BILINEAR SPHERICAL MAXIMAL FUNCTION

J. A. BARRIONEVO, LOUKAS GRAFAKOS, DANQING HE, PETR HONZÍK,
AND LUCAS OLIVEIRA

ABSTRACT. We obtain boundedness for the bilinear spherical maximal function in a range of exponents that includes the Banach triangle and a range of L^p with $p < 1$. We also obtain counterexamples that are asymptotically optimal with our positive results on certain indices as the dimension tends to infinity.

1. INTRODUCTION

Let σ be surface measure on the unit sphere. The spherical maximal function

$$\mathscr{M}(f)(x) = \sup_{t>0} \Big| \int_{|y|=1} f(x-ty)d\sigma(y) \Big|, \tag{1}$$

was first studied by Stein [19] who provided a counterexample showing that it is unbounded on $L^p(\mathbb{R}^n)$ for $p \le \frac{n}{n-1}$ and obtained the a priori inequality $\|\mathscr{M}(f)\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$ when $n \ge 3$, $p \in (\frac{n}{n-1}, \infty)$ for smooth functions f; see also the account in [20, Chapter XI]. The extension of this result to the case $n = 2$ was established about a decade later by Bourgain [1].

In addition to Stein and Bourgain, other authors have studied the spherical maximal function; for instance see [5], [3], [17], [16], and [18]. Among the techniques used in these works, we highlight that of Rubio de Francia [17], in which the L^p boundedness of (1) is reduced to certain L^2 estimates obtained by Plancherel's theorem. Extensions of the spherical maximal function to different settings have also been established by several authors: for instance see [4], [2] [12], [7] and [15].

In this work we study the bi(sub)linear spherical maximal function defined in (2), which was introduced and first studied by [8]. In the bilinear setting the role of the crucial $L^2 \to L^2$ estimate is played by an $L^2 \times L^2 \to L^1$, and obviously Plancherel's identity cannot be used on L^1. We overcome the lack of orthogonality on L^1 via a wavelet technique introduced by three of the authors in [10] in the study of certain bilinear operators; on this approach

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see [11], [14]. It is worth mentioning a related interesting recent paper [13], where the authors studied the bilinear circular average when $n = 1$. Our object of study here is the bi(sub)linear spherical maximal function

$$\mathcal{M}(f,g)(x) = \sup_{t>0} \left| \int_{\mathbb{S}^{2n-1}} f(x-ty)g(x-tz)\,d\sigma(y,z) \right| \tag{2}$$

initially defined for Schwartz functions f, g on $\mathbb{R}^n$. Here σ is surface measure on the $2n-1$-dimensional sphere. We are concerned with bounds for $\mathcal{M}$ from a product of Lebesgue spaces $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to another Lebesgue space $L^p(\mathbb{R}^n)$, where $1/p = 1/p_1 + 1/p_2$. The main result of this article is the following:

Theorem 1. *Let $n \geq 8$ and let $\delta_n = (2n-15)/10$. Then the bilinear maximal operator $\mathcal{M}$, when restricted to Schwartz functions, is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for all indices $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ in the open rhombus with vertices the points $\vec{P}_0 = (\frac{1}{\infty}, \frac{1}{\infty}, \frac{1}{\infty})$, $\vec{P}_1 = (1, \frac{1}{\infty}, 1)$, $\vec{P}_2 = (\frac{1}{\infty}, 1, 1)$ and $\vec{P}_3 = (\frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{1+\delta_n})$.*

Once Theorem 1 is known, it follows that $\mathcal{M}$ admits a bounded extension from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for indices in the open rhombus of Theorem 1 (for such indices we have $p_1, p_2 < \infty$). Indeed, given $\{f_j\}_j$ Schwartz functions converging to f in L^{p_1} and $\{g_k\}_k$ Schwartz functions converging to g in L^{p_2}, we have that

$$\|\mathcal{M}(f_j, g_j) - \mathcal{M}(f_{j'}, g_{j'})\|_{L^p} \leq \left\|\mathcal{M}(f_j - f_{j'}, g_j) + \mathcal{M}(f_j, g_j - g_{j'})\right\|_{L^p}.$$

It follows from this that the sequence $\{\mathcal{M}(f_j, g_j)\}_j$ is Cauchy in $L^p(\mathbb{R}^n)$ and hence it converges to a value which we also call $\mathcal{M}(f,g)$. This is the bounded extension of $\mathcal{M}$ from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In order to pass to the maximal function defined on $L^{p_1} \times L^{p_2}$, it is also possible to used the technique desribed in [20, page 508].

Concerning dimensions smaller than 8, we have positive answers in the Banach range in next section.

2. The Banach range in dimensions $n \geq 2$

Proposition 2. *Let $n \geq 2$. Then $\mathcal{M}$ maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 < p_1, p_2 \leq \infty$, and $1 < p \leq \infty$.*

Proof. We show that $\mathcal{M}$ is bounded on the intervals $[\vec{P}_0, \vec{P}_1)$ and $[\vec{P}_0, \vec{P}_2)$, where $\vec{P}_1$ and $\vec{P}_2$ are as in Theorem 1. Then the claimed assertion follows by interpolation. If one function, for instance the second one g, lies in L^∞, matters reduce to the $L^p(\mathbb{R}^n)$ boundedness of the maximal operator

$$\mathcal{M}^0(f)(x) = \sup_{t>0} \int_{\mathbb{S}^{2n-1}} |f(x-ty)|\,d\sigma(y,z),$$

since $\mathcal{M}(f,g)(x) \le \|g\|_{L^\infty}\mathcal{M}^0(f)(x)$. This expression inside the supremum is a Fourier multiplier operator of the form

$$\int_{\mathbb{R}^{2n}} \widehat{|f|}(\xi)\delta_0(\eta)\widehat{d\sigma}(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta = \int_{\mathbb{R}^n} \widehat{|f|}(\xi)\widehat{d\sigma}(t\xi,0)e^{2\pi i x\cdot\xi}d\xi$$

where δ_0 is the Dirac mass and

$$\widehat{d\sigma}(t(\xi,0)) = 2\pi\frac{J_{n-1}(2\pi t|(\xi,0)|)}{|t(\xi,0)|^{n-1}}.$$

The multiplier $\widehat{d\sigma}(\xi,0)$ is smooth everywhere and decays like $|\xi|^{-(n-\frac{1}{2})}$ as $|\xi|\to\infty$ and its gradient has a similar decay.

