

A NON-DC FUNCTION WHICH IS DC ALONG ALL CONVEX CURVES

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ABSTRACT. A problem asked by the authors in 1989 concerns the natural question, whether one can deduce that a continuous function f on an open convex set $D \subset \mathbb{R}^n$ is DC (i.e., is a difference of two convex functions) from the behavior of f “along some special curves φ ”. I.M. Prudnikov published in 2014 a theorem (working with convex curves φ in the plane), which would give a positive answer in \mathbb{R}^2 to our problem. However, in the present note we construct an example showing that this theorem is not correct, and thus our problem remains open in each \mathbb{R}^n , $n > 1$.

INTRODUCTION

A function f on an open convex set $D \subset \mathbb{R}^n$ is called a DC (or d.c.) function if it is a difference of two convex functions. For more information about DC functions and their applications see e.g. [3], [6], [1].

DC functions of one variable have a very simple internal characterization: they are precisely indefinite integrals of functions with locally bounded variation. However, for $n \geq 2$, no simple and useful internal characterization of DC functions is known.

Already A.D. Aleksandrov (who first studied and used DC functions of more variables) in 1949 asked whether a function on \mathbb{R}^2 which is (in a natural sense) DC on each line must be DC; now it is well-known that this is not the case (see e.g. [7, p. 35]). So the following rough question arises:

Is it possible to characterize DC functions on $D \subset \mathbb{R}^n$ “in the language of curves” only?

Two precise versions of this question were formulated in [7, Problems 6 and 7, p. 45]; Problem 7 is reproduced in Remark 2.2 below. A paper [4] by I.M. Prudnikov contains a claim (Theorem 1) which would give a positive answer to [7, Problem 7] for real functions in \mathbb{R}^2 (see Remark 2.2). However, we construct an example showing that [4, Theorem 1] is not correct, and thus our [7, Problem 7] remains open in each \mathbb{R}^n , $n > 1$, even for real functions.

Let us describe the claim of [4, Theorem 1]:

Date: May 10, 2017.

2000 Mathematics Subject Classification. 26B25; 26B40.

Key words and phrases. DC function, d.c. function, characterization of DC functions.

For a two-dimensional compact convex set $K \subset \mathbb{R}^2$ let $r_K : [0, T_K] \rightarrow \partial K$ be a simple closed curve parametrized by the arc length. Let $D \subset \mathbb{R}^2$ be an open bounded convex set and $f : D \rightarrow \mathbb{R}$ a Lipschitz function.

[4, Theorem 1] asserts, that f is DC if and only if the following condition holds:

- (C) There exists a constant $c = c(D, f) > 0$ such that for each two-dimensional convex compact $K \subset D$ the variation $V(\Phi', [0, T_K])$ is less than c , where $\Phi(t) := f(r_K(t)), t \in [0, T_K]$. (The variation is calculated by using only partitions consisting of points of differentiability of Φ .)

(Note that the assumption of convexity of K is omitted in the English translation of [4].)

The main aim of the present note is to show (by a counterexample) that condition (C) does not imply that f is DC. It is difficult to specify which step in the proof of [4, Theorem 1] is incorrect, since many arguments in this proof are sketched only.

1. PRELIMINARIES

We consider \mathbb{R}^d equipped with the standard Euclidean norm $\|\cdot\|$. Given $x \in \mathbb{R}^d$ and $r > 0$, by $B(x, r)$ we denote the closed ball of center x and radius r .

Let $I \subset \mathbb{R}$ be a non-degenerate interval, and $h : I \rightarrow \mathbb{R}^d$. If I is compact the variation $V(h, I)$ of h on I is defined in the standard way via finite partitions of I . For an arbitrary I , we put

$$V(h, I) = \sup\{V(h, [a, b]) : a, b \in I, a < b\}.$$

It is elementary to see that we always have

$$\begin{aligned} (1) \quad V(h, [a, b]) &\leq V(h, [a, b]) + \limsup_{s \rightarrow b^-} \|h(s) - h(b)\| \\ &\leq V(h, (a, b)) + \limsup_{s \rightarrow a^+} \|h(s) - h(a)\| + \limsup_{s \rightarrow b^-} \|h(s) - h(b)\|. \end{aligned}$$

Let us recall the following definition from [2]. Given a continuous mapping $F : [a, b] \rightarrow \mathbb{R}^d$ and a finite partition $D = \{a = t_0 < t_1 < \dots < t_n = b\}$ of $[a, b]$, we define

$$P(F, D) := \sum_{i=1}^{n-1} \left\| \frac{F(t_{i+1}) - F(t_i)}{\|F(t_{i+1}) - F(t_i)\|} - \frac{F(t_i) - F(t_{i-1})}{\|F(t_i) - F(t_{i-1})\|} \right\|$$

if the quantity of the right-hand side makes sense, otherwise $P(F, D) := 0$. The quantity

$$P_a^b F := \sup_D P(F, D)$$

is called the *turn* of F on $[a, b]$.

Analogously, the *convexity* of F over $[a, b]$ is the quantity

$$K_a^b F := \sup_D K(F, D)$$

where

$$K(F, D) := \sum_{i=1}^{n-1} \left\| \frac{F(t_{i+1}) - F(t_i)}{t_{i+1} - t_i} - \frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}} \right\|.$$

We shall sometimes use the alternative notation $K(F, [a, b]) := K_a^b F$.

