OPTIMAL DECAY ESTIMATES FOR SOLUTIONS TO DAMPED SECOND ORDER ODE'S

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Abstract. In this paper we derive optimal decay estimates for solutions to second order ordinary differential equations with weak damping. The main assumptions are Kurdyka-Łojasiewicz gradient inequality and its inverse.

1. Introduction

In this paper we study long-time behavior for solutions of damped second order ordinary differential equations

$$\ddot{u} + g(\dot{u}) + \nabla E(u) = 0, \tag{SOP}$$

where $E \in C^2(\Omega)$, Ω being an open connected subset of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function satisfying $\langle g(v), v\rangle \geq 0$ on $\mathbb{R}^n$. This last condition means that the term $g(\dot{u})$ in (SOP) has a damping effect. It is easy to see that energy

$$\mathcal{E}(u, \dot{u}) = \frac{1}{2}\|\dot{u}\|^2 + E(u)$$

is nonincreasing along solutions. In fact, if u is a classical solution to (SOP), then

$$\frac{d}{dt}\mathcal{E}(u(t), \dot{u}(t)) = -\langle g(v), v\rangle \leq 0.$$

If $u : [0, +\infty) \to \Omega$ is a global solution and φ belongs to the ω-limit set of u, then $\mathcal{E}(u(t), \dot{u}(t)) \to \mathcal{E}(\varphi, 0) = E(\varphi)$ as $t \to +\infty$. In this paper, we derive the exact rate of convergence of $\mathcal{E}(u(t), \dot{u}(t))$ to $E(\varphi)$.

Our main assumption is the Kurdyka-Łojasiwicz gradient inequality (see [10])

$$\Theta(|E(u) - E(\varphi)|) \leq \|\nabla E(u)\|. \tag{KLI}$$

For linear g, the optimal decay estimate was derived in [2]. For nonlinear g (typically satisfying $g'(0) = 0$) some decay estimates were shown in [8], [7], [3]. Here we derive better decay estimates under additional assumptions on E and we show that these estimates are optimal. We will assume that

Date: March 17, 2017.

Key words and phrases. Kurdyka-Łojasiewicz inequality, rate of convergence to equilibrium, second order equation with damping.

E satisfies an inverse to (KLI) and some estimates on the second gradient and that g has certain behavior near zero. The present result generalizes the one from [5, Theorem 20] where we worked with the Łojasiewicz gradient inequality, i.e. (KLI) with $\Theta(s) = s^{1-\theta}$ for a constant $\theta \in (0, \frac{1}{2}]$ (see [11]). It also generalizes the result by Haraux (see [9]) and Abdelli, Anguiano, Haraux (see [1]). The present result applies e.g. to functions E and g having the growth near origin as

$$s^a \ln^{r_1}(1/s) \ln^{r_2}(\ln(1/s)) \dots \ln^{r_k}(\ln \dots \ln(1/s)) \tag{1}$$

for some constants $a, r_1, \dots, r_k$. It also applies to functions E with a non-strict local minimum in φ.

The paper is organized as follows. In Section 2 we present our notations, basic definitions and the main result. Section 3 contains the proof of the main result.

2. Notations and the main result

By $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ we denote the usual norm and scalar product on $\mathbb{R}^d$. For nonnegative functions f, $g : G \subset \mathbb{R}^d \to \mathbb{R}$ we write $g(x) = O(f(x))$ on G if there exists $C > 0$ such that $g(x) \leq Cf(x)$ for all $x \in G$. We say that $g(x) = O(f(x))$ for $x \to a$ if $g(x) = O(f(x))$ on a neighborhood of a. If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, we write $f \sim g$.

We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$ and $f(s) > 0$ for $s > 0$

- is *admissible* if f is nondecreasing and there exists $c > 0$ such that $sf'_\pm(s) \leq cf(s)$ for all $s > 0$,
- has *property (K)* if for every $K > 0$ there exists $C(K) > 0$ such that $f(Ks) \leq C(K)f(s)$ holds for all $s > 0$,
- is *C-sublinear* if there exists $C > 0$ such that $f(t + s) \leq C(f(t) + f(s))$ holds for all $t, s > 0$.

It is easy to see that admissible functions are C-sublinear and have property (K) (for proof see Appendix of [4]). Further, for nondecreasing functions property (K) is equivalent to C-sublinearity. Moreover, every concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is admissible and satisfies $sf'_\pm(s) \leq f(s)$.