The following result is in [17, Theorem B] (see also [6]):

Theorem A. *Let $m(\xi)$ be a $\mathcal{C}^{[n/2]+1}(\mathbb{R}^n)$ function that satisfies $|\partial^\gamma m(\xi)| \le (1+|\xi|)^{-a}$ for all $|\gamma| \le [n/2]+1$ with $a \ge (n+1)/2$. Then the maximal operator*

$$f \mapsto \sup_{t>0}\left|\left(\widehat{f}(\xi)m(t\xi)\right)^\vee\right|$$

maps $L^p(\mathbb{R}^n)$ to itself for $1<p<\infty$.

In order to have $n-\frac{1}{2} \ge \frac{n+1}{2}$ we must assume that $n\ge 2$. It follows from Theorem A that $\mathcal{M}^0$ is bounded on L^p when $1<p\le\infty$ and $n\ge 2$. This completes the proof of Proposition 2.

□

3. The point $(2,2,1)$

Next we turn to the main estimate of this article which concerns the point $L^2\times L^2 \to L^1$, i.e., the estimate $\|\mathcal{M}(f,g)\|_{L^1} \le \|f\|_{L^2}\|g\|_{L^2}$.

Proposition 3. *If ψ is in $\mathcal{C}_0^\infty(\mathbb{R}^{2n})$, then the maximal function*

$$M(f,g)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\psi(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\right|$$

satisfies that for any $1<p_1,p_2<\infty$ and $1/p = 1/p_1+1/p_2$, there exists a constant C independent f and g such that

$$\|M(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

The proof of Proposition 3 is standard and is omitted. Next, we decompose $\mathcal{M}$. We fix $\varphi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2n})$ such that $\chi_{B(0,1)} \le \varphi_0 \le \chi_{B(0,2)}$ and we let $\varphi(\xi,\eta) = \varphi_0((\xi,\eta)) - \varphi_0(2(\xi,\eta))$. For $j\ge 1$ define

$$m_j(\xi,\eta) = \widehat{d\sigma}(\xi,\eta)\varphi(2^{-j}(\xi,\eta))$$

and for $j=0$ define $m_0(\xi,\eta)=\widehat{d\sigma}(\xi,\eta)\varphi_0(\xi,\eta)$. Then we have

$$\widehat{d\sigma}=m=\sum_{j\geq 0} m_j$$

where $\widehat{d\sigma}(\xi,\eta)=2\pi\frac{J_{n-1}(2\pi(\xi,\eta))}{|(\xi,\eta)|^{n-1}}$. Setting

$$\mathcal{M}_j(f,g)(x)=\sup_{t>0}\left|\int_{\mathbb{R}^{2n}}\widehat{f}(\xi)\widehat{g}(\eta)m_j(t\xi,t\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta\right|,$$

we have the pointwise estimate

$$\mathcal{M}(f,g)(x)\leq\sum_{j\geq 0}\mathcal{M}_j(f,g)(x),\qquad x\in\mathbb{R}^n. \tag{3}$$

Proposition 4. *For $n\geq 8$, there exist positive constants C and $\delta_n=\frac{n}{5}-\frac{3}{2}$ such that for all $j\geq 1$ and all functions $f,g\in L^2(\mathbb{R}^n)$ we have*

$$\|\mathcal{M}_j(f,g)\|_{L^1}\leq C\,j\,2^{-\delta_n j}\|f\|_{L^2}\|g\|_{L^2}. \tag{4}$$

Proposition 4 will be proved in the next section. In the remaining of this section we state and prove a lemma needed for its proof.

Lemma 5. *Suppose that $\sigma_1(\xi,\eta)$ is defined on $\mathbb{R}^{2n}$ and for some $\delta>0$ it satisfies:*

(i) for any multiindex $|\alpha|\leq M=4n$, there exists a positive constant C_α independent of j such that $\|\partial^\alpha(\sigma_1(\xi,\eta))\|_{L^\infty}\leq C_\alpha 2^{-j\delta}$,

(ii) supp $\sigma_1\subset\{(\xi,\eta)\in\mathbb{R}^{2n}:|(\xi,\eta)|\sim 2^j, c_1 2^{-j}\leq\frac{|\xi|}{|\eta|}\leq c_2 2^j\}$.

Then $T(f,g)(x):=\int_0^\infty|T_{\sigma_t}(f,g)(x)|\frac{dt}{t}$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with bound at most a multiple of $j\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}$, where $\sigma_t(\xi,\eta)=\sigma_1(t\xi,t\eta)$.

