{\text{edges}\in\mathcal{E}_R} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_S} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_M} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_B} A_{j\ell} \\
&= \sum_{\text{edges}\in\mathcal{E}} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_B} A_{j\ell}
\end{aligned}
\tag{5.13}
$$

where $\widetilde{A}_{j\ell} := A_{j\ell} + A_{\ell j}$ and $\mathcal{E} = \mathcal{E}_R \cup \mathcal{E}_S \cup \mathcal{E}_M$.

Proof:

$$
\begin{aligned}
\sum_{T_j\in\mathcal{T}_h}\sum_{\ell\in N_j} A_{j\ell} &= \sum_{T_j\in\mathcal{T}_R}\sum_{\ell\in N_j} A_{j\ell} + \sum_{T_j\in\mathcal{T}_S}\sum_{\ell\in N_j} A_{j\ell} \\
&= \sum_{T_j\in\mathcal{T}_R}\sum_{\ell\in N_{jR}} A_{j\ell} + \sum_{T_j\in\mathcal{T}_R}\sum_{\ell\in N_{jS}} A_{j\ell} + \sum_{T_j\in\mathcal{T}_R}\sum_{\ell\in N_{jG}} A_{j\ell} \\
&\quad + \sum_{T_j\in\mathcal{T}_S}\sum_{\ell\in N_{jR}} A_{j\ell} + \sum_{T_j\in\mathcal{T}_S}\sum_{l\in N_{jS}} A_{j\ell} + \sum_{T_j\in\mathcal{T}_S}\sum_{\ell\in N_{G}} A_{j\ell} \\
&= \sum_{\text{edges}\in\mathcal{E}_R} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_S} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_M} \widetilde{A}_{j\ell} + \sum_{\text{edges}\in\mathcal{E}_B} A_{j\ell}
\end{aligned}
$$

where $\widetilde{A}_{j\ell} := A_{j\ell} + A_{\ell j}$. □

REMARK 5.2. We often abbreviate $\sum_{\mathcal{E}} \equiv \sum_{\text{edges}\in\mathcal{E}}$ and so on.

In what follows we prove the discrete energy estimate for the higher order scheme.

LEMMA 5.3. *Let u_h be the numerical solution defined by the scheme* (3.2)–(3.4) *with a numerical flux satisfying* (3.11), (3.18)–(3.19), (4.1)–(4.3) *and let Assumption 2.4 hold. Then there is a constant $c > 0$ such that for all $\varepsilon > 0$ and all $h > 0$*

$$
\varepsilon\gamma \sum_{\mathcal{E}\cup\mathcal{E}_B} (u_j - u_\ell)^2 + c_0 \sum_{T_j\in\mathcal{T}_h} u_j^2 |T_j| \le c \sum_{T_j\in\mathcal{T}_h} f_j^2 |T_j|. \tag{5.14}
$$

Proof: We multiply $(L_h u_h)_j = f_j$ by $|T_j| u_j$ and sum up over all j such that $T_j \in \mathcal{T}_h$. Then we obtain

$$
-\varepsilon \sum_j \sum_{\ell\in N_j} (u_\ell - u_j)\gamma_{j\ell} u_j + \sum_j \sum_{\ell\in N_j} g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) u_j + \sum_j c_j u_j^2 |T_j| = \sum_j f_j u_j |T_j|. \tag{5.15}
$$

For the first term on the left hand side in (5.15) we get using $\gamma_{j\ell} = \gamma_{\ell j}$ and (3.3),

$$
\begin{aligned}
-\varepsilon \sum_j \sum_{\ell\in N_j} (u_\ell - u_j)\gamma_{j\ell} u_j &= -\varepsilon \sum_j \sum_{\ell\in N_{jI}} (u_\ell - u_j)\gamma_{j\ell} u_j - \varepsilon \sum_j \sum_{\ell\in N_{jG}} (u_\ell - u_j)\gamma_{j\ell} u_j \\
&= -\varepsilon \sum_{\mathcal{E}} \big((u_\ell - u_j)\gamma_{j\ell} u_j + (u_j - u_\ell)\gamma_{\ell j} u_\ell\big) + \varepsilon \sum_j \sum_{\ell\in N_{jG}} 2u_j^2 \gamma_{j\ell} \\
&= \varepsilon \sum_{\mathcal{E}} (u_\ell - u_j)^2 \gamma_{j\ell} + \frac{\varepsilon}{2} \sum_j \sum_{\ell\in N_{jG}} (u_\ell - u_j)^2 \gamma_{j\ell} \\
&\ge \frac{\varepsilon\gamma}{2} \sum_{\mathcal{E}\cup\mathcal{E}_B} (u_\ell - u_j)^2 .
\end{aligned}
\tag{5.16}
$$

For the second term on the left hand side of (5.15) we get

$$\begin{aligned}\sum_j \sum_{\ell\in N_j} g_{j\ell}(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j})u_j \;&=\; \sum_j\sum_{\ell\in N_j}(b_{j\ell}^+u_j + b_{j\ell}^-u_\ell)u_j\\ &\quad+\sum_{\mathcal{T}_R}\sum_{\ell\in N_{jR}}\big(b_{j\ell}^+\alpha_j^u G_j^u(x_{j\ell}-x_j)+b_{j\ell}^-\alpha_\ell^u G_\ell^u(x_{j\ell}-x_\ell)\big)u_j\\ &=: B_1+B_2\end{aligned}$$

Due to (3.13) we have $\sum_{\ell\in N_j}(b_{j\ell}^+ + b_{j\ell}^-)u_j^2 = u_j^2\sum_{\ell\in N_j} b_{j\ell} = 0$ for all j, and we can proceed, using also (3.14) and (3.3), as follows:

$$\begin{aligned}B_1 &= \sum_j\sum_{\ell\in N_j}\Big(b_{j\ell}^+u_j^2 + b_{j\ell}^-u_\ell u_j\Big) = \sum_j\sum_{\ell\in N_j}\Big(b_{j\ell}^+u_j^2+b_{j\ell}^-u_\ell u_j - \frac12(b_{j\ell}^+ + b_{j\ell}^-)u_j^2\Big)\\ &= \frac12\sum_j\sum_{\ell\in N_j}\Big(b_{j\ell}u_\ell + b_{j\ell}^+(u_j-u_\ell) - b_{j\ell}^-(u_j-u_\ell)\Big)u_j = \frac12\sum_j\sum_{N_{jI}}\cdots + \frac12\sum_j\sum_{N_{jG}}\cdots\\ &= \frac12\sum_{\mathcal{E}}(b_{j\ell}u_\ell u_j + b_{\ell j}u_j u_\ell)\\ &\quad+\frac12\sum_{\mathcal{E}}\Big(b_{j\ell}^+(u_j-u_\ell)u_j - b_{j\ell}^-(u_j-u_\ell)u_j - b_{j\ell}^-(u_\ell-u_j)u_\ell + b_{j\ell}^+(u_\ell-u_j)u_\ell\Big)\\ &\quad+\frac12\sum_j\sum_{N_{jG}}\cdots\\ &= 0+\frac12\sum_{\mathcal{E}}\Big(b_{j\ell}^+(u_j-u_\ell)^2 - b_{j\ell}^-(u_j-u_\ell)^2\Big) + \frac12\sum_j\sum_{N_{jG}}(-3b_{j\ell}^-u_j + b_{j\ell}^+u_j)u_j\\ &\ge \frac12\sum_{\mathcal{E}}\Big(b_{j\ell}^+(u_j-u_\ell)^2 - b_{j\ell}^-(u_j-u_\ell)^2\Big) = \frac12\sum_{\mathcal{E}}(b_{j\ell}^+ - b_{j\ell}^-)(u_j-u_\ell)^2\,. \end{aligned} \tag{5.17}$$

Now let us treat the term B_2.

$$\begin{aligned}B_2 &= \sum_{\mathcal{T}_R}\sum_{\ell\in N_{jR}}\Big(b_{j\ell}^+\alpha_j^uG_j^u(x_{j\ell}-x_j) + b_{j\ell}^-\alpha_\ell^uG_\ell^u(x_{j\ell}-x_\ell)\Big)u_j\\ &= \sum_{\mathcal{E}_R}\Big(\big(b_{j\ell}^+\alpha_j^uG_j^u(x_{j\ell}-x_j) + b_{j\ell}^-\alpha_\ell^uG_\ell^u(x_{j\ell}-x_\ell)\big)u_j\\ &\quad+\big(b_{\ell j}^+\alpha_\ell^uG_\ell^u(x_{\ell j}-x_\ell) + b_{\ell j}^-\alpha_j^uG_j^u(x_{\ell j}-x_j)\big)u_\ell\Big)\\ &= \sum_{\mathcal{E}_R}\Big(b_{j\ell}^+\alpha_j^uG_j^u(x_{j\ell}-x_j)(u_j-u_\ell) + b_{j\ell}^-\alpha_\ell^uG_\ell^u(x_{j\ell}-x_\ell)(u_j-u_\ell)\Big)\\ &\ge -\sum_{\mathcal{E}_R}\Big(b_{j\ell}^+\alpha_j^u|G_j^u(x_{j\ell}-x_j)||u_j-u_\ell| + |b_{j\ell}^-|\alpha_\ell^u|G_\ell^u(x_{j\ell}-x_\ell)||u_j-u_\ell|\Big)\,.\end{aligned}$$

Now since

$$\alpha_j^u|G_j^u(x_{j\ell}-x_j)| = \frac12\alpha_j^u|G_j^u(x_\ell-x_j)| = \frac12|(L_j^u(x_\ell)-u_j)| \le \frac12|u_\ell-u_j|$$

(see Lemma 3.5), we can continue

$$\ge -\frac12\sum_{\mathcal{E}_R}\Big(b_{j\ell}^+|u_j-u_\ell|^2 + |b_{j\ell}^-||u_j-u_\ell|^2\Big) = -\frac12\sum_{\mathcal{E}_R}(b_{j\ell}^+ - b_{j\ell}^-)(u_j-u_\ell)^2\,. \tag{5.18}$$

Finally, using the estimate $\sum_j f_ju_j|T_j| \le \frac{c_0}{2}\sum_j u_j^2|T_j| + c\sum_j f_j^2|T_j|$ and (5.15)–(5.18), we get (5.14). ∎

In the next lemma we give the basic local estimates for the exact solution v.