Let us introduce the inverse Kurdyka-Łojasiewicz inequality

$$\Theta_1(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \geq \|\nabla\mathcal{E}(u)\| \tag{IKLI}$$

and an inequality for second gradient

$$\|\nabla^2 E(u)\| \leq \Gamma(\|\nabla E(u)\|). \tag{2}$$

When we say that inequality (KLI) (resp. (IKLI), (2)) holds on a set U it means that the inequality holds for all $u \in U$ with a given fixed φ and Θ (resp. Θ_1, Γ).

By a solution to (SOP) we always mean a classical solution defined on $[0,+\infty)$. By $R(u) = \{u(t) : t \geq 0\}$ we denote the *range of u*. We say that a solution is precompact if $R(u)$ is precompact in Ω (the domain of E). The ω-limit set of u is

$$\omega(u) = \{\varphi \in \Omega : \exists t_n \nearrow +\infty,\ u(t_n) \to \varphi\}.$$

By c, C, $\tilde{c}$, $\tilde{C}$ we denote generic constants, their values can change from line to line or from expression to expression.

The main result of the present paper is the following.

Theorem 1. *Let u be a precompact solution to* (SOP) *and $\varphi \in \omega(u)$. Let $E(\cdot) \geq E(\varphi)$ on $R(u)$ and let E satisfy* (KLI), (IKLI) *and* (2) *on $R(u)$ with admissible functions Θ, Θ_1 and Γ, such that $\Theta(s) \sim \Theta_1(s)$ and $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s)$ for $s \to 0+$. Let g satisfies*

$$\langle g(v), v\rangle \geq ch(\|v\|)\|v\|^2, \qquad \|g(v)\| \leq Ch(\|v\|)\|v\| \tag{3}$$

with an admissible function h satisfying

$$\Theta(s) \geq c\sqrt{s}\, h(\sqrt{s}) \tag{4}$$

for some $c > 0$ and all $s \geq 0$. Let us denote

$$\chi(s) = sh(\sqrt{s}), \qquad \Phi_\chi = \int \frac{1}{\chi(s)} ds \tag{5}$$

and assume that $\psi(s) = s^2 h(s)$ is convex. Then

$$c(-\Phi_\chi)^{-1}(Ct) \leq \mathcal{E}(u(t), \dot{u}(t)) - \mathcal{E}(\varphi, 0) \leq C(-\Phi_\chi)^{-1}(ct)$$

for some c, $C > 0$ and all t large enough.

Let us first mention that if $E(u) = \|u\|^p$, $p \geq 2$, then (KLI), (IKLI) hold with $\Theta(s) \sim \Theta_1(s) = Cs^{1-\theta}$, $\theta = \frac{1}{p}$ and (2) holds with $\Gamma(s) = Cs^{\frac{1-2\theta}{1-\theta}}$. If $h(s) = s^\alpha$, $\alpha \in (0,1)$, then condition (4) becomes $\alpha \geq 1 - 2\theta$ and $(-\Phi_\chi)^{-1}(ct) = Ct^{-\frac{2}{\alpha}}$. In this case, we obtain the same result as [5, Theorem 20] and also [9].

Remarks. *1. If $(\Phi_\chi)^{-1}$ has property (K), then the statement of Theorem 1 can be written as $\mathcal{E}(u(t), \dot{u}(t)) - E(\varphi) \sim (-\Phi_\chi)^{-1}(t)$.*

2. We can see that the energy decay depends on h only. In particular, it is independent of Θ.

3. It is enough to assume that all the assumptions except $\langle g(v), v\rangle > 0$ for all $v \neq 0$ hold on a small neigborhood of zero, resp. a small neighborhood of $\omega(u)$.

4. It follows from (KLI) *and* [2, Proposition 2.8] *that $\Theta(s) = O(\sqrt{s})$. Hence, by* (4) *function h must be bounded on a neighborhood of zero and $\Phi_\chi(t) \to -\infty$ as $t \to 0+$. So, it is not important which primitive function Φ_χ we take and we have $(-\Phi_\chi)^{-1}(t) \to 0$ as $t \to +\infty$.*

5. Theorem 1 does not imply that $u(t) \to \varphi$ as $t \to +\infty$. In fact, in [6, Theorem 4] *we have shown that $u(t) \to \varphi$ if h is large enough, in particular if $\int_0^\varepsilon \frac{1}{\Theta(s)h(\Theta(s))} < +\infty$. If this condition is not satisfied, it may happen that $\omega(u)$ contains more than one point.*