Proof of Lemma 5. A crucial tool in the proof of Lemma 5 is the following result [10, Corollary 8]:

Proposition B. *Let $m\in L^2(\mathbb{R}^{2n})$ and $C_M>0$ satisfy $\|\partial^\alpha m\|_{L^\infty}\leq C_M$ for each multiindex $|\alpha|\leq M=16n$. Then the bilinear operator T_m associated with the multiplier m satisfies*

$$\|T_m\|_{L^2\times L^2\to L^1}\leq C C_M^{1/5}\|m\|_{L^2}^{4/5}.$$

Using Proposition B, setting $\widehat{f^j}=\widehat{f}\chi_{\{c_1\leq|\xi|\leq c_2 2^{j+1}\}}$, by the support of σ_1 we obtain that

$$\|T_{\sigma_1}(f,g)\|_{L^1}\leq C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}\|f^j\|_{L^2}\|g^j\|_{L^2}.$$

Notice that $T_{\sigma_t}(f,g)(x) = t^{-2n} T_{\sigma_1}(f_t, g_t)(\frac{x}{t})$, where $\widehat{f_t}(\xi) = \widehat{f}(\xi/t)$. Then

$$\begin{aligned}\|T_{\sigma_t}(f,g)\|_{L^1} &\leq C\|\sigma_1\|_{L^2}^{4/5} 2^{-j\delta/5} t^{-n} \|\widehat{f}(\xi/t)\chi_{E_{j,0}}\|_{L^2} \|\widehat{g}(\eta/t)\chi_{E_{j,0}}\|_{L^2} \\ &= C\|\sigma_1\|_{L^2}^{4/5} 2^{-j\delta/5} \|\widehat{f}\chi_{E_{j,t}}\|_{L^2} \|\widehat{g}\chi_{E_{j,t}}\|_{L^2},\end{aligned}$$

where $E_{j,t} = \{\xi \in \mathbb{R}^n : \frac{c_1}{t} \leq |\xi| \leq \frac{2^j c_2}{t}\}$.

As a result we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_0^\infty |T_{\sigma_t}(f,g)| \frac{dt}{t} dx \\ \leq & C\|\sigma_1\|_{L^2}^{4/5} 2^{-j\delta/5} \int_0^\infty \|\widehat{f}\chi_{E_{j,t}}\|_{L^2} \|\widehat{g}\chi_{E_{j,t}}\|_{L^2} \frac{dt}{t} \\ \leq & C\|\sigma_1\|_{L^2}^{4/5} 2^{-j\delta/5} \left(\int_0^\infty \int_{\mathbb{R}^n} |\widehat{f}\chi_{E_{j,t}}|^2 d\xi \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\mathbb{R}^n} |\widehat{g}\chi_{E_{j,t}}|^2 d\xi \frac{dt}{t} \right)^{\frac{1}{2}}.\end{aligned}$$

We control the last term as follows:

$$\int_0^\infty \int_{\mathbb{R}^n} |\widehat{f}\chi_{E_{j,t}}|^2 d\xi \frac{dt}{t} \leq C \int_{\mathbb{R}^n} \int_{1/|\xi|}^{2^j/|\xi|} \frac{dt}{t} |\widehat{f}(\xi)|^2 d\xi \leq Cj\|f\|_{L^2}^2$$

and thus we deduce

$$\|T(f,g)(x)\|_{L^1} \leq C\|\sigma_1\|_{L^2}^{4/5} 2^{-j\delta/5} j \|f\|_{L^2} \|g\|_{L^2}.$$

This completes the proof of Lemma 5. □

4. Proof of Proposition 4

Proof. Estimate (4) is automatically holds for finitely many terms in view of Proposition 3, so we fix a large j and define

$$T_{j,t}(f,g)(x) = \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta) m_j(t\xi, t\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta.$$

Take a smooth function ρ on $\mathbb{R}$ such that $\chi_{[\varepsilon-1, 1-\varepsilon]} \leq \rho \leq \chi_{[-1,1]}$. Define $m_j^1(\xi,\eta) = m_j(\xi,\eta)\rho(\frac{1}{j}(\log_2 \frac{|\xi|}{|\eta|}))$, then we have a smooth decomposition of m_j with $m_j = m_j^1 + m_j^2$. On the support of m_j^1 we have $C^{-1}2^{-j}|\xi| \leq |\eta| \leq C2^j|\xi|$ and on the support of m_j^2 we have $2^{j(1-\varepsilon)}|\xi| \lesssim |\eta|$ or $2^{j(1-\varepsilon)}|\eta| \lesssim |\xi|$. We define

$$\mathcal{M}_j^i(f,g) = \sup_{t>0} |T_{j,t}^i(f,g)|, \quad i \in \{1,2\},$$

where $T_{j,t}^1$ and $T_{j,t}^2$ correspond to multipliers $m_j^1(t(\xi,\eta))$ and $m_j^2(t(\xi,\eta))$ respectively, such that $T_{j,t} = T_{j,t}^1 + T_{j,t}^2$. Then for f, g Schwartz functions we have

$$\mathcal{M}_j^1(f,g)(x) = \sup_{t>0} |T_{j,t}^1(f,g)(x)|$$

$$= \sup_{t>0} \left| \int_0^t s \frac{dT^1_{j,s}(f,g)}{ds} \frac{ds}{s} \right|$$
$$\leq \int_0^\infty |\widetilde{T}^1_{j,s}(f,g)(x)| \frac{ds}{s},$$

where $\widetilde{T}^1_{j,s}$ has bilinear multiplier $\widetilde{m}^1_j(s\xi, s\eta) = (s\xi, s\eta) \cdot (\nabla m^1_j)(s\xi, s\eta)$, a diagonal multiplier with nice decay, which can be used to establish the boundedness of the diagonal part with the aid of Lemma 5.