LEMMA 5.4. *For $v \in W^{2,2}(\Omega)$ the following estimates hold.*

a) For all $S_{j\ell}$, and for all $x, y \in T_j \cup T_\ell$ we have

$$|v(x) - v(y)| \le c\,h\,|v|_{W^{2,2}(Q_{j\ell})} + c|v|_{W^{1,2}(Q_{j\ell})}\,. \tag{5.19}$$

Here, $Q_{j\ell}$ is a rectangular domain containing $T_j \cup T_\ell$ and a finite fixed number of neighboring triangles. The number of these triangles is independent of h (for more details see the proof).

b) For all $S_{j\ell}$ we have

$$|\kappa_{j\ell}(v)| := \Big|\frac{v_\ell - v_j}{\overline{d_{j\ell}}} - \frac{1}{|S_{j\ell}|}\int_{S_{j\ell}} \partial_n v\Big| \le c\,|v|_{W^{2,2}(Q_j)}\,. \tag{5.20}$$

Here, Q_j is a finite union of triangles containing the rectangular $S_{j\ell} \times [\overline{x_{j\ell}}, \overline{x_j}]$.

c) For $S_{j\ell} \in \mathcal{E}_R$, and if the solution v moreover satisfies $v \in W^{3,2}(\Omega)$, we have

$$|\kappa_{j\ell}(v)| := \Big|\frac{v_\ell - v_j}{\overline{d_{j\ell}}} - \frac{1}{S_{j\ell}}\int_{S_{j\ell}} \partial_n v\Big| \le c\,h\,|v|_{W^{3,2}(T_j \cup T_\ell)}\,. \tag{5.21}$$

d) For all $T_j \in \mathcal{T}_h$ we have

$$\Big|\int_{T_j} \big(v(\overline{x_j}) - v(x)\big)\,dx\Big| \le c\,h^{3-\delta}\,\|v\|_{W^{2,2}(T_j)}\,. \tag{5.22}$$

for all $\delta > 0$.

Proof of a): Suppose $T_j \in \mathcal{T}_h$ and that either $T_\ell \in T_h$ (in this case we have $S_{j\ell} \in \mathcal{E}$) or $T_\ell \in \mathcal{T}_G$ (in this case we have $S_{j\ell} \in \mathcal{E}_B$). Without loss of generality we can suppose that there is a local cartesian coordinate system (x, y) such that the origin of this system coincides with $x_{j\ell}$, the x axis is aligned with the line through $\overline{x_\ell}$ and $\overline{x_j}$ and that the vertices of $S_{j\ell}$ are given by $(0, -\alpha)$ and $(0, \alpha)$. Let also $x = (x_1, x_2)$, $y = (y_1, y_2) \in T_j \cup T_\ell$. We write

$$v(x) - v(y) = \underbrace{v(x_1, x_2) - v(x_1, y_2)}_{=:R_1} + \underbrace{v(x_1, y_2) - v(y_1, y_2)}_{=:R_1}\,. \tag{5.23}$$

First we will estimate R_1. To this point, consider the rectangular $Q_j = [-3h, 3h] \times [-\alpha, \alpha]$ and a smooth cut-off 1D function ϕ such that $\phi = 0$ on $(-3h, -2h)$ and on $(2h, 3h)$, and $\phi = 1$ on $(-h, h)$, with $0 \le \phi \le 1$ and $|\phi'| \le \frac{c}{h}$. Then we have

$$\begin{aligned}
R_1 &= \int_{y_2}^{x_2} \partial_2 v(x_1, s)\,ds \\
&= \int_{y_2}^{x_2} \Big[\partial_2 v(x_1, s)\phi(x_1) - \partial_2 v(-3h, s)\phi(-3h)\Big]\,ds \\
&= \int_{y_2}^{x_2}\int_{-3h}^{x_1} \partial_t\big(\partial_2 v(t, s)\phi(t)\big)\,dt\,ds \\
&= \int_{y_2}^{x_2}\int_{-3h}^{x_1} \partial_t\partial_2 v(t, s)\phi(t)\,dt\,ds + \int_{y_2}^{x_2}\int_{-3h}^{x_1} \partial_2 v(t, s)\phi'(t)\,dt\,ds\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|R_1| &\le \int_{\min(x_2,y_2)}^{\max(x_2,y_2)}\int_{-3h}^{3h} \Big(\big|\partial_{12} v(t, s)\phi(t))\big| + \big|\partial_2 v(t, s)\phi'(t))\big|\Big)\,dt\,ds \\
&\le \int_{Q_j} |\partial_{12} v(t, s)|\,dt\,ds + \frac{c}{h}\int_{Q_j} |\partial_2 v(t, s)|\,dt\,ds \\
&\le ch|v|_{W^{2,2}(Q_j)} + |v|_{W^{1,2}(Q_j)}.
\end{aligned}$$

For R_2 we proceed in a similar way, using the cut-off function defined on $[-h,h]\times-[3\alpha,3\alpha]$.

Proof of b): Suppose $T_j\in\mathcal{T}_h$ and that either $T_\ell\in T_h$ (in this case we have $S_{j\ell}\in\mathcal{E}$) or $T_\ell\in\mathcal{T}_G$ (in this case we have $S_{j\ell}\in\mathcal{E}_B$). Consider the same local cartesian coordinate system as in case a). Then let $\overline{x_\ell}=:(a,0)$ and $\overline{x_j}=:(b,0)$, $a<b$. Using the notation $v_j:=v(\overline{x_j})$, $v_\ell:=v(\overline{x_\ell})$ we obtain in both cases, using $\gamma_{j\ell}=\frac{|S_{j\ell}|}{d_{j\ell}}=\frac{2\alpha}{b-a}$,

$$
\begin{aligned}
(v_\ell-v_j)\gamma_{j\ell}-&\int_{S_{j\ell}}\partial_n v\\
&=\big(v(a,0)-v(b,0)\big)\frac{2\alpha}{b-a}+\int_{-\alpha}^{\alpha}\partial_1 v(0,y)\,dy\\
&=\frac{2\alpha}{b-a}\int_b^a\partial_1 v(x,0)\,dx+\int_{-\alpha}^{\alpha}\partial_1 v(0,y)\,dy\\
&=\frac{1}{b-a}\int_{-\alpha}^{\alpha}\int_a^b-\partial_1 v(x,0)\,dx\,dy+\frac{1}{b-a}\int_a^b\int_{-\alpha}^{\alpha}\partial_1 v(0,y)\,dy\,dx\\
&=\frac{1}{b-a}\int_{-\alpha}^{\alpha}\int_a^b\Big[\partial_1 v(0,y)-\partial_1 v(x,y)+\partial_1 v(x,y)-\partial_1 v(x,0)\Big]\,dx\,dy\\
&=\frac{1}{b-a}\int_{-\alpha}^{\alpha}\int_a^b\Big[\int_x^0\partial_{11}v(t,y)\,dt+\int_0^y\partial_{12}v(x,s)\,ds\Big]\,dx\,dy.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\Big|(v_\ell-v_j)\gamma_{j\ell}-\int_{S_{j\ell}}\partial_n v\Big|&\le\frac{1}{b-a}\int_{-\alpha}^{\alpha}\int_a^b\int_a^b|\partial_{11}v(t,y)|\,dt\,dx\,dy\\
&+\frac{1}{b-a}\int_{-\alpha}^{\alpha}\int_a^b\int_{-\alpha}^{\alpha}|\partial_{12}v(x,s)|\,ds\,dx\,dy\\
&\le\int_{\tilde{Q}_j}|\partial_{11}v(x,s)|\,ds\,dx+\frac{2\alpha}{b-a}\int_{\tilde{Q}_j}|\partial_{12}v(x,s)|\,ds\,dx\\
&\le ch|v|_{W^{2,2}(Q_j)}
\end{aligned}
$$

where Q_j is a union of triangles which contains the rectangular $[a,b]\times[-\alpha,\alpha]$.

Proof of c): Since the constant $c>0$ should not depend on T_j, T_ℓ, h, we use the following transformation mapping. Let $T:=T_{j\ell}=T_j\cup T_\ell$ be a fixed rhombus in the computational domain and $\widehat{T}$ the reference rhombus with vertices $P_1=(1,0)$, $P_2=(0,1)$, $P_3=(-1,0)$, $P_4=(0,-1)$. Suppose $\widehat{T}$ is mapped onto T by a 1-1 mapping

$$
\begin{aligned}
&F:\widehat{T}\to T,\\
&F(\widehat{x})=A\widehat{x}+a,
\end{aligned}
$$

where $A\in\mathbb{R}^{2\times2}$ is an invertible matrix and $a\in\mathbb{R}^2$, such that

- the oriented segment S of length 2 connecting points P_3 and P_1 is mapped onto the oriented segment $S_{j\ell}$. I.e., denoting $\widehat{\xi}:=(2,0)^T$, we have $|A\widehat{\xi}|=|S_{j\ell}|$.
- if $n=n_{j\ell}=\frac{x_\ell-x_j}{|x_\ell-x_j|}$ is the outward (with respect to T_j) unit normal vector to $S_{j\ell}$, then

$$\widehat{n}\equiv\widehat{n}(\widehat{x}):=A^{-1}n(F(\widehat{x}))=A^{-1}n=\frac{A^{-1}x_\ell-A^{-1}x_j}{|x_\ell-x_j|}=\frac{\widehat{x}_\ell-\widehat{x}_j}{|x_\ell-x_j|}$$

and therefore

$$|A\widehat{n}|=1.$$

- The properties of A imply

$$\|A\| \le c\,h, \quad \|A^{-1}\| \le c\,h^{-1}, \quad |\det A|^{-1} \le c\,h^{-2}.$$