6. If φ is an asymptotically stable equilibrium for the gradient system $\dot{u}+\nabla E(u) = 0$ (e.g. if E has a strict local minimum in φ and is convex on a neighborhood of φ) and (KLI), (IKLI) *hold on a neighborhood of φ, then by* [5, Corollary 5] *we have $\|x-\varphi\| \sim \Phi_\Theta(E(x)-E(\varphi))$ on a neighborhood of φ where $\Phi_\Theta(t) = \int_0^t \frac{1}{\Theta}$. In this case, for any solution starting in a neighborhood of φ we have*

$$c(-\Phi_\chi)^{-1}(Ct) \le \|v(t)\|^2 + \Phi_\Theta^{-1}(\|u(t)-\varphi\|) \le C(-\Phi_\chi)^{-1}(ct)$$

and, especially,

$$\|u(t)-\varphi\| \le \Phi_\Theta(C(-\Phi_\chi)^{-1}(ct)),$$

so $u(t) \to \varphi$. We do not have the estimate for $\|u(t)-\varphi\|$ from below since, at least in one-dimensional case, the solution oscilates and $u(t_n) = \varphi$ for a sequence $t_n \nearrow +\infty$ (see [9]*).*

Example 2. Let us consider $E(u) = F(\|u\|)$ with a real function F having a strict local minimum $F(0) = 0$ and satisfying on a right neighborhood of zero $CF(s) \ge sF'(s) \ge (1+\varepsilon)F(s)$ and $sF''(s) \sim F'(s)$. Moreover, we assume that $(F')^{-1}$ has property (K). (It is easy to show that any analytic function $F(s) = \sum_{k=2m}^\infty a_k s^k$, $a_{2m} > 0$ and any function of the form (1) with $a > 2$, $r_i \in \mathbb{R}$ or $a = 2$, $r_1 = \cdots = r_{j-1} = 0$, $r_j < 0$, $r_{j+1}, \ldots, r_k \in \mathbb{R}$ satisfies these assumptions.) Then (KLI), (IKLI) holds with $\Theta(s) = \frac{s}{F^{-1}(s)}$, since

$$\Theta(E(u)) = \Theta(F(\|u\|)) = \frac{F(\|u\|)}{\|u\|} \sim F'(\|u\|) = \|\nabla E(u)\|.$$

Further, (2) holds with $\Gamma(s) = \frac{s}{(F')^{-1}(s)}$ since

$$\|\nabla^2 E(u)\| = F''(\|u\|) \sim \frac{F'(\|u\|)}{\|u\|} = \Gamma(F'(\|u\|)) = \Gamma(\|\nabla E(u)\|).$$

Furher, we have

$$\Theta'(F(s)) = \frac{\frac{d}{ds}\Theta(F(s))}{F'(s)} = \frac{\frac{d}{ds}\frac{F(s)}{s}}{F'(s)} = \frac{F'(s)s - F(s)}{s^2 F'(s)} = \frac{1}{s}\left(1 - \frac{F(s)}{sF'(s)}\right) \sim \frac{1}{s},$$

so

$$\Theta(F(s))\Theta'(F(s)) \sim \frac{1}{s}\Theta(F(s)) \sim \frac{1}{s^2}F(s)$$

and

$$\Gamma(\Theta(F(s))) = \frac{\Theta(F(s))}{(F')^{-1}(\Theta(F(s)))} = \frac{F(s)}{s(F')^{-1}(\frac{F(s)}{s})} \sim \frac{F(s)}{s(F')^{-1}(F'(s))} = \frac{F(s)}{s^2},$$

hence $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s)$. Then, for any g satisfying (3) with a function h small enough (such that (4) holds) Theorem 1 can be applied and we obtain the exact energy decay which depends on h only and not on F. In particular, if $h(s) = s^\alpha$ we have $\mathcal{E}(u(t), v(t)) \sim t^{-\frac{2}{\alpha}}$ and if h is of the form (1), we have by [4, Lemmas 6.5, 6.6]

$$\mathcal{E}(u(t), v(t)) \sim t^{-\frac{2}{a}} \ln^{-\frac{r_1}{a}}(\ln 1/t) \dots \ln^{-\frac{r_k}{a}}(\ln \dots \ln 1/t).$$

Let us mention that if h is equal to (1) and such that $cs \leq h(s) \leq c$ near zero (i.e. $a \in [0, 1]$ and if $a \in \{0, 1\}$ we have a sign condition on the first nonzero number r_i), then $\psi(s) = s^2 h(s)$ is convex near zero.