Recall that

$$m^1_j(\xi,\eta) = \varphi(2^{-j}(\xi,\eta))2\pi \frac{J_{n-1}(2\pi(\xi,\eta))}{|(\xi,\eta)|^{n-1}} \rho(\tfrac{1}{j}(\log_2 \tfrac{|\xi|}{|\eta|}))$$

for $j \geq 1$ and a calculation shows that $|\partial_1(m^1_j)|$ is controlled by the sum of three terms bounded by $C2^{-j(2n-1)/2}$, $C2^{-j(2n+1)/2}$ and $C\frac{1}{j}2^{-j(2n-1)/2}$ respectively. Indeed, when the derivative falls on ϕ, we can bound it by $C2^{-j}2^{-j(n-1/2)} = C2^{-j(n+1/2)}$. If the derivative falls on the second part, using properties of Bessel functions (see, e.g., [9, Appendix B.2]), we obtain the bound $C\frac{J_n(2\pi(\xi,\eta))}{|(\xi,\eta)|^n}|\xi_1| \leq C2^{-j(n-1/2)}$. For the last case, we can bound it by $C2^{-j(n-1/2)}j^{-1}\frac{1}{|\xi|}\frac{\xi_1}{|\xi|} \leq C2^{-j(n-1/2)}j^{-1}2^{-\varepsilon j}$. As a consequence we have $|\partial_1(m^1_j)| \leq C2^{-j(2n-1)/2}$. Then we can show that $|\partial_1(\widetilde{m}^1_j)| \leq C2^{-j(2n-3)/2}$ and similar arguments give that for any multiindex α we have $|\partial^\alpha \widetilde{m}^1_j| \leq C2^{-j(2n-3)/2}$. Moreover, from this we can show that

$$\|\widetilde{m}^1_j\|_2 \leq C\left(\int_{|(\xi,\eta)|\sim 2^j} |2^{-j(n-\frac{3}{2})}|^2 d\xi d\eta \right)^{\frac{1}{2}} \leq C2^{-j(n-\frac{3}{2})}2^{jn} \leq C2^{\frac{3}{2}j}.$$

Applying Lemma 5 to the function $\widetilde{m}^1_j(\xi,\eta) = (\xi,\eta)\cdot(\nabla m^1_j)(\xi,\eta)$ which satisfies the hypotheses with $\delta = (2n-3)/2$, we obtain

$$\|\mathcal{M}^1_j(f,g)\|_{L^1} \leq Cj\|\widetilde{m}^1_j\|^{\frac{4}{5}}_{L^2} 2^{-j\frac{\delta}{5}}\|f\|_{L^2}\|g\|_{L^2} = Cj2^{j(\frac{3}{2}-\frac{n}{5})}\|f\|_{L^2}\|g\|_{L^2}. \tag{5}$$

It remains to obtain an analogous estimate for $\mathcal{M}^2_j$.

For the off-diagonal part m^2_j we use a different decomposition involving g-functions. For $f,g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\mathcal{M}^2_j(f,g)(x) &= \big(\sup_{t>0} |T^2_{j,t}(f,g)(x)|^2\big)^{\frac{1}{2}} \\
&= \left(\sup_{t>0} \left| 2\int_0^t T^2_{j,s}(f,g)(x)\, s\frac{dT^2_{j,s}(f,g)(x)}{ds}\frac{ds}{s} \right| \right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left\{ \left(\int_0^\infty |T^2_{j,s}(f,g)|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \left(\int_0^\infty |\widetilde{T}^2_{j,s}(f,g)|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}
\end{aligned}$$

$$= \sqrt{2}\left(G_j(f,g)(x)\widetilde{G}_j(f,g)\right)^{\frac{1}{2}}. \tag{6}$$

Here $\widetilde{T}^2_{j,s}(f,g)$ has symbol $\widetilde{m}^2_j(s\xi,s\eta) = (s\xi,s\eta)\cdot(\nabla m^2_j)(s\xi,s\eta)$ and

$$\begin{aligned} G_j(f,g)(x) &= \left(\int_0^\infty |T^2_{j,s}(f,g)|^2 \frac{ds}{s}\right)^{\frac{1}{2}} \\ \widetilde{G}_j(f,g)(x) &= \left(\int_0^\infty |\widetilde{T}^2_{j,s}(f,g)|^2 \frac{ds}{s}\right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 6. *If a $\sigma_1(\xi,\eta)$ on $\mathbb{R}^{2n}$ satisfies*

(i) for any multiindex $|\alpha| \le M = 4n$, there exists a positive constant C_α independent of j such that $\|\partial^\alpha(\sigma_1(\xi,\eta))\|_{L^\infty} \le C_\alpha 2^{-j\delta}$,

(ii) supp $\sigma_1 \subset \{(\xi,\eta)\in\mathbb{R}^{2n} : |(\xi,\eta)| \sim 2^j, |\xi| \ge 2^{j(1-\varepsilon)}|\eta|$, or $|\eta| \ge 2^{j(1-\varepsilon)}|\xi|\}$,

then $T(f,g)(x) := (\int_0^\infty |T_{\sigma_t}(f,g)(x)|^2 \frac{dt}{t})^{1/2}$ is bounded from $L^2\times L^2$ to L^1 with bound at most a multiple of $2^{-j(\delta-\varepsilon)}$, where $\sigma_t(\xi,\eta) = \sigma_1(t\xi,t\eta)$.