A map with these properties exists (see Heinrich [27, Section 5.2.2], and Ciarlet [11]), since all involved triangles are equilateral. Using the above notation, the following transformation rule holds:

$$\int_{S_{j\ell}} v\, d\sigma(x) = \int_S (v \circ F) \frac{|A\widehat{\xi}|}{|\widehat{\xi}|}\, d\sigma(\widehat{x}).$$

Furthermore for $\widehat{v}(\widehat{x}) := v(F(\widehat{x}))$. we have

$$\nabla_{\widehat{x}} \widehat{v}(\widehat{x}) = A^T \cdot \nabla v(x)\,, \qquad x = F(\widehat{x})\,,$$

and therefore

$$\begin{aligned}
\frac{1}{|S_{j\ell}|} \int_{S_{j\ell}} \partial_n v &= \frac{1}{|S_{j\ell}|} \int_{S_{j\ell}} n(x) \nabla v(x)\, d\sigma \\
&= \frac{1}{|A\widehat{\xi}|} \int_S A\widehat{n} (A^T)^{-1} \nabla_{\widehat{x}} \widehat{v} \frac{|A\widehat{\xi}|}{|\widehat{\xi}|}\, d\sigma(\widehat{x}) = \frac{A\widehat{n}}{|\widehat{\xi}|} \int_S (A^T)^{-1} \nabla_{\widehat{x}} \widehat{v}\, d\sigma(\widehat{x}).
\end{aligned}$$

We again denote $\kappa_{j\ell}(v) := \frac{v_\ell - v_j}{d_{j\ell}} - \frac{1}{|S_{j\ell}|} \int_{S_{j\ell}} \partial_n v$. (cf. (5.20)) Note that since $W^{2,2}$ is imbedded into $C^{0,\alpha}$ for any $\alpha \in (0,1)$, $\kappa_{j\ell}(v)$ is well defined. Hence we can define

$$H(\widehat{v}) := \frac{\widehat{v}(\widehat{x}_\ell) - \widehat{v}(\widehat{x}_j)}{|A(\widehat{x}_\ell - \widehat{x}_j)|} - \frac{A\widehat{n}(A^T)^{-1}}{|\widehat{\xi}|} \int_S \nabla_{\widehat{x}} \widehat{v}(\widehat{x})\, d\sigma(\widehat{x}).$$

We see that

$$H(\widehat{v}) = \kappa_{j\ell}(v).$$

Hence,

$$\begin{aligned}
|H(\widehat{v})| &\le \frac{|\widehat{v}(\widehat{x}_\ell) - \widehat{v}(\widehat{x}_j)|}{|A(\widehat{x}_\ell - \widehat{x}_j)|} + \frac{|A\widehat{n}||(A^T)^{-1}|}{|\widehat{\xi}|} \int_S |\nabla_{\widehat{x}} \widehat{v}|\, d\sigma(\widehat{x}) \\
&\le c\,h^{-1} |\widehat{v}|_{C(\widehat{T})} + c\,h^{-1} \|\nabla \widehat{v}\|_{L^1(S)} \\
&\le c\,h^{-1} \|\widehat{v}\|_{W^{2,2}(\widehat{T})} \le c\,h^{-1} \|\widehat{v}\|_{W^{k,2}(\widehat{T})}
\end{aligned}$$

where $k \ge 2$. Next we show that H vanishes for all polynomials p of degree $\le k$. Let us assume without restriction that the vortices of the two neighboring triangles which build the rhombus are, as before, given by $P_1 := (1,0), P_2 := (0,1), P_3 := (-1,0), P_4 := (0,-1)$. Let

$$p(y_1, y_2) = by_1^2 + cy_2^2 + dy_1 + ey_2 + f$$

and let $(x_1, 0)$, $(-x_1, 0)$ be the centers of gravity of the triangles $P_1P_2P_4$ and $P_2P_3P_4$, respectively. Then we have

$$\begin{aligned}
H(\widehat{p}) = \kappa_{j\ell}(p) &= \frac{p(x_1,0) - p(-x_1,0)}{|(x_1,0) - (-x_1,0)|} - \frac{1}{2} \int_{-\frac{1}{2}}^{-\frac{1}{2}} n \nabla p \\
&= \frac{2d \cdot x_1}{2 \cdot x_1} - \frac{1}{2} \int_{-\frac{1}{2}}^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} d \\ 2cy_2 + e \end{pmatrix} dy_2 \\
&= d - \frac{1}{2} \cdot 2d = 0.
\end{aligned}$$

Due to the properties of H we can use the Bramble–Hilbert lemma (see for example [11]) to obtain

$$|H(\widehat{v})| \le c\,\|H\|_{2,2}^{*}|\widehat{v}|_{W^{k,2}(\widehat{T})} \le c\,h^{-1}|\widehat{v}|_{W^{k,2}(\widehat{T})}\,.$$

Finally,

$$\begin{aligned}|\kappa_{j\ell}(v)| = |H(\widehat{v})| &\le c\,h^{-1}|\widehat{v}|_{W^{k,2}(\widehat{T})}\\ &\le c\,h^{-1}|\det A|^{-1/2}\|A\|^{k}|v|_{W^{k,2}(T)}\\ &\le c\,h^{-1}\,h^{-1}h^{k}|v|_{W^{k,2}(T)}\end{aligned}$$

which proves for $k = 2$ again the assertion b) of the Lemma (along singular edges) and for $k = 3$ the assertion c) of the Lemma.

Proof of d): Let $\overline{x_j}$ be the intersection of the perpendicular bisectors in T_j and

$$G(v) := \int_{T_j} v\,dx - v(\overline{x_j})|T_j|.$$

Since the last term is an integration formula which is exact for constant functions and since the imbedding

$$W^{1,2+r}(T_j) \hookrightarrow L^{\infty}(T_j)$$

exists for any $r > 0$, we can apply [20, Satz 7.4] or [1, Lemma 4.3.8] and obtain

$$G(v) \le ch^{3-\frac{2}{2+r}}|v|_{W^{1,2+r}(T_j)} \le ch^{3-\delta}||v||_{W^{2,2}(T_j)}$$

for arbitrary $\delta > 0$.
This proves d). □

LEMMA 5.5. *As a consequence, we have for all $v \in W^{2,2}(\Omega)$, $T_j \in \mathcal{T}_R$ and $E(v) := \sum_{\ell\in N_j}(v_\ell - v_j)$,*

$$|E(v)| \le c\,h\,|v|_{W^{2,2}(\widetilde{T}_j)}\,, \tag{5.24}$$

where $\widetilde{T}_j := T_j \cup \Big(\bigcup_{\ell\in N_j} T_\ell\Big)$. For $T_j \in \mathcal{T}_S$, we immediately get by (5.19)*,*

$$|E(v)| \le c\,h\,|v|_{W^{2,2}(Q_j)} + c|v|_{W^{1,2}(Q_j)}. \tag{5.25}$$

Here, Q_j is a rectangular domain containing $\widetilde{T}_j$ and a finite fixed number of neighboring triangles (cf. (5.19)*).*

Proof:

$$E(v) = E(v_j) = \sum_{\ell\in N_j}(v_\ell - v_j) = (v_\ell - v_j) + (v_k - v_j) + (v_m - v_j)\,. \tag{5.26}$$

Consider now $T := T_j \cup T_\ell \cup T_k \cup T_m$ and a reference triangle $\hat{T} = \hat{T}_1 \cup \hat{T}_2 \cup \hat{T}_3 \cup \hat{T}_4$, such that $\hat{T}$ is mapped onto T by a 1-1 mapping

$$\begin{aligned}&F : \hat{T} \to T,\\ &F(\hat{x}) = A\hat{x} + a\end{aligned}$$

and $F(\hat{T}_i) = T_{p_i}$ for $i = 1, 2, 3, 4$, $p_i = j, \ell, k, m$, respectively. As in the proof of Lemma 5.4, c) $A \in \mathbb{R}^{2\times 2}$ is an invertible matrix and $a \in \mathbb{R}^2$. Define

$$\begin{aligned}&\hat{E} : W^{2,2}(\hat{T}) \to \mathbb{R}\,,\\ &\hat{E}(\hat{v}) = (\hat{v}_\ell - \hat{v}_j) + (\hat{v}_k - \hat{v}_j) + (\hat{v}_m - \hat{v}_j)\,.\end{aligned}$$

Then

$$|\hat{E}(\hat{v})| \le c\,\|\hat{v}\|_{C(\hat{T})} \le c\,\|\hat{v}\|_{W^{2,2}(\hat{T})}\,.$$

Moreover, $\hat{E}$ vanishes for all polynomials $\hat{p}$ of degree ≤ 1. Indeed, we have for such $\hat{p}$ that

$$\hat{E}(\hat{p}) = \nabla\hat{p}\big[(\hat{x}_\ell - \hat{x}_j) + (\hat{x}_k - \hat{x}_j) + (\hat{x}_m - \hat{x}_j)\big] = 0\,.$$

Using then the Bramble–Hilbert lemma, one gets

$$|\hat{E}(\hat{v})| \le c\,|\hat{v}|_{W^{2,2}(\hat{T})}\,.$$

Finally,

$$|E(v)| = |\hat{E}(\hat{v})| \le c\,|\hat{v}|_{W^{2,2}(\hat{T})} \le c\,h\,|v|_{W^{2,2}(T)}. \tag{5.27}$$

□

6. The basic strategy in proving the main result. The main technical step in the whole proof is to consider the term $(L_h(I_hv) - L_hu_h, z_h) := \sum_j (L_h(I_hv) - L_hu_h)_j|T_j|z_j$ in the following form.