3. Proof of Theorem 1

Let us write $v(t)$ instead of $\dot{u}(t)$ and $\mathcal{E}(t)$ instead of $\mathcal{E}(u(t), v(t))$. We also often write u, v instead of $u(t)$, $v(t)$.

First of all, since u is precompact $\{E(u(t)) : t \geq 0\}$ is bounded. Therefore, $\{\mathcal{E}(t) : t \geq 0\}$ is bounded, hence v is bounded and by (SOP) also $\ddot{u} = \dot{v}$ is bounded. Since

$$\int_0^t \langle g(v), v\rangle = \mathcal{E}(0) - \mathcal{E}(t) \leq K,$$

we have $\langle g(v), v\rangle \in L^1((0, +\infty))$. Then boundedness of $\dot{v}$ yields convergence of $\langle g(v(t)), v(t)\rangle$ to 0. Hence $v(t) \to 0$ as $t \to +\infty$ and it follows that $\mathcal{E}(t) \to \mathcal{E}(\varphi, 0)$. So, we can assume without loss of generality that $E(\varphi) = 0, \mathcal{E}(\varphi, 0) = 0$.

In the rest of the proof we will work with

$$H(t) = \mathcal{E}(t) + \varepsilon B(E(u(t)))\langle \nabla E(u(t)), v\rangle,$$

where

$$B(s) = \begin{cases} \frac{1}{\Theta(s)^2} s h(\sqrt{s}) & s > 0 \\ 0 & s = 0 \end{cases}$$

and $\varepsilon > 0$ is small enough. Let us mention that B can be unbounded in a neighborhood of zero, but due to (4) we have $\Theta(s)B(s) \leq C\sqrt{s}$, hence H is continuous even in the points where $E(u(t)) = 0$ and in these points we have $H(t) = \mathcal{E}(t)$. Let us denote $M := \{t \geq 0 : E(u(t)) > 0\}$ and $M^c = \{t \geq 0 : E(u(t)) = 0\}$.

We show that $H(t) \sim \mathcal{E}(t)$. On M^c it is trivial. On M we apply (IKLI), Cauchy-Schwarz and Young inequalities and $\Theta(s)B(s) \leq C\sqrt{s}$ and we obtain

$$\begin{aligned} |\varepsilon B(E(u))\langle \nabla E(u(t)), v\rangle| &\leq \varepsilon C B(E(u))\Theta(E(u))\|v\| \\ &\leq \varepsilon C B(E(u))^2 \Theta(E(u))^2 + \varepsilon C\|v\|^2 \\ &\leq \varepsilon C \mathcal{E}(t), \end{aligned}$$

hence

$$(1-\varepsilon C)\mathcal{E}(t) \le H(t) \le (1+\varepsilon C)\mathcal{E}(t)$$

and taking $\varepsilon > 0$ small enough we obtain $H(t) \sim \mathcal{E}(t)$.

The next step is to show that

$$0 \le -H'(t) \sim h(\|v\|)\|v\|^2 + E(u)h\left(\sqrt{E(u)}\right). \tag{6}$$

Let us first estimate $B'(s)$. For any $s > 0$ we have

$$B'(s) = \frac{B(s)}{s}\left(1 + \frac{h'(\sqrt{s})\sqrt{s}}{h(\sqrt{s})} - 2\frac{s\Theta'(s)}{\Theta(s)}\right) \in \left[\frac{B(s)}{s}(1-2C), \frac{B(s)}{s}(1+C)\right],$$

where the equality follows by definition of B and the rest from admissibility of h and θ (the two fractions in round bracket are nonnegative and bounded above by a constant). Hence, $|sB'(s)| \le CB(s)$.