Proof. Recall that supp $m^2_j \subset \{(\xi,\eta) : 2^{j(1-\varepsilon)}|\xi| \lesssim |\eta|$ or $2^{j(1-\varepsilon)}|\eta| \lesssim |\xi|\}$. We consider only the part $\{|\xi| \ge 2^{j(1-\varepsilon)}|\eta|\}$ because the other part is similar. By [10, Section 5] we have

$$|T_{\sigma_1}(f,g)(x)| \le C2^{\varepsilon j}2^{-j\delta}M(g)(x)|T_m(f)(x)|,$$

where M is the Hardy-Littlewood maximal function and T_m is a linear operator that satisfies $\|T_m(f)\|_{L^2} \le C\|\widehat{f}\chi_{\{|\xi|\sim 2^j\}}\|_{L^2}$. Then

$$|T_{\sigma_t}(f,g)(x)| \le 2^{-j(\delta-\varepsilon)}t^{-n}M(g)(x)T_m(f_t)(x/t),$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n}\left(\int_0^\infty |T_{\sigma_t}(f,g)(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx \\ &\qquad \le C2^{-j(\delta-\varepsilon)}\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2n}M(g)(x)^2|T_m(f_t)(x/t)|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx \\ &\qquad \le C2^{-j(\delta-\varepsilon)}\|M(g)\|_{L^2}\left(\int_{\mathbb{R}^n}\int_0^\infty \left|t^{-n}T_m(f_t)(x/t)\right|^2\frac{dt}{t}dx\right)^{\frac{1}{2}} \\ &\qquad \le C2^{-j(\delta-\varepsilon)}\|g\|_{L^2}\left(\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2\int_{2^{j-1}/|\xi|}^{2^{j+1}/|\xi|}\frac{dt}{t}d\xi\right)^{\frac{1}{2}} \\ &\qquad \le C2^{-j(\delta-\varepsilon)}\|g\|_{L^2}\|f\|_{L^2}. \end{aligned}$$

This completes the proof of Lemma 6. □

We now return to the proof of Proposition 4. Notice that both $m_j^2(\xi,\eta)$ and $\widetilde{m}_j^2(\xi,\eta)$ satisfy conditions of Lemma 6 with δ being either $(2n-1)/2$ or $(2n-3)/2$ respectively, so

$$\|G_j(f,g)\|_{L^1} \le C2^{-j(2n-1)/2}\|f\|_{L^2}\|g\|_{L^2}$$
$$\|\widetilde{G}_j(f,g)\|_{L^1} \le C2^{-j(2n-3)/2}\|f\|_{L^2}\|g\|_{L^2}.$$

Using (6) we deduce

$$\|\mathcal{M}_j^2(f,g)\|_{L^1} \le \|G_j(f,g)\|_{L^1}^{1/2}\|\widetilde{G}_j(f,g)\|_{L^1}^{1/2} \le C2^{-j(n-1)}\|f\|_{L^2}\|g\|_{L^2}. \tag{7}$$

Combining (5) and (7) yields Proposition 4 with $\delta_n = \frac{n}{5} - \frac{3}{2}$. □

5. Interpolation

By Proposition 3 (for term $j \le c_0$) and Proposition 4 (for $j \ge c_0$), for any $\delta_n' < \delta_n$, as a consequence of (3) we obtain

$$\|\mathcal{M}(f,g)\|_{L^1} \le \sum_{j=0}^{\infty} C_{\delta_0'} 2^{-\delta_n' j}\|f\|_{L^2}\|g\|_{L^2} \le C_{\delta_0'}\|f\|_{L^2}\|g\|_{L^2}.$$

This establishes the boundedness of $\mathcal{M}$ from $L^2 \times L^2$ to L^1 claimed in Theorem 1 (recall $n \ge 8$). It remains to obtain estimates for other values of p_1, p_2. This is achieved via bilinear interpolation.

Notice that when one index among p_1 and p_2 is equal to 1, we have that $\mathcal{M}_j$ maps $L^{p_1} \times L^{p_2}$ to $L^{p,\infty}$ with norm $\lesssim 2^j$. Indeed, this follows from the estimate

$$|\varphi_j^\vee * (d\sigma)(y,z)| \le C_N 2^j(1+|(y,z)|)^{-2N} \le C_N 2^j (1+|y|)^{-N}(1+|z|)^{-N}$$

which can be found, for instance, in [9, estimate (6.5.12)]. Thus we have

$$\mathcal{M}_j(f,g)(x) \le C2^j M(f)M(g)$$

where M is the Hardy-Littlewood maximal function. We pick two points

$$\vec{Q}_1 = (1/1, 1/(1+\varepsilon), (2+\varepsilon)/(1+\varepsilon))$$
$$\vec{Q}_2 = (1/(1+\varepsilon), 1/1, (2+\varepsilon)/(1+\varepsilon))$$

and we also consider the point $\vec{Q}_0 = (1/2, 1/2, 1)$. We interpolate the known estimates for $\mathcal{M}_j$ at these three points. Letting ε go to 0, we obtain that for $p > \frac{2+2\delta_n}{1+2\delta_n}$ we have that $\mathcal{M}_j$ maps $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$ with a geometrically decreasing bound in j. Recall that $\delta_n = (2n-15)/10 > 0$, so we need $n \ge 8$.

Thus summing over j gives boundedness for $\mathcal{M}$ from $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$ when $p > \frac{2+2\delta_n}{1+2\delta_n}$. By interpolation we obtain boundedness for $\mathcal{M}$ in the interior of a rhombus with vertices the points $(1/\infty, 1/\infty, 1/\infty)$,

$(\frac{2n-3/2}{2n-1}, \frac{1}{\infty}, \frac{2n-3/2}{2n-1})$, $(\frac{1}{\infty}, \frac{2n-3/2}{2n-1}, \frac{2n-3/2}{2n-1})$ and $(\frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{2+2\delta_n}, \frac{2+4\delta_n}{2+2\delta_n})$. The proof of Theorem 1 is now complete.

We remark that is the largest region for which we presently know boundedness for $\mathcal{M}$ in dimensions $n \geq 8$.