Lemma 6.1.

$$(L_h(I_hv) - L_hu_h, z_h) = (\Psi_H, z_h) + (\Psi_K, z_h) + (\Psi_N, z_h)\,, \tag{6.1}$$

where

$$\Psi_{Hj} = -\frac{\varepsilon}{|T_j|}\sum_{\ell\in N_j}|S_{j\ell}|\Big(\frac{v_\ell - v_j}{\overline{d_{j\ell}}} - \frac{1}{|S_{j\ell}|}\int_{S_{j\ell}}\partial_n v\Big)\,,$$

$$\Psi_{Kj} = \frac{1}{|T_j|}\sum_{\ell\in N_j}\Big(g_{j\ell}(\mathcal{V}_{j\ell},\mathcal{V}_{\ell j}) - \int_{S_{j\ell}} n_{j\ell}bv\Big)\,,$$

$$\Psi_{Nj} = \frac{1}{|T_j|}\int_{T_j} c(v_j - v)$$

where $\mathcal{V}_{j\ell} = \mathcal{U}_{j\ell}(v_j, v_\ell)$, $\mathcal{V}_{\ell j} = \mathcal{U}_{\ell j}(v_\ell, v_j)$, *and*

$$(\Psi_A, z_h) := \sum_j \Psi_{Aj} z_j |T_j|, \quad \textit{for} \quad A = H, K, N.$$

Proof: We have due to (3.4) and (4.5),

$$(L_hI_hv)_j = -\frac{\varepsilon}{|T_j|}\sum_{\ell\in N_j}|S_{j\ell}|\left(\frac{v_\ell - v_j}{\overline{d_{j\ell}}}\right) + \frac{1}{|T_j|}\sum_{\ell\in N_j} g_{j\ell}(\mathcal{V}_{j\ell},\mathcal{V}_{\ell j}) + \frac{1}{|T_j|}\int_{T_j} cv_j.$$

Moreover,

$$(L_hu_h)_j = f_j = \frac{1}{|T_j|}\int_{T_j} f = \frac{1}{|T_j|}\int_{T_j} Lv\,,$$

while (2.1) implies

$$\frac{1}{|T_j|}\int_{T_j} Lv = -\frac{\varepsilon}{|T_j|}\sum_{\ell\in N_j}\int_{S_{j\ell}}\partial_n v + \frac{1}{|T_j|}\sum_{\ell\in N_j}\int_{S_{j\ell}} bvn_{j\ell} + \frac{1}{|T_j|}\int_{T_j} cv\,,$$

which completes the proof. □

In what follows, we split the sums in (6.1) into two parts,

$$\sum_j = \sum_{T_j\in\mathcal{T}_R} + \sum_{T_j\in\mathcal{T}_S} .$$

In the "regular" part of the sum we have regular triangles and the higher order approximation, in the "singular" part of the sum ("on the strips") we have general triangles and the first order approximation. We thus get

$$(L_h(I_h v) - L_h u_h, z_h) = (\Psi_H, z_h)_R + (\Psi_H, z_h)_S + (\Psi_K, z_h)_R + (\Psi_K, z_h)_S + (\Psi_N, z_h)_R + (\Psi_N, z_h)_S$$

and will proceed by estimating the terms on the right-hand side both from above and from below.

7. The estimates from above.

7.1. The estimate of (Ψ_N, z_h) from above. For the estimate of (Ψ_N, z_h) (approximating the zero-order term) from above we obtain the following results.

LEMMA 7.1. *We have on the regular triangles*

$$\sum_{T_j\in\mathcal{T}_R} |\Psi_{Nj}|^2|T_j| \le ch^4||v||^2_{2,2,R}\,, \tag{7.1}$$

$$(\Psi_N, z_h)_R \le ch^4\|v\|^2_{2,2,R} + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_R} z_j^2|T_j| \tag{7.2}$$

where $\|v\|^2_{2,2,R} := \sum_{T_j\in\mathcal{T}_R}\|v\|^2_{W^{2,2}(T_j)}$.

Proof: Using [1, Lemma 4.3.8] we obtain that $\int_{T_j}(v(\overline{x_j}) - v(x))dx \le ch^3|v|_{W^{2,2}(T_j)}$ on the regular part $\mathcal{T}_R$ of the grid, and therefore

$$\begin{aligned}
\Psi_{Nj} &= \frac{1}{|T_j|}\int_{T_j}(v(\overline{x_j}) - v(x))c(x)\,dx\\
&= \frac{1}{|T_j|}\Big(c(\overline{x_j})\int_{T_j}\big(v(\overline{x_j}) - v(x)\big)dx + \int_{T_j}\big(v(\overline{x_j}) - v(x)\big)\big(c(x) - c(\overline{x_j})\big)\,dx\Big)\\
&\le ch||v||_{W^{2,2}(T_j)} + \sup_{x\in T_j}\big|v(\overline{x_j}) - v(x)\big|\,\frac{1}{|T_j|}\int_{T_j}\big|c(x) - c(\overline{x_j})\big|\,dx\\
&\le ch||v||_{W^{2,2}(Q_{j\ell})}
\end{aligned}$$

using (5.19) and the smoothness of $c(x)$. Then (7.1) follows, since

$$\sum_{T_j\in\mathcal{T}_R}|\Psi_{Nj}|^2|T_j| \le c\sum_{T_j\in\mathcal{T}_R}h^2\|v\|^2_{W^{2,2}(Q_{j\ell})}|T_j| \le ch^4\|v\|^2_{2,2,R}.$$

Now this implies

$$\begin{aligned}
(\Psi_N, z_h)_R &\le \sum_{T_j\in\mathcal{T}_R}|\Psi_{Nj}|\,|z_j|\,|T_j| \le c\sum_{T_j\in\mathcal{T}_R}|\Psi_{Nj}|^2\,|T_j| + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_R}|T_j||z_j|^2\\
&\le ch^4\|v\|^2_{2,2,R} + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_R}|T_j||z_j|^2
\end{aligned}$$

which is (7.2). □

Lemma 7.2. *We have on singular triangles*

$$\sum_{T_j \in \mathcal{T}_S} |\Psi_j|^2 |T_j| \leq ch^{4-2\delta} ||v||^2_{2,2,S} \tag{7.3}$$

$$(\Psi_N, z_h)_S \leq ch^{4-2\delta} \|v\|^2_{2,2,S} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_S} z_j^2 |T_j| \tag{7.4}$$

for any $\delta > 0$, *where* $\|v\|^2_{2,2,S} := \sum_{T_j \in \mathcal{T}_S} \|v\|^2_{W^{2,2}(T_j)}$.

Proof: Similarly as in the proof of Lemma 7.1 we have

$$\Psi_{Nj} = \frac{1}{|T_j|} \Big(c(x_j) \int_{T_j} \big(v(\overline{x_j}) - v(x)\big) dx + \int_{T_j} \big(v(x_j) - v(x)\big)\big(c(x) - c(\overline{x_j})\big) dx \Big).$$

The second term can be estimated as in the proof of Lemma 7.1. For the first one we use (5.22) and obtain

$$\begin{aligned}\Psi_{Nj} &= \frac{1}{|T_j|} \Big(c(x_j) \int_{T_j} (v(\overline{x_j}) - v(x)) dx + \int_{T_j} (v(\overline{x_j}) - v(x))(c(x) - c(\overline{x_j}) dx \Big). \\ &\leq c\, h^{1-\delta} \, \|v\|_{W^{2,2}(Q_{j\ell})} \, .\end{aligned}$$

This implies

$$\sum_S |\psi_{Nj}|^2 |T_j| \leq c \sum_S h^{2(1-\delta)} ||v||^2_{W^{2,2}(Q_{j\ell})} |T_j| \leq ch^{4-2\delta} \|v\|^2_{2,2,S}$$

and therefore, as before,

$$\begin{aligned}(\Psi_N, z_h)_S &\leq c \sum_{T_j \in \mathcal{T}_S} |\Psi_{Nj}|^2 \, |T_j| + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_S} |T_j| |z_j|^2 \\ &\leq ch^{4-2\delta} \|v\|^2_{2,2,S} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_S} |T_j| |z_j|^2.\end{aligned}$$

This finishes the proof of the lemma. □

Putting the results of Lemmata 7.1, 7.2 together, we obtain

Lemma 7.3. *We have (on the whole domain)*

$$(\Psi_N, z_h) \leq ch^{4-2\delta} \|v\|^2_{2,2} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_h} z_j^2 |T_j| \, . \tag{7.5}$$

7.2. The estimates of (Ψ_H, z_h) **from above.** For the estimate of (Ψ_H, z_h) (approximating the diffusion part) from above we obtain the following result.

Lemma 7.4. *For* $v \in W^{2,2}(\Omega)$ *we have*

$$(\Psi_H, z_h) \leq c \, \varepsilon \, h \, \|v\|_{2,2} \Big(\sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell) \Big)^{1/2} \tag{7.6}$$

$$\leq c \, \varepsilon \, h^2 \, \|v\|^2_{2,2} + \frac{\varepsilon \gamma}{8} \sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell)^2 \, . \tag{7.7}$$

If moreover $v \in W^{3,2}(\Omega)$, *we have*

$$(\Psi_H, z_h) \le c\,\varepsilon\, h^3\, \|v\|^2_{3,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 \,. \tag{7.8}$$

Proof: We keep using the notation $\kappa_{j\ell} := \kappa_{j\ell}(v) := \frac{v_\ell - v_j}{d_{j\ell}} - \frac{1}{|S_{j\ell}|}\int_{S_{j\ell}} \partial_n v$, (cf. (5.20)). Note that $\kappa_{j\ell} = -\kappa_{\ell j}$. We have

$$(\Psi_H, z_h) = \sum_j \Psi_{Hj} z_j |T_j| = -\varepsilon \sum_j \sum_{N_{jI}} \kappa_{j\ell}|S_{j\ell}| z_j - \varepsilon \sum_j \sum_{N_{jG}} \kappa_{j\ell}|S_{j\ell}| z_j \,. \tag{7.9}$$

Now,

$$\begin{aligned} -\varepsilon \sum_j \sum_{N_{jI}} \kappa_{j\ell}|S_{j\ell}| z_j \;&=\; -\varepsilon \sum_{\mathcal{E}} |S_{j\ell}|\big(\kappa_{j\ell} z_j + \kappa_{\ell j} z_\ell\big) = \varepsilon \sum_{\mathcal{E}} |S_{j\ell}|\kappa_{j\ell}\big(z_\ell - z_j\big) \qquad (7.10)\\ &\le\; c\,\varepsilon\, h \Big(\sum_{\mathcal{E}} |\kappa_{j\ell}|^2\Big)^{1/2} \Big(\sum_{\mathcal{E}} (z_\ell - z_j)^2\Big)^{1/2} \\ &\overset{(5.20)}{\le} c\,\varepsilon\, h \Big(\sum_{\mathcal{E}} |v|^2_{W^{2,2}(Q_j)}\Big)^{1/2} \Big(\sum_{\mathcal{E}} (z_\ell - z_j)^2\Big)^{1/2} \\ &\le\; c\,\varepsilon\, h |v|_{W^{2,2}(\Omega)} \Big(\sum_{\mathcal{E}} (z_\ell - z_j)^2\Big)^{1/2}, \\ -\varepsilon \sum_j \sum_{N_{jG}} \kappa_{j\ell}|S_{j\ell}| z_j \;&=\; -\frac{\varepsilon}{2} \sum_j \sum_{N_{jG}} \kappa_{j\ell}|S_{j\ell}|(z_j - z_\ell) = -\frac{\varepsilon}{2} \sum_{\mathcal{E}_B} \kappa_{j\ell}|S_{j\ell}|(z_j - z_\ell)\,, \end{aligned}$$

the last-but-one equality holding due to (4.9). The last term can now be estimated as in (7.10) and (7.6) follows.