Let $t \in M$. Let us compute $H'(t)$ and use the fact that u solves (SOP) to get

$$\begin{aligned} H'(t) = &-\langle g(v), v\rangle - \varepsilon B(E(u))\|\nabla E(u)\|^2 \\ &+ \varepsilon B'(E(u))\langle \nabla E(u), v\rangle^2 \\ &+ \varepsilon B(E(u))\langle \nabla^2 E(u)v, v\rangle \\ &+ \varepsilon B(E(u))\langle \nabla E(u), -g(v)\rangle. \end{aligned} \tag{7}$$

Due to (3) we have $\langle g(v), v\rangle \sim h(\|v\|)\|v\|^2$ and by definition of B, (KLI) and (IKLI) we immediately have $B(E(u))\|\nabla E(u)\|^2 \sim E(u)h(\sqrt{E(u)})$. So,

$$\langle g(v), v\rangle + \varepsilon B(E(u))\|\nabla E(u)\|^2 \sim h(\|v\|)\|v\|^2 + \varepsilon C E(u)h\left(\sqrt{E(u)}\right).$$

We show that the second, third and fourth lines of (7) are smaller than this term, then (6) is proved.

The second line of (7) is less than

$$\varepsilon C\frac{B(E(u))}{E(u)}\Theta(E(u))^2\|v\|^2 \le \varepsilon C h\left(\sqrt{E(u)}\right)\|v\|^2.$$

Since Γ has property (K) and satisfies $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s) \le Cs^{-1}\Theta(s)^2$ and due to (IKLI) and definition of B, the third line in (7) is less than

$$\varepsilon C B(E(u))\Gamma(\|\nabla E(u)\|)\|v\|^2 \le \varepsilon C h\left(\sqrt{E(u)}\right)\|v\|^2.$$

If $E(u) \le 4C\|v\|^2$, then (h satisfies property (K)) we have $h\left(\sqrt{E(u)}\right)\|v\|^2 \le \tilde{C}h(\|v\|)\|v\|^2$ and if $E(u) \ge 4C\|v\|^2$, then $h\left(\sqrt{E(u)}\right)\|v\|^2 \le \frac{1}{4C}h\left(\sqrt{E(u)}\right)E(u)$. So, in either case we have that lines two and three in (7) are less than

$$\varepsilon C h(\|v\|)\|v\|^2 + \frac{1}{4}\varepsilon h\left(\sqrt{E(u)}\right)E(u),$$

so they are less than the first line in (7) since we can make εC small by taking ε small enough. The last line in (7) is (by definition of B and (4)) less than

$$\begin{aligned}\varepsilon C B(E(u))\|\nabla E\| h(\|v\|)\|v\| &\leq \varepsilon C \frac{1}{\Theta(E(u))} E(u) h\left(\sqrt{E(u)}\right) h(\|v\|)\|v\| \\ &\leq \varepsilon C \sqrt{E(u)} h(\|v\|)\|v\|.\end{aligned}$$

Applying the Young inequality with $\psi(s) = s^2 h(s)$ and the convex conjugate $\tilde{\psi}$ we get

$$\begin{aligned}\varepsilon C \sqrt{E(u)} h(\|v\|)\|v\| &\leq \frac{1}{4}\varepsilon \psi\left(\sqrt{E(u)}\right) + \varepsilon C \tilde{\psi}(\|v\| h(\|v\|)) \\ &\leq \frac{1}{4}\varepsilon E(u) h\left(\sqrt{E(u)}\right) + \varepsilon C h(\|v\|)\|v\|^2\end{aligned}$$

since $\tilde{\psi}(sh(s)) \leq Cs^2 h(s)$ due to Lemma 3 below. Now, (6) is proven on M. If $E(u(t)) \to 0$ for $t \to t_0$, we can see that $H'(t) \to -\langle g(v(t_0)), v(t_0)\rangle = \mathcal{E}'(t_0)$ (due to the estimates above, all terms on the right-hand side of (6) except the first one tend to zero). By continuity of H, we have $H' = \mathcal{E}'$ on M^c, in particular (6) holds on M^c.

We show that $\chi(H(t)) \sim -H'(t)$. In fact,

$$\begin{aligned}\chi(H(t)) &\leq \chi(C(\|v\|^2 + E(u)))) \\ &\leq C(\chi(\|v\|^2) + \chi(E(u))) \\ &= C\left(h(\|v\|)\|v\|^2 + E(u) h\left(\sqrt{E(u)}\right)\right) \\ &\leq -CH'(t),\end{aligned}$$

where we applied monotonicity in the first line, C-sublinearity and property (K) in the second line (χ has these properties by Lemma 4 below), definition of χ in the third line and (6) in the last inequality. On the other hand, by Lemma 4 also the inverse inequalities in C-sublinearity and property (K) are valid, so we have

$$\begin{aligned}\chi(H(t)) &\geq \chi(c(\|v\|^2 + E(u)))) \\ &\geq c(\chi(\|v\|^2) + \chi(E(u))) \\ &= c\left(h(\|v\|)\|v\|^2 + E(u) h\left(\sqrt{E(u)}\right)\right) \\ &\geq -cH'(t),\end{aligned}$$

so $\chi(H(t)) \sim -H'(t)$ is proved.