6. COUNTEREXMAPLES

In this section we construct counterexamples indicating the unboundedness of the bilinear spherical maximal operator in a certain range. Our examples are inspired by Stein [19] but the situation is more complicated.

Proposition 7. *The bilinear spherical maximal operator $\mathcal{M}$ is unbounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 \leq p_1, p_2 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $n \geq 1$, and $p \leq \frac{n}{2n-1}$. In particular, $\mathcal{M}$ is unbounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$ when $n = 1$.*

Remark 1. We note that $\frac{1+\delta_n}{1+2\delta_n} - \frac{n}{2n-1} = \frac{1+\frac{n}{5}-\frac{3}{2}}{1+\frac{2n}{5}-3} - \frac{n}{2n-1} \approx \frac{1}{n} \to 0$ as $n \to \infty$. This means that the gap between the range of boundedness and unboundedness tends to 0 as the dimension increases to infinity.

Proof. We first consider the case $n = 1$ where it is easy to demonstrate the main idea.

Define functions on $\mathbb{R}$ by setting $f(y) = |y|^{-1/p_1}(\log \frac{1}{|y|})^{-2/p_1}\chi_{|y|\leq 1/2}$ and $g(y) = |y|^{-1/p_2}(\log \frac{1}{|y|})^{-2/p_2}\chi_{|y|\leq 1/2}$. Then $f \in L^{p_1}(\mathbb{R})$, $g \in L^{p_2}(\mathbb{R})$ and we will estimate from below $M_{\sqrt{2}R}(f,g)(R)$ for large R, where

$$M_t(f,g)(x) = \int_{\mathbb{S}^1} |f(x-ty)g(x-tz)|d\sigma(y,z).$$

In view o the support properties of f and g we have $|y - \frac{1}{\sqrt{2}}| \leq \frac{1}{2\sqrt{2}R}$, and $|z - \frac{1}{\sqrt{2}}| \leq \frac{1}{2\sqrt{2}R}$. We also have that $y^2 + z^2 = 1$ since $(y,z) \in \mathbb{S}^1$.

Therefore we rewrite $M_{\sqrt{2}R}(f,g)(R)$ as

$$\int_{\frac{\sqrt{2}}{2}-\frac{1}{2\sqrt{2}R}}^{\frac{\sqrt{2}}{2}+\frac{1}{2\sqrt{2}R}} |R(1-\sqrt{2}y)|^{-\frac{1}{p_1}}(-\log|R(1-\sqrt{2}y)|)^{-\frac{2}{p_1}} |R(1-\sqrt{2}z)|^{-\frac{1}{p_2}}(-\log|R(1-\sqrt{2}z)|)^{-\frac{2}{p_2}} \frac{dy}{\sqrt{1-y^2}}, \tag{8}$$

with $z = \sqrt{1-y^2}$.

Notice that $|R(1-\sqrt{2}z)| = R|\frac{1-2z^2}{1+\sqrt{2}z}| \leq R|1-2y^2| \leq 3R|1-\sqrt{2}y|$ since[1] $z \approx y \approx \sqrt{2}/2$. As a result, with the help of (9) [Lemma 8], the expression

[1] Here $a \approx b$ means that $|a-b|$ is very small.

in (8) is greater than

$$\int_{\frac{\sqrt{2}}{2}-\frac{1}{100R}}^{\frac{\sqrt{2}}{2}+\frac{1}{100R}} R^{-\frac{1}{p}}|(1-\sqrt{2}y)|^{-\frac{1}{p}}(-\log|R(1-\sqrt{2}y)|)^{-\frac{2}{p}}dy$$

$$=2R^{-1}\int_0^{\frac{1}{100}} t^{-1/p}(\log\tfrac{1}{t})^{-2/p}dt = \begin{cases} C_pR^{-1} & \text{if } p\geq 1 \\ \infty & \text{if } p<1. \end{cases}$$

Thus $\mathcal{M}(f,g)\notin L^p(\mathbb{R})$ for $p<1$ and also $\mathcal{M}(f,g)(x)\geq C/x$ for x large if $p=1$. It follows that $\mathcal{M}(f,g)\notin L^1(\mathbb{R})$ for $p=1$, hence the statement of the proposition holds.

We now consider the higher-dimensional case $n\geq 2$. We define $f(y)=|y|^{-n/p_1}(\log\frac{1}{|y|})^{-2/p_1}\chi_{|y|\leq 1/100}$ and $g(y)=|y|^{-n/p_2}(\log\frac{1}{|y|})^{-2/p_2}\chi_{|y|\leq 1/2}$. We have that f lies in $L^{p_1}(\mathbb{R}^n)$ and g lies in $L^{p_2}(\mathbb{R}^n)$. The mapping $(y,z)\mapsto(Ay,Az)$ with $A\in SO_n$ is an isometry on $\mathbb{S}^{2n-1}$, hence we have $M_t(f,g)(x)=M_t(f,g)(|x|e_1)$, where $e_1=(1,0,\dots,0)\in\mathbb{R}^n$. Thus we may take $x=Re_1\in\mathbb{R}^n$ with R large.