In order to prove (7.8) we consider $(\Psi_H, z_h) = (\Psi_H, z_h)_R + (\Psi_H, z_h)_S$. The second term can be estimated as in (7.10). For the first one we use (5.21) and proceed as in (7.10):

$$\begin{aligned} (\Psi_H, z_h) &= (\Psi_H, z_h)_R + (\Psi_H, z_h)_S \\ &\le c\,\varepsilon\, h^4\, \|v\|^2_{3,2} + \frac{\varepsilon\gamma}{16} \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 \;+ c\,\varepsilon\, h^2\, \|v\|^2_{2,2,S} + \frac{\varepsilon\gamma}{16} \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 \,. \\ &\le c\,\varepsilon\, h^4\, \|v\|^2_{3,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 \;+ c\,\varepsilon\, h^3\, \|v\|^2_{3,2,\Omega}\,, \end{aligned}$$

using (2.5), and the proof follows. □

7.3. The estimates of (Ψ_K, z_h) from above. For (Ψ_K, z_h) (approximating the convective term) we obtain the following lemma.

LEMMA 7.5. *We have*

$$(\Psi_K, z_h)_R \le c\,\frac{h^3}{\varepsilon}\|v\|^2_{2,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2. \tag{7.11}$$

Proof: Considering first $S_{j\ell} \in \mathcal{E}_R$, we use that in this case $g_{j\ell}(\mathcal{V}_{j\ell}, \mathcal{V}_{\ell j}) = g_{j\ell}(L^v_j(x_{j\ell}), L^v_\ell(x_{j\ell}))$ to write

$$\begin{aligned} g_{j\ell}(L^v_j(x_{j\ell}), L^v_\ell(x_{j\ell})) \;-\; &\int_{S_{j\ell}} n_{j\ell} b v \\ &= \int_{S_{j\ell}} (n_{j\ell}b)^+\big(L^v_j(x_{j\ell}) - v\big) + \int_{S_{j\ell}} (n_{j\ell}b)^-\big(L^v_\ell(x_{j\ell}) - v\big) \\ &=: T + U\,. \end{aligned} \tag{7.12}$$

We consider T only, the term U can be treated similarly. We have

$$\begin{aligned} T &= \Big(n_{j\ell} b(x_{j\ell})\Big)^+ \int_{S_{j\ell}} \big(L_j^v(x_{j\ell}) - v\big) + \int_{S_{j\ell}} \Big[(n_{j\ell} b)^+ - \Big(n_{j\ell} b(x_{j\ell})\Big)^+\Big] \big(L_j^v(x_{j\ell}) - v\big) \\ &=: T_1 + T_2\,, \end{aligned} \tag{7.13}$$

with $L_j^v(x_{j\ell}) = v_j + \alpha_j^v G_j^v(x_{j\ell} - x_j)$.
First let us consider the case $\alpha_j^v = 1$. Estimating T_1, we will consider two cases. Firstly, if the stencil of G_j^v contains ℓ, we get

$$L_j^v(x_{j\ell}) - v = v_j + (v_{j\ell} - v_j) - v = v_{j\ell} - v\,.$$

Then,

$$\int_{S_{j\ell}} (v_{j\ell} - v)\, d\sigma(x) = \int_S (\hat{v}_{j\ell} - \hat{v}(\hat{x})) \frac{|A\hat{\xi}|}{|\hat{\xi}|}\, d\sigma(\hat{x}) =: H(\hat{v})$$

for $\hat{v} \in W^{2,2}(\hat{T})$, using the same mapping $F : \hat{T} \to T$ as in the proof of Lemma 5.4. It follows that

$$H(\hat{p}) = 0 \quad \text{for all polynomials } \hat{p} \text{ of degree} \leq 1$$

and

$$|H(\hat{v})| \leq c\, h\, \|\hat{v}\|_{C(\hat{T})} \leq c\, h\, \|\hat{v}\|_{W^{2,2}(\hat{T})}\,.$$

Therefore, using the Bramble–Hilbert lemma,

$$|H(\hat{v})| \leq c\, h\, |\hat{v}|_{W^{2,2}(\hat{T})} \leq c\, h^2\, |v|_{W^{2,2}(T)}\,.$$

Consequently,

$$|T_1| \leq c\, h^2 |v|_{W^{2,2}(T)}\,.$$

Secondly, but still for $\alpha_j^v = 1$, if the stencil of G_j^v does not contain ℓ, we have

$$\begin{aligned} L_j^v(x_{j\ell}) - v &= v_j - G_j^v\Big((x_{jk} - x_j) + (x_{jm} - x_j)\Big) - v \\ &= (v_j - v_{jk}) + (v_j - v_{jm}) + (v_j - v)\,. \end{aligned}$$

Using the same transformation as before, we can show that

$$H(\hat{v}) := \int_S \Big((\hat{v}_j - \hat{v}_{jk}) + (\hat{v}_j - \hat{v}_{jm}) + (\hat{v}_j - \hat{v}(\hat{x}))\Big) \frac{|A\hat{\xi}|}{|\hat{\xi}|}\, d\sigma(\hat{x})$$

satisfies

$$|H(\hat{v})| \leq c\, h\, \|\hat{v}\|_{W^{2,2}(\hat{T})}\,,$$

while for all polynomials $\hat{p}$ of degree ≤ 1,

$$\begin{aligned} H(\hat{p}) &= \int_{S_{j\ell}} \big(\nabla p(x_j - x_{jk}) + \nabla p(x_j - x_{jm}) + p(x_j) - p(x)\big)\, d\sigma(x) \\ &= \int_{S_{j\ell}} \big(p(x_{j\ell}) - p(x)\big)\, d\sigma(x) = 0\,. \end{aligned}$$

Therefore, by the same argument as before,

$$|T_1| \leq c\, h^2 |v|_{W^{2,2}(T)}\,.$$

Finally, for general α_j^v we obtain

$$\begin{aligned} T_1 &= \Big(n_{j\ell} b(x_{j\ell})\Big)^+ \int_{S_{j\ell}} \big(L_j^v(x_{j\ell}) - v\big) \\ &= \Big(n_{j\ell} b(x_{j\ell})\Big)^+ \int_{S_{j\ell}} \big(v_j + \alpha_j^v G_j^v(x_{j\ell} - x_j) - v\big) \\ &\le \Big(n_{j\ell} b(x_{j\ell})\Big)^+ \int_{S_{j\ell}} \Big(v_j + G_j^v(x_{j\ell} - x_j) - v\Big) + \Big(n_{j\ell} b(x_{j\ell})\Big)^+ \int_{S_{j\ell}} |\alpha_j^v - 1|\, |G_j^v(x_{j\ell} - x_j)|\,. \end{aligned} \tag{7.14}$$

The integrand of the first integral corresponds to the case of $\alpha_j^v = 1$ studied above and therefore the first integral can be estimated by $ch^2|v|_{W^{2,2}(T)}$. The second integral can be estimated as follows using (3.21):

$$\int_{S_{j\ell}} |\alpha_j^v - 1||G_j^v(x_{j\ell} - x_j)| \le \frac{1}{2}\int_{S_{j\ell}} |\alpha_j^v - 1||v_\ell - v_j| \le \frac{1}{2}\int_{S_{j\ell}} |\alpha_j^v - 1|\Big|\sum_{l\in N_j}(v_\ell - v_j)\Big|$$

since $\alpha_j^v - 1 \neq 0$ implies that there is a local extremum in v_j. We then continue

$$\int_{S_{j\ell}} |\alpha_j^v - 1||G_j^v(x_{j\ell} - x_j)| \le c\int_{S_{j\ell}} |E(v_j)| \le c\, h^2 |v|_{W^{2,2}(\widetilde{T}_j)}$$

where we used (5.24). We conclude that in any case we have using (3.20)

$$|T_1| \le c\, h^2 \|v\|_{W^{2,2}(\widetilde{T}_j)}\,.$$

The term T_2 in (7.13) can be estimated as follows:

$$|T_2| \le c\,|S_{j\ell}|\, h \sup_{S_{j\ell}} |L_j^v(x_{j\ell}) - v| \le ch^2\|v\|_{W^{2,2}(T)}. \tag{7.15}$$

The term U in (7.12) can be treated in the same way. This finishes the first part of the proof.