Let $T = \sup\{t \geq 0 : \ H(t) > 0\}$. For any $t \in (0, T)$ we have proved

$$-\frac{d}{dt}\Phi_\chi(H(t)) = -\frac{H'(t)}{\chi(H(t))} \in [c, C].$$

Integrating this relation from t_0 to t we obtain

$$c(t-t_0) - \Phi_\chi(H(t_0)) \le -\Phi_\chi(H(t)) \le C(t-t_0) - \Phi_\chi(H(t_0)). \tag{8}$$

If $T < +\infty$, then we can see that $-\Phi_\chi(H(t))$ is bounded on $(0,T)$, hence $0 < \lim_{t\to T-} H(t) = H(T)$, contradiction. Therefore, $T = +\infty$, (8) holds for all $t > 0$ and for t large enough we have

$$\tilde{c}t \le c(t-t_0) - \Phi_\chi(H(t_0)) \le -\Phi_\chi(H(t)) \le C(t-t_0) - \Phi_\chi(H(t_0)) \le \tilde{C}t.$$

Hence

$$c(-\Phi_\chi)^{-1}(\tilde{C}t) \le H(t) \sim \mathcal{E}(u(t),v(t)) \le C(-\Phi_\chi)^{-1}(\tilde{c}t),$$

which completes the proof of Theorem 1.

Lemma 3. *Let $\psi(s) = s^2h(s)$ and $\tilde{\psi}(r) = \sup\{rs - \psi(s) : s \ge 0\}$ be the convex conjugate to ψ. Then there exists $C > 0$ such that $\tilde{\psi}(sh(s)) \le Cs^2h(s)$ for all $s \ge 0$.*

Proof. Since ψ is convex, the one-sided derivatives $\psi'_\pm(s) = s^2h'_\pm(s) + 2sh(s)$ are nondecreasing functions and the interval $[\psi'_-(s), \psi'_+(s)]$ is nonempty. Take $s_0 > 0$ arbitrarily and take $r \in [\psi'_-(s_0), \psi'_+(s_0)]$. Then the function $s \mapsto rs - \psi(s)$ attains it maximum in s_0, hence $\tilde{\psi}(r) = rs_0 - s_0^2h(s_0)$. Since $r \ge \psi'_-(s_0) = s_0^2h'_-(s_0) + 2s_0h(s_0) \ge s_0h(s_0)$ and $\tilde{\psi}$ is increasing, we have $\tilde{\psi}(s_0h(s_0)) \le \tilde{\psi}(r) = rs_0 - s_0^2h(s_0) \le \psi'_+(s_0)s_0 - s_0^2h(s_0) = s_0^3h'_+(s_0) + 2s_0^2h(s_0) - s_0^2h(s_0) \le (c+2-1)s_0^2h(s_0)$. □

Lemma 4. *Function $\chi(s) = sh(\sqrt{s})$ is C-sublinear and it has property (K). Moreover, $\chi(s+t) \ge \frac{1}{2}(\chi(s) + \chi(t))$ for all $s, t > 0$ and for every $c > 0$ there exists $\tilde{c} > 0$ such that $\chi(cs) \ge \tilde{c}\chi(s)$.*

Proof. Since h has property (K), we have for a fixed $K > 0$

$$\chi(Ks) = Ksh(\sqrt{K}\sqrt{s}) \le KsC(\sqrt{K})h(\sqrt{s}) = KC(\sqrt{K})\chi(s).$$

So, χ has property (K) and since it is increasing, it is also C-sublinear. Since χ is increasing, we also have $\chi(s+t) \ge \chi(s)$, $\chi(s+t) \ge \chi(t)$ and therefore $\chi(s+t) \ge \frac{1}{2}(\chi(s) + \chi(t))$. From property (K) we have for any fixed $c > 0$

$$\chi(s) = \chi(\frac{1}{c}cs) \le C(\frac{1}{c})\chi(cs) = \frac{1}{\tilde{c}}\chi(cs)$$

and the last property is proven. □

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