By the change of variables identity (10) [Lemma 9], we have

$$\begin{aligned} &M_{\sqrt{2}R}(f,g)(Re_1) \\ &\quad=\int_{\mathbb{S}^{2n-1}} f(Re_1-\sqrt{2}Ry)g(Re_1-\sqrt{2}Rz)d\sigma(y,z) \\ &\quad=\int_{B_n(\frac{1}{\sqrt{2}}e_1,\frac{1}{100R})} |\sqrt{R}y-Re_1|^{-\frac{n}{p_1}}(-\log|Re_1-\sqrt{2}Ry|)^{-\frac{2}{p_1}} \\ &\qquad\quad\int_E |\sqrt{2}Rz-Re_1|^{-\frac{n}{p_2}}(-\log|Re_1-\sqrt{2}Rz|)^{-\frac{2}{p_2}}d\sigma^r_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}, \end{aligned}$$

where $B_n(a,r)$ is a ball in $\mathbb{R}^n$ centered at a with radius r, and E is the $(n-1)$-dimensional manifold $\mathbb{S}^{n-1}_{\sqrt{1-|y|^2}}\cap B_n(\frac{1}{\sqrt{2}}e_1,\frac{1}{2\sqrt{2}R})$ with $\mathbb{S}^{n-1}_r$ being the sphere in $\mathbb{R}^n$ with radius r and $d\sigma^r_{n-1}$ the measure on $\mathbb{S}^{n-1}_r$.

We next focus on the inner integral, namely

$$I=\int_E |\sqrt{2}Rz-Re_1|^{-\frac{n}{p_2}}(-\log|Re_1-\sqrt{2}Rz|)^{-\frac{2}{p_2}}d\sigma^r_{n-1}(z).$$

Take a point $z_0\in\mathbb{S}^{n-1}_{\sqrt{1-|y|^2}}\cap\partial\big(B_n(\frac{1}{\sqrt{2}}e_1,\frac{1}{2\sqrt{2}R})\big)$, and let θ be the angle between vectors z_0 and e_1, which the largest one between $z\in E$ and e_1. Here ∂B is the boundary of a set B. Then θ is small if R is large and[2] $|E|\sim(\sqrt{1-|y|^2}\theta)^{n-1}\sim\theta^{n-1}$. Noticing that $\theta^2\sim\sin^2\theta=1-\cos^2\theta\sim$

[2] $A\sim B$ means that the ratio A/B is bounded above and below

$1-\cos\theta$ and that

$$1-|y|^2+\tfrac{1}{2}-\sqrt{2}\sqrt{1-|y|^2}\cos\theta=\tfrac{1}{8R^2},$$

we obtain that $\theta^2\sim\frac{1}{8R^2}-(\sqrt{1-|y|^2}-\frac{1}{\sqrt{2}})^2$. Then we write

$$\Big|\sqrt{1-|y|^2}-\tfrac{1}{\sqrt{2}}\Big|=\Big|\frac{1-|y|^2-\frac{1}{2}}{\sqrt{1-|y|^2}+\frac{1}{\sqrt{2}}}\Big|\le 2|\tfrac{1}{2}-|y|^2|\le\tfrac{1}{25R}.$$

Consequently $\theta\ge C/R$.

Collecting the previous calculations, we can bound I from below by

$$\int_0^\theta\int_{\mathbb{S}^{n-2}_{t\sin\alpha}}|\sqrt{2}Rz-Re_1|^{-\frac{n}{p_2}}(-\log|Re_1-\sqrt{2}Rz|)^{-\frac{2}{p_2}}d\sigma_{n-2}^{t\sin\alpha}(z)d\alpha,$$

where $t=|z|=\sqrt{1-|y|^2}\approx\frac{1}{\sqrt{2}}$, and $z_1=\cos\alpha$. By symmetry, let us consider just that case $t<\frac{1}{\sqrt{2}}$. Let β be the angle such that $|\sqrt{2}z-e_1|=2|\sqrt{2}t-1|$, then $2t^2+1-2\sqrt{2}t\cos\beta=4|\sqrt{2}t-1|^2$, which implies that $\beta^2\sim1-\cos\beta\sim2\sqrt{2}t-2t^2-1+4(\sqrt{2}t-1)^2=3(\sqrt{2}t-1)^2$. So $\beta\sim1-\sqrt{2}t$. When $\alpha=0$, we have trivially that $|\sqrt{2}z-e_1|=|\sqrt{2}t-1|$. So for $\alpha\in[0,\beta]$, we have $|\sqrt{2}z-e_1|\sim2|\sqrt{2}t-1|\le2\big|2|z|^2-1\big|=2\big|2|y|^2-1\big|\le6\big|\sqrt{2}|y|-1\big|\le6|\sqrt{2}y-e_1|$. Consequently using the fact that $1-\sqrt{2}t\le C\theta$ and (9) again we obtain

$$\begin{aligned}
I&\ge C\int_0^\theta\int_{\mathbb{S}^{n-2}_{t\sin\alpha}}\frac{|\sqrt{2}Rz-Re_1|^{1-n}}{|\sqrt{2}Rz-Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1-\sqrt{2}Rz|)^{\frac{2}{p_2}}}\,d\sigma_{n-2}^{t\sin\alpha}(z)\,d\alpha\\
&\ge\frac{CR^{1-n}|\sqrt{2}t-1|^{1-n}}{|\sqrt{2}Ry-Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1-\sqrt{2}Ry|)^{\frac{2}{p_2}}}\int_0^{C(1-\sqrt{2}t)}\sin^{n-2}\alpha d\alpha\\
&\ge CR^{1-n}\frac{|\sqrt{2}t-1|^{1-n}|1-\sqrt{2}t|^{n-1}}{|\sqrt{2}Ry-Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1-\sqrt{2}Ry|)^{\frac{2}{p_2}}}\\
&=CR^{1-n}|\sqrt{2}Ry-Re_1|^{-\frac{n}{p_2}+n-1}(-\log|Re_1-\sqrt{2}Ry|)^{-\frac{2}{p_2}}.
\end{aligned}$$