Now we will consider the general situation for any $S_{j\ell}$. We have

$$\begin{aligned} (\Psi_K, z_h) &= \sum_j \sum_{N_j} \Big(g_{j\ell}(\mathcal{V}_{j\ell}, \mathcal{V}_{\ell j}) - \int_{S_{j\ell}} n_{j\ell} b v\Big) z_j \\ &= \sum_{\mathcal{T}_R}\sum_{N_j} \cdots + \sum_{\mathcal{T}_S}\sum_{N_j} \cdots \\ &= \sum_{\mathcal{T}_R}\sum_{N_{jR}} \cdots + \sum_{\mathcal{T}_R}\sum_{N_{jS}} \cdots + \sum_{\mathcal{T}_R}\sum_{N_{jG}} \cdots \\ &\quad + \sum_{\mathcal{T}_S}\sum_{N_{jR}} \cdots + \sum_{\mathcal{T}_S}\sum_{N_{jS}} \cdots + \sum_{\mathcal{T}_S}\sum_{N_{jG}} \cdots\,. \end{aligned} \tag{7.16}$$

Using the first part of the proof, we get for the first case

$$\begin{aligned} \sum_{\mathcal{T}_R}\sum_{N_{jR}} \cdots &= \sum_{\mathcal{T}_R}\sum_{N_{jR}} \Big(g_{j\ell}(L_j^v(x_{j\ell}), L_\ell^v(x_{j\ell})) - \int_{S_{j\ell}} n_{j\ell} b v\Big) z_j \\ &= \sum_{\mathcal{E}_R} \Big(g_{j\ell}(L_j^v(x_{j\ell}), L_\ell^v(x_{j\ell})) - \int_{S_{j\ell}} n_{j\ell} b v\Big)(z_j - z_\ell) \\ &\le c \sum_{\mathcal{E}_R} h^2 \|v\|_{W^{2,2}(T_{j\ell})}(z_j - z_\ell) \\ &\le c\,\frac{h^4}{\varepsilon}\,\|v\|^2_{W^{2,2}(\Omega)} + \frac{\varepsilon\gamma}{8}\sum_{\mathcal{E}_R}(z_j - z_\ell)^2. \end{aligned} \tag{7.17}$$

Secondly,

$$\begin{aligned}\sum_{\mathcal{T}_R}\sum_{N_{jS}}\cdots+\sum_{\mathcal{T}_S}\sum_{N_{jR}}\cdots &= \sum_{\mathcal{T}_R}\sum_{N_{jS}}\Big(g_{j\ell}(v_j,v_\ell)-\int_{S_{j\ell}} n_{j\ell}bv\Big)z_j \\ &\quad+\sum_{\mathcal{T}_S}\sum_{N_{jR}}\Big(g_{j\ell}(v_j,v_\ell)-\int_{S_{j\ell}} n_{j\ell}bv\Big)z_j \\ &= \sum_{\mathcal{E}_M}\Big(g_{j\ell}(v_j,v_\ell)-\int_{S_{j\ell}} n_{j\ell}bv\Big)(z_j-z_\ell)\end{aligned} \tag{7.18}$$

Since $g_{j\ell}(v_j,v_\ell)=b^+_{j\ell}v_j+b^-_{j\ell}v_\ell$ for $S_{j\ell}\in\mathcal{E}_M$, we proceed by estimating

$$\begin{aligned}&\Big|b^+_{j\ell}v_j+b^-_{j\ell}v_\ell-\int_{S_{j\ell}} n_{j\ell}bv\Big| \\ &\le\Big|\int_{S_{j\ell}}(bn_{j\ell})^+\underbrace{(v_j-v)}_{\text{use (5.19)}}+(bn_{j\ell})^-\underbrace{(v_\ell-v)}_{\text{use (5.19)}}\Big| \\ &\le c\int_{S_{j\ell}}|bn_{j\ell}|(h\|v\|_{W^{2,2}(Q_{j\ell})}+\|v\|_{W^{1,2}(Q_{j\ell})}) \\ &\le c\,h^2\,\|v\|_{W^{2,2}(Q_{j\ell})}+c\,h\,\|v\|_{W^{1,2}(Q_{j\ell})}\,,\end{aligned} \tag{7.19}$$

and continue in (7.18) to obtain

$$\begin{aligned}&\sum_{\mathcal{T}_R}\sum_{N_{jS}}\cdots+\sum_{\mathcal{T}_S}\sum_{N_{jR}}\cdots \\ &\le c\sum_{\mathcal{E}_M}h^2\|v\|_{W^{2,2}(Q_{j\ell})}(z_j-z_\ell)+c\sum_{\mathcal{E}_M}h\|v\|_{W^{1,2}(Q_{j\ell})}(z_j-z_\ell) \qquad (7.20)\\ &\le c\frac{h^4}{\varepsilon}\|v\|^2_{W^{2,2}(\mathcal{T}_S)}+c\frac{h^2}{\varepsilon}\|v\|^2_{W^{1,2}(\mathcal{T}_S)}+\frac{\varepsilon\gamma}{8}\sum_{\mathcal{E}_M}(z_j-z_\ell)^2\,, \qquad (7.21)\end{aligned}$$

and, since according to (2.5) we have $\|v\|^2_{W^{1,2}(\mathcal{T}_S)}\le ch\|v\|^2_{W^{2,2}(\Omega)}$, we obtain in this case finally

$$\sum_{\mathcal{T}_R}\sum_{N_{jS}}\cdots+\sum_{\mathcal{T}_S}\sum_{N_{jR}}\cdots\le c\frac{h^3}{\varepsilon}\|v\|^2_{2,2}+\frac{\varepsilon\gamma}{8}\sum_{\mathcal{E}_M}(z_j-z_\ell)^2\,. \tag{7.22}$$

Thirdly, we consider

$$\begin{aligned}\sum_{\mathcal{T}_S}\sum_{N_{jS}}\cdots &= \sum_{\mathcal{T}_S}\sum_{N_{jS}}\Big(g_{j\ell}(v_j,v_\ell)-\int_{S_{j\ell}} n_{j\ell}bv\Big)z_j \\ &= \sum_{\mathcal{E}_S}\Big(g_{j\ell}(v_j,v_\ell)-\int_{S_{j\ell}} n_{j\ell}bv\Big)(z_j-z_\ell)\end{aligned} \tag{7.23}$$

and proceed as in (7.19), (7.22) to get

$$\sum_{\mathcal{T}_S}\sum_{N_{jS}}\cdots\le c\frac{h^3}{\varepsilon}\|v\|^2_{2,2}+\frac{\varepsilon\gamma}{8}\sum_{\mathcal{E}_S}(z_j-z_\ell)^2\,. \tag{7.24}$$

Finally we will estimate the terms with the ghost cells. We have

$$\sum_{\mathcal{T}_R}\sum_{N_{jG}}\cdots = \sum_{\mathcal{T}_R}\sum_{N_{jG}}\Big(g_{j\ell}(v_j,v_\ell) - \int_{S_{j\ell}} n_{j\ell}bv\Big)z_j$$
$$= \frac{1}{2}\sum_{\mathcal{T}_R}\sum_{N_{jG}}\Big(g_{j\ell}(v_j,v_\ell) - \int_{S_{j\ell}} n_{j\ell}bv\Big)(z_j - z_\ell)$$

and therefore we can proceed as before to get $c\frac{h^3}{\varepsilon}\|v\|_{2,2}^2 + \frac{\varepsilon\gamma}{8}\sum_{\mathcal{E}_B}(z_j - z_\ell)^2$. The other terms in (7.16) can be treated similarly, and the result follows. ☐

7.4. The final estimate from above. Putting together the estimates (7.5), (7.7), (7.8), (7.11), we get the following result.

THEOREM 7.6. **(Estimate from above)** *Under the assumptions of Theorem 4.1 there exists a constant $c > 0$ independent of ε such that for $v \in W^{2,2}(\Omega)$ we have*

$$(L_h I_h v - L_h u_h, z_h) \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 + \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}\cup\mathcal{E}_B}(z_j - z_\ell)^2 + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_h} z_j^2|T_j|\,. \tag{7.25}$$

If moreover $v \in W^{3,2}(\Omega)$, we have

$$(L_h I_h v - L_h u_h, z_h) \le c\Big(\varepsilon h^3 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{3,2}^2 + \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}\cup\mathcal{E}_B}(z_j - z_\ell)^2 + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_h} z_j^2|T_j|\,. \tag{7.26}$$

8. The estimates from below. In this part of the paper we will prove an estimate from below.

THEOREM 8.1. **(Estimate from below)** *Under the assumptions of Theorem 4.1 there exists a constant $c > 0$ independent of ε such that for $v \in W^{2,2}(\Omega)$ we have*

$$\begin{aligned}(L_h I_h v &- L_h u_h, z_h)\\ &\ge \frac{\varepsilon\gamma}{2}\sum_{\mathcal{E}}(z_\ell - z_j)^2 + 2\varepsilon\gamma\sum_{\mathcal{E}_B} z_j^2 + c_0\sum_j z_j^2|T_j| + \frac{1}{2}\sum_{\mathcal{E}_M\cup\mathcal{E}_S}(b_{j\ell}^+ - b_{j\ell}^-)(z_j - z_\ell)^2\\ &\quad - c\,\frac{h^4}{\varepsilon}\|v\|_{W^{2,2}(\Omega)}^2 - c\,\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_R} R_j^2|T_j|\end{aligned} \tag{8.1}$$

where $\gamma = \min\gamma_{j\ell}$ and $R_j := \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}$.