Using this estimate we see that

$$\begin{aligned}
&M_{\sqrt{2}R}(f,g)(Re_1)\\
&\quad\ge CR^{1-n}\int_{B_n(\frac{1}{\sqrt{2}}e_1,\frac{1}{100R})}|Re_1-\sqrt{2}Ry|^{-\frac{n}{p}+n-1}(-\log|Re_1-\sqrt{2}Ry|)^{-\frac{2}{p}}dy\\
&\quad=CR^{1-2n}\int_{B_n(0,\frac{1}{100})}|x|^{-\frac{n}{p}+n-1}(-\log|x|)^{-\frac{2}{p}}dx
\end{aligned}$$

$$= CR^{1-2n}\int_0^{\frac{1}{100}} r^{-\frac{n}{p}+2n-2}(-\log r)^{-\frac{2}{p}}dr$$
$$= \begin{cases} CR^{-2n+1} & \text{if } p = \frac{n}{2n-1} \\ \infty & \text{if } p < \frac{n}{2n-1} \end{cases}.$$

Hence $\mathcal{M}(f,g)$ is not in L^p for $p < \frac{n}{2n-1}$ and $\mathcal{M}(f,g)(x) \geq C|x|^{1-2n}$ for all $|x|$ large enough, hence it is also not in $L^{\frac{n}{2n-1}}(\mathbb{R}^n)$ when $p = \frac{n}{2n-1}$. □

Lastly, we prove a couple of points left open.

Lemma 8. *Let $r_1, r_2 > 0$, $t,\ s \leq \frac{1}{10}$, and $t \leq Cs$ for some $C \geq 1$. Then there exists an absolute constant C' (depending on C, r_1, r_2) such that*

$$s^{-r_1}(\log \tfrac{1}{s})^{-r_2} \leq C' t^{-r_1}(\log \tfrac{1}{t})^{-r_2}. \tag{9}$$

Proof. Define $F(x) = x^{r_1}(\log x)^{-r_2}$. Differentiating F, we see that F is increasing when x is large enough and so,

$$F(\tfrac{1}{s}) = s^{-r_1}(\log \tfrac{1}{s})^{-r_2} \leq C^{r_1}(Cs)^{-r_1}(\log \tfrac{1}{Cs})^{-r_2} = C^{r_1}F(\tfrac{1}{Cs}) \leq C'F(\tfrac{1}{t}),$$

which is a restatement of (9).

□

Lemma 9. *For functions $F(y,z)$ defined in $\mathbb{R}^{2n}$ with $y,\ z \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{S}^{2n-1}} F(y,z)d\sigma(y,z) = \int_{B_n}\int_{\mathbb{S}^{n-1}_{r_y}} F(y,z)d\sigma^{r_y}_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}, \tag{10}$$

where B_n is the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}_{r_y}$ is the sphere in $\mathbb{R}^n$ centered at 0 with radius $r_y = \sqrt{1-|y|^2}$.

Proof. We begin by writing $\int_{\mathbb{S}^{2n-1}} F(y,z)d\sigma(y,z)$ as

$$\int_{B_{2n-1}} \left[F(y,z',z_n) + F(y,z',-z_n)\right]\frac{dydz'}{\sqrt{1-|y|^2-|z'|^2}}, \tag{11}$$

where $z = (z', z_n)$, and $z_n = \sqrt{1-|y|^2-|z'|^2}$; see [9, Appendix D.5].

Writing $z/r_y = \omega = (\omega', \omega_{n-1}) \in \mathbb{R}^{n-1}\times\mathbb{R}$, we express the right hand side of (10) as

$$\int_{B_n}\int_{\mathbb{S}^{n-1}_{r_y}} F(y,z)d\sigma^{r_y}_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}$$
$$= \int_{B_n} r_y^{n-1}\int_{\mathbb{S}^{n-1}} F(y,r_y\omega)d\sigma_{n-1}(\omega)\frac{dy}{\sqrt{1-|y|^2}}$$
$$= \int_{B_n} r_y^{n-1}\int_{B_{n-1}} \left[F(y,r_y\omega',r_y\omega_n) + F(y,r_y\omega',-r_y\omega_n)\right]\frac{d\omega'}{\sqrt{1-|\omega'|^2}}\frac{dy}{\sqrt{1-|y|^2}}$$

$$= \int_{B_n} r_y^{n-1} \int_{r_y B_{n-1}} \left[F(y,z',z_n) + F(y,z',-z_n) \right] \frac{r_y^{1-n} dz'}{\sqrt{1-|\omega'|^2}} \frac{dy}{\sqrt{1-|y|^2}}$$

$$= \int_{B_n} \int_{r_y B_{n-1}} \left[F(y,z',z_n) + F(y,z',-z_n) \right] \frac{dy dz'}{\sqrt{1-|y|^2-|z'|^2}},$$

as one can easily verify that $\sqrt{1-|\omega'|^2}\sqrt{1-|y|^2} = \sqrt{1-|y|^2-|z'|^2}$. Using that B_{2n-1} is equal to the disjoint union of the sets $\{(y, r_y v) : v \in B_{n-1}\}$ over all $y \in B_n$, we see that the last double integral is equal to the expression in (11), as claimed. □

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DEPARTMENT OF PURE AND APPLIED MATHEMATICS, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL PORTO ALEGRE, RS, BRAZIL 91509-900
E-mail address: josea@mat.ufrgs.br

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211, USA
E-mail address: grafakosl@missouri.edu

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P. R. CHINA
E-mail address: hedanqing@mail.sysu.edu.cn

MFF UK, SOKOLOVSKA 83, PRAHA 7, CZECH REPUBLIC
E-mail address: honzik@gmail.com

DEPARTMENT OF PURE AND APPLIED MATHEMATICS, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL PORTO ALEGRE, RS, BRAZIL 91509-900
E-mail address: lucas.oliveira@ufrgs.br