Proof: Using the notation $\sum_j \equiv \sum_{T_j\in\mathcal{T}_h}$ and the definition of L_h one can write

$$(L_h I_h v - L_h u_h, z_h) = A_1 + A_2 + A_3$$

where, denoting as before $\mathcal{V}_{j\ell} = \mathcal{U}_{j\ell}(v_j, v_\ell)$, $\mathcal{V}_{\ell j} = \mathcal{U}_{\ell j}(v_\ell, v_j)$,

$$\begin{aligned}A_1 &= \sum_j\Big(-\varepsilon\sum_{\ell\in N_j}(v_\ell - v_j)\gamma_{j\ell} + \varepsilon\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}\Big)z_j \ge \varepsilon\gamma\sum_{\mathcal{E}}(z_\ell - z_j)^2 + 2\varepsilon\gamma\sum_{\mathcal{E}_B} z_j^2,\\ A_2 &= \sum_j\sum_{\ell\in N_j}\Big(g_{j\ell}(\mathcal{V}_{j\ell},\mathcal{V}_{\ell j})) - g_{j\ell}(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j})\Big)z_j\,,\\ A_3 &= \sum_j c_j(v_j - u_j)z_j|T_j| \ge c_0\sum_j z_j^2|T_j|.\end{aligned} \tag{8.2}$$

It remains to treat A_2. Recall that

$$g_{j\ell}(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j}) = \begin{cases} b^+_{j\ell} L^u_j(x_{j\ell}) + b^-_{j\ell} L^u_\ell(x_{j\ell}) & \text{for } S_{j\ell} \in \mathcal{E}_R\,, \\ \\ b^+_{j\ell} u_j + b^-_{j\ell} u_\ell & \text{for } S_{j\ell} \in \mathcal{E}_S \cup \mathcal{E}_M \cup \mathcal{E}_B\,. \end{cases} \tag{8.3}$$

For $S_{j\ell} \in \mathcal{E}_R$ we can use the fact that

$$L^u_j(x_{j\ell}) - u_j = \frac{1}{2}\big(L^u_j(x_\ell) - u_j\big) = \frac{1}{2}\big(L^u_j(x_\ell) - u_\ell\big) + \frac{1}{2}(u_\ell - u_j)$$

(and similarly for $L^u_\ell(x_{j\ell}) - u_\ell$), and write

$$\begin{aligned} g_{j\ell}(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j}) &= b^+_{j\ell} u_j + b^-_{j\ell} u_\ell + b^+_{j\ell}\big(L^u_j(x_{j\ell}) - u_j\big) + b^-_{j\ell}\big(L^u_\ell(x_{j\ell}) - u_\ell\big) \\ &= b^+_{j\ell} u_j + b^-_{j\ell} u_\ell + \frac{1}{2} b^+_{j\ell}\big(L^u_j(x_\ell) - u_\ell\big) + \frac{1}{2} b^+_{j\ell}(u_\ell - u_j) \\ &\qquad + \frac{1}{2} b^-_{j\ell}\big(L^u_\ell(x_j) - u_j\big) + \frac{1}{2} b^-_{j\ell}(u_j - u_\ell) \end{aligned}$$

and similarly for $g_{j\ell}(\mathcal{V}_{j\ell},\mathcal{V}_{\ell j})$. Therefore,

$$\begin{aligned} A_2 &= \sum_j \sum_{\ell\in N_j} \Big(g_{j\ell}\big(\mathcal{V}_{j\ell},\mathcal{V}_{\ell j})\big) - g_{j\ell}\big(\mathcal{U}_{j\ell},\mathcal{U}_{\ell j}\big)\Big) z_j \\ &= \sum_j \sum_{\ell\in N_j} \big(b^+_{j\ell} v_j + b^-_{j\ell} v_\ell\big) z_j + \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(L^v_j(x_\ell) - v_\ell\big) + b^-_{j\ell}\big(L^v_\ell(x_j) - v_j\big)\Big) z_j \\ &\qquad + \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(v_\ell - v_j\big) + b^-_{j\ell}\big(v_j - v_\ell\big)\Big) z_j \\ &\quad - \sum_j \sum_{\ell\in N_j} \big(b^+_{j\ell} u_j + b^-_{j\ell} u_\ell\big) z_j - \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(L^u_j(x_\ell) - u_\ell\big) + b^-_{j\ell}\big(L^u_\ell(x_j) - u_j\big)\Big) z_j \\ &\qquad - \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(u_\ell - u_j\big) + b^-_{j\ell}\big(u_j - u_\ell\big)\Big) z_j\,, \end{aligned}$$

and hence,

$$\begin{aligned} A_2 \ &= \ \sum_j \sum_{\ell\in N_j} \big(b^+_{j\ell} z_j + b^-_{j\ell} z_\ell\big) z_j \\ &\quad + \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(L^v_j(x_\ell) - v_\ell\big) + b^-_{j\ell}\big(L^v_\ell(x_j) - v_j\big)\Big) z_j \\ &\quad - \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}\big(L^u_j(x_\ell) - u_\ell\big) + b^-_{j\ell}\big(L^u_\ell(x_j) - u_j\big)\Big) z_j \\ &\quad + \frac{1}{2} \sum_{\mathcal{T}_R} \sum_{N_{jR}} \Big(b^+_{j\ell}(z_\ell - z_j) + b^-_{j\ell}(z_j - z_\ell)\Big) z_j \\ &=: W_1 + W_2 + W_3 + W_4\,. \end{aligned}$$

For W_1 we procced by the same way as we did treating the term B_1 on page 12 to get (cf. (5.17))

$$W_1 \geq \frac{1}{2} \sum_{\mathcal{E}} \big(b^+_{j\ell} - b^-_{j\ell}\big)(z_j - z_\ell)^2\,, \tag{8.4}$$

while for W_4 we get

$$W_4 = \frac{1}{2}\sum_{\mathcal{T}_R}\sum_{N_{jR}}\Big((b^+_{j\ell} - b^-_{j\ell})(z_\ell - z_j)\Big)z_j = \frac{1}{2}\sum_{\mathcal{E}_R}(b^+_{j\ell} - b^-_{j\ell})\Big((z_\ell - z_j)z_j + (z_j - z_\ell)z_\ell\Big)$$
$$= -\frac{1}{2}\sum_{\mathcal{E}_R}(b^+_{j\ell} - b^-_{j\ell})(z_j - z_\ell)^2\,.$$

Therefore,

$$W_1 + W_4 \ge \frac{1}{2}\sum_{\mathcal{E}_S\cup\mathcal{E}_M}(b^+_{j\ell} - b^-_{j\ell})(z_j - z_\ell)^2\,. \tag{8.5}$$

In order to estimate terms W_2 and W_3 we use the definition (3.18) to see that

$$u_\ell - L^u_j(x_\ell) = u_\ell - u_j - \alpha^u_j G^u_j(x_\ell - x_j)\,.$$

Let us discuss the three possible cases. If $\alpha^u_j = 1$ and stencil of G^u_j contains x_ℓ then $u_\ell - L^u_j(x_\ell) = 0$. If $\alpha^u_j = 1$ and stencil of G^u_j does not contain x_ℓ, we have

$$\begin{aligned} u_\ell - L^u_j(x_l) &= u_\ell - u_j - G^u_j(x_l - x_j) \\ &= u_\ell - u_j + G^u_j(x_m - x_j + x_k - x_j) \\ &= u_\ell - u_j + u_m - u_j + u_k - u_j \\ &= \left(\frac{1}{|T_j|}\sum_{k\in N_j}(u_k - u_j)\gamma_{jk}\right)\frac{|T_j|}{\gamma_j} =: R_j\frac{|T_j|}{\gamma_j}. \end{aligned} \tag{8.6}$$

In the (8.6) we take the advantage of the fact that for $T_j \in \mathcal{T}_R$ we have $\gamma_{jk} = \text{const}_j =: \gamma_j$ for all $k \in N_j$.

Finally, if $\alpha^u_j = 0$, then $u_\ell - L^u_j(x_l) = u_\ell - u_j$. However, $\alpha^u_j = 0$ implies that there is local extremum in u_j, or more precisely, that u_j is extremal out of the values $u_j, u_\ell, \ell \in N_j$. Therefore we have in this case

$$|u_\ell - u_j| \le \Big|\sum_{k\in N_j}(u_k - u_j)\Big| = |R_j|\frac{|T_j|}{\gamma_j}\,.$$

We conclude that in any case,

$$|u_\ell - L^u_j(x_\ell)| \le |R_j|\frac{|T_j|}{\gamma_j} \tag{8.7}$$

and the same estimate holds when replacing j by ℓ and vice versa.
By the same considerations we get that

$$|v_\ell - L^v_j(x_\ell)| \le \Big|\sum_{\ell\in N_j}(v_\ell - v_j)\Big|. \tag{8.8}$$

In order to estimate the last term, we define

$$E(v) = E(v_j) := \sum_{\ell\in N_j}(v_\ell - v_j) = (v_\ell - v_j) + (v_k - v_j) + (v_m - v_j)\,. \tag{8.9}$$

Consider now $T := T_j \cup T_\ell \cup T_k \cup T_m$ and a reference triangle $\hat{T} = \hat{T}_1 \cup \hat{T}_2 \cup \hat{T}_3 \cup \hat{T}_4$, such that $\hat{T}$ is mapped onto T by a 1-1 mapping

$$F : \hat{T} \to T,$$
$$F(\hat{x}) = A\hat{x} + a$$

and $F(\hat{T}_i) = T_{p_i}$ for $i = 1, 2, 3, 4$, $p_i = j, \ell, k, m$, respectively. Note that $\hat{T}$ is also a regular, larger triangle. As in the proof of Lemma 5.4 part c), $A \in \mathbb{R}^{2\times 2}$ is an invertible matrix and $a \in \mathbb{R}^2$. Define

$$\begin{aligned}&\hat{E} : W^{2,2}(\hat{T}) \to \mathbb{R}\,,\\ &\hat{E}(\hat{v}) = (\hat{v}_\ell - \hat{v}_j) + (\hat{v}_k - \hat{v}_j) + (\hat{v}_m - \hat{v}_j)\,.\end{aligned}$$

Then

$$|\hat{E}(\hat{v})| \le c\,\|\hat{v}\|_{C(\hat{T})} \le c\,\|\hat{v}\|_{W^{2,2}(\hat{T})}\,.$$

Moreover, $\hat{E}$ vanishes for all polynomials $\hat{p}$ of degree ≤ 1. Indeed, we have for such $\hat{p}$ that

$$\hat{E}(\hat{p}) = \nabla\hat{p}\big[(\hat{x}_\ell - \hat{x}_j) + (\hat{x}_k - \hat{x}_j) + (\hat{x}_m - \hat{x}_j)\big] = 0\,.$$

Using then the Bramble–Hilbert lemma, one gets

$$|\hat{E}(\hat{v})| \le c\,|\hat{v}|_{W^{2,2}(\hat{T})}\,.$$

Finally,

$$|E(v)| = |\hat{E}(\hat{v})| \le c\,|\hat{v}|_{W^{2,2}(\hat{T})} \le c\,h\,|v|_{W^{2,2}(T)} \tag{8.10}$$

using the properties of F similar to those in the proof of Lemma 5.4, part c). Therefore we obtain using (8.8), (5.27),

$$|v_\ell - L_j^v(x_\ell)| \le c\,h\,|v|_{W^{2,2}(T)} \tag{8.11}$$

with $c > 0$ independent of h and $T = T_j \cup \bigcup_{j\in N_\ell} T_\ell$, and the same estimate holds for $|v_j - L_\ell^v(x_j)|$.

Using now (8.11) we see that

$$\begin{aligned}
W_2 &= \frac{1}{2}\sum_{\mathcal{T}_R}\sum_{N_{jR}}\Big(b_{j\ell}^+\big(L_j^v(x_\ell) - v_\ell\big) + b_{j\ell}^-\big(L_\ell^v(x_j) - v_j\big)\Big)z_j\\
&= \frac{1}{2}\sum_{\mathcal{E}_R}\Big(b_{j\ell}^+\big(L_j^v(x_\ell) - v_\ell\big) + b_{j\ell}^-\big(L_\ell^v(x_j) - v_j\big)\Big)(z_j - z_l)\\
&\ge -\frac{1}{2}\sum_{\mathcal{E}_R}\Big(b_{j\ell}^+|L_j^v(x_l) - v_\ell||z_j - z_\ell| + |b_{j\ell}^-||L_\ell^v(x_j) - v_j||z_j - z_\ell|\Big)\\
&\ge -c\,h^2\sum_{\mathcal{E}_R}|v|_{W^{2,2}(T)}|z_j - z_\ell|\\
&\ge -c\,\frac{h^4}{\varepsilon}\sum_{\mathcal{E}_R}|v|^2_{W^{2,2}(T)} - \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}_R}(z_j - z_\ell)^2\\
&\ge -c\,\frac{h^4}{\varepsilon}\|v\|^2_{W^{2,2}(\Omega)} - \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}_R}(z_j - z_\ell)^2\,.
\end{aligned} \tag{8.12}$$

Similarly, (8.7) implies that

$$\begin{aligned}
W_3 &\ge -\frac{1}{2}\sum_{\mathcal{E}_R}\Big(b_{j\ell}^+|L_j^u(x_l) - u_\ell||z_j - z_\ell| + |b_{j\ell}^-||L_\ell^u(x_j) - u_j||z_j - z_\ell|\Big)\\
&\ge -c\,h^2\sum_{\mathcal{E}_R}\Big(|R_j|\sqrt{|T_j|} + |R_\ell|\sqrt{|T_\ell|}\Big)|z_j - z_\ell|\\
&\ge -c\,\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_R}R_j^2|T_j| - \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}_R}(z_j - z_\ell)^2\,.
\end{aligned} \tag{8.13}$$

Now, estimates (8.5), (8.12), (8.13) imply

$$A_2 \geq \frac{1}{2}\sum_{\mathcal{E}_S\cup\mathcal{E}_M}(b^+_{j\ell}-b^-_{j\ell})(z_j-z_\ell)^2 - \frac{\varepsilon\gamma}{2}\sum_{\mathcal{E}_R}(z_j-z_\ell)^2 - c\,\frac{h^4}{\varepsilon}\|v\|^2_{W^{2,2}(\Omega)} - c\,\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_R}R_j^2|T_j|\,,$$

which together with (8.2) finishes the proof. ☐

Lemma 8.2. *Let* $R_j = \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell-u_j)\gamma_{j\ell}$ *for* $T_j\in\mathcal{T}_R$ *(cf. (8.6)). Then there is a constant* $c>0$ *independent of* ε *and* h*, such that*

$$\sum_{\mathcal{T}_R} R_j^2|T_j| \;\leq\; \frac{c}{\varepsilon^3}\,\|f\|^2_{L^2(\Omega)}. \tag{8.14}$$

Proof: From the definition of the numerical scheme in (3.4) we obtain

$$\begin{aligned} R_j &= \frac{1}{\varepsilon|T_j|}\Big(\varepsilon\sum_{\ell\in N_j}(u_\ell-u_j)\gamma_{j\ell}\Big)\\ &\qquad \text{(see (3.4) and use } (L_hu_h)_j=f_j \text{ with } T_j\in\mathcal{T}_R)\\ &= \frac{1}{\varepsilon|T_j|}\Big(\sum_{\ell\in N_j} g_{j\ell}\big(L^u_j(x_{j\ell}),L^u_\ell(x_{j\ell})\big)+c_ju_j|T_j|-f_j|T_j|\Big). \end{aligned} \tag{8.15}$$

We have for $T_j\in\mathcal{T}_R$,

$$\begin{aligned} &\Big|\sum_{\ell\in N_j} g_{j\ell}\big(L^u_j(x_{j\ell}),L^u_\ell(x_{j\ell})\big)\Big| = \Big|\sum_{\ell\in N_j}\Big[b^+_{j\ell}\big(L^u_j(x_{j\ell})\big)+b^-_{j\ell}\big(L^u_\ell(x_{j\ell})\big)\Big]\Big|\\ &= \Big|\sum_{\ell\in N_j}\Big[b^+_{j\ell}\big(L^u_j(x_{j\ell})-u_j\big)+b^-_{j\ell}\big(L^u_\ell(x_{j\ell})-u_\ell\big)+u_jb^+_{j\ell}+u_\ell b^-_{j\ell}\Big]\Big| \qquad (8.16)\\ &\qquad\text{using (3.20)}\\ &\leq ch\sum_{\ell\in N_j}|u_\ell-u_j| + \Big|\sum_{\ell\in N_j}(b^+_{j\ell}+b^-_{j\ell})u_j + b^-_{j\ell}(u_\ell-u_j)\Big|\\ &\leq ch\sum_{\ell\in N_j}|u_\ell-u_j|, \end{aligned}$$

since $\sum_{\ell\in N_j}(b^+_{j\ell}+b^-_{j\ell})u_j=0$. Therefore

$$\begin{aligned} |R_j| &\leq \frac{c}{\varepsilon h^2}\Big(h\sum_{\ell\in N_j}|u_\ell-u_j|+|u_j||T_j|+|f_j||T_j|\Big)\\ &\leq \frac{c}{\varepsilon}\Big(\frac{1}{h}\sum_{\ell\in N_j}|u_\ell-u_j|+|u_j|+|f_j|\Big) \end{aligned} \tag{8.17}$$

and

$$\begin{aligned} \sum_{T_j\in\mathcal{T}_R} R_j^2|T_j| &\leq \frac{c}{\varepsilon^2}\sum_{T_j\in\mathcal{T}_R}\Big(\sum_{N_{jR}}|u_\ell-u_j|^2+u_j^2|T_j|+f_j^2|T_j|\Big) \qquad (8.18)\\ &\leq \frac{c}{\varepsilon^3}\Big(\varepsilon\sum_{\mathcal{E}_R}|u_\ell-u_j|^2\Big)+\frac{c}{\varepsilon^2}\sum_j\Big(u_j^2|T_j|+f_j^2|T_j|\Big)\\ &\leq \frac{c}{\varepsilon^3}\sum_j f_j^2|T_j|, \end{aligned}$$

using (5.14). This implies (8.14). ☐

REMARK 8.3. The terms

$$\|v\|^2_{W^{2,2}(\Omega)} \quad \text{and} \quad \sum_{\mathcal{T}_R} R_j^2 |T_j| \tag{8.19}$$

contained on the right-hand side of the estimate (8.1) depend of course on ε. As we will see in the following Lemma, the sum $\sum_{\mathcal{T}_R} R_j^2|T_j|$ is of the same order in ε as is the norm $\|v\|^2_{W^{2,2}(\Omega)}$.

LEMMA 8.4. *Let $v \in W^{2,2}(\Omega)$ be the solution of* (2.1)–(2.2). *Then*

$$\|v\|^2_{W^{2,2}(\Omega)} \le \frac{c}{\varepsilon^3} \|f\|^2_{L^2(\Omega)}\,, \tag{8.20}$$

Proof: We have the energy estimate

$$\varepsilon \int_\Omega |\nabla v|^2 \,+\, c_0 \int_\Omega |v|^2 \,\le\, C \int_\Omega |f|^2 \tag{8.21}$$

and therefore

$$\|v\|^2_{W^{2,2}(\Omega)} \le C \int_\Omega (\Delta v)^2 \le \frac{C}{\varepsilon^2} \int_\Omega \Big(b\nabla v + cv - f\Big)^2 \le \frac{C}{\varepsilon^3} \int_\Omega f^2\,.$$

REMARK 8.5. The order of ε in (8.20) is sharp. This follows from the following example. For $x, y \in \mathbb{R}$ let

$$a := \left(1 - \exp\left(-\frac{1}{\varepsilon}\right)\right)^{-1}, \quad v_1(x) := a\left(1 - \exp\left(-\frac{1-x}{\varepsilon}\right)\right) - 1 + x, \quad v_2(y) := y(1-y),$$
$$v(x,y) := v_1(x)\, v_2(y), \quad f(x,y) := v(x,y) + v_1(x)(2\varepsilon + 1 - 2y) + v_2(y).$$

Then we have

$$\begin{aligned} -\varepsilon\Delta v + \partial_x v + \partial_y v + v &= f \qquad \text{in} \quad \Omega :=]0,1[\times]0,1[,\\ v &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned}$$

and

$$\|\partial_x^2 v\|_{L^2(\Omega)} > \varepsilon^{-\frac{3}{2}} \quad \text{and} \quad \|f\|_{L^2(\Omega)} \le c \quad \text{for all } \varepsilon > 0. \tag{8.22}$$

It is not difficult to verify these results.

9. The final estimate. Putting together the estimates (7.25), (8.1) and using the definition of $\|z_h\|_\varepsilon$ (see (4.6)), we obtain the main estimates (4.7) and (4.8) of Theorem 4.1. The result for the first order scheme (4.10) can be obtained using only the parts of the estimates (7.25), (8.1) which corresponds to the first order parts of the scheme.

10. Acknowledgement. The authors want to thank the unknown referees for their valuable comments.

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