Lemma 1.1. *Let $K \subset \mathbb{R}^2$ be a two-dimensional compact convex set, and $r: [0, \ell] \rightarrow \partial K$ a parametrization by the arc-length of ∂K as a simple closed curve. Then r admits the right derivative $r'_+(t)$ at each $t \in [0, \ell)$, and the left derivative $r'_-(t)$ at each $t \in (0, \ell]$. Moreover,*

$$K_0^\ell r = P_0^\ell r \leq 2\pi.$$

Proof. We can (and do) assume that r is counterclockwise oriented. Given a partition $D = \{0 = t_0 < t_1 < \dots < t_n = \ell\}$, convexity of K implies that the vectors $v_i := \frac{r(t_i) - r(t_{i-1})}{\|r(t_i) - r(t_{i-1})\|}$, $i = 1, \dots, n$, are “ordered in the counterclockwise way” on the unit circumference in the plane, and $P(r, D)$ is the length of the corresponding simple (not necessarily closed) polygonal curve inscribed in the unit circle. Thus clearly $P_0^\ell r \leq 2\pi$ (= the length of the unit circumference). By [2, Theorem 4.10], r has finite “turn of tangents” $T_0^\ell r$ (for definition see [2, pp. 25–26]) which satisfies $T_0^\ell r = P_0^\ell r$. By [2, Lemma 4.4], r admits at each point both one-sided derivatives $r'_\pm(t)$ and they are equal to the corresponding “one-sided tangents” $\tau_\pm(r, t)$ (for definition see [2, p. 25]). Finally, by [2, Proposition 5.7], r has finite convexity and $K_0^\ell r = T_0^\ell r = P_0^\ell r \leq 2\pi$ holds. \square

2. THE COUNTEREXAMPLE

There exists a Lipschitz real-valued function f on $D := (-2, 2)^2$ such that f is not DC and satisfies condition (C).

Our construction will proceed in three steps. In the first step we will construct a DC mapping (“curve”) $\varphi: \mathbb{R} \rightarrow D$. The second step will produce a Lipschitz function $f: D \rightarrow \mathbb{R}$ such that $f \circ \varphi$ is not DC (which implies that f is not DC). In the third, most difficult step, we will prove that condition (C) is satisfied.

First step: construction of φ .

Choose a sequence $a_n > 0$, $n \in \mathbb{N}$, with $\sum_{n=1}^{\infty} a_n = 1/2$. Then we can clearly choose $h_n > 0$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} h_n = 1$, and $\sum_{n=1}^{\infty} h_n/a_n < \infty$.

Further set $b_1 := 0$, $b_n := 2 \sum_{i=1}^{n-1} a_i$ for $n \geq 2$ and $c_n := b_n + a_n$, $n \in \mathbb{N}$. Thus $0 = b_1 < c_1 < b_2 < c_2 < \dots < 1$, $\lim b_n = \lim c_n = 1$ and $c_n - b_n = b_{n+1} - c_n = a_n$, $n \in \mathbb{N}$.

Let g be the (unique) function on \mathbb{R} such that

- (i) $g(b_1) = 0$, $g(c_1) = 0$, $g(b_n) = g(c_n) = \sum_{i=1}^{n-1} h_i$ for $n \geq 2$, $g(1) = \sum_{i=1}^{\infty} h_i = 1$,
- (ii) g is affine on each $[b_n, c_n]$ and $[c_n, b_{n+1}]$, $n \in \mathbb{N}$, and
- (iii) g is constant on $(-\infty, 0]$ and $[1, \infty)$.

Set

$$\varphi(t) := \begin{cases} (t, g(t)) & \text{for } t \in [0, 1], \\ (0, 0) & \text{for } t < 0, \\ (1, 1) & \text{for } t > 1. \end{cases}$$

Obviously, g is nondecreasing and $\varphi(\mathbb{R}) \subset D$. Since $\sup\{h_n/a_n : n \in \mathbb{N}\} < \infty$, it is easy to see that both g and φ are Lipschitz. Further, the right derivative $\varphi'_+(t)$ equals: $(1, 0)$ for $t \in [b_n, c_n)$; $(1, h_n/a_n)$ for $t \in [c_n, b_{n+1})$; and $(0, 0)$ for $t \notin [0, 1)$. Clearly $\lim_{s \rightarrow 1^-} \varphi'_+(s) = (1, 0)$ since $h_n/a_n \rightarrow 0$. Now, (1) and the definition of φ easily yield

$$V(\varphi'_+, \mathbb{R}) \leq V(\varphi'_+, [0, 1)) + \|(1, 0)\| = 2 \sum_{n=1}^{\infty} h_n/a_n + 2 < \infty.$$

By [7, Theorem 2.3] we obtain that φ is DC.

Second step: construction of f .

Let $t_0^{(n)} = b_n < t_1^{(n)} < t_2^{(n)} < \dots < t_n^{(n)} < c_n = t_{n+1}^{(n)}$ be an equidistant partition of the interval $[b_n, c_n]$ ($n \in \mathbb{N}$); so $t_{j+1}^{(n)} - t_j^{(n)} = (c_n - b_n)/(n+1) = a_n/(n+1)$.

Choose $r_n > 0$, $n \in \mathbb{N}$, so small that

$$(2) \quad r_n < \frac{a_n}{4(n+1)}, \quad r_n < \frac{h_n}{2} \quad \text{and} \quad r_n < \frac{h_{n-1}}{2} \quad (n \geq 2),$$

$$(3) \quad r_n + \frac{4r_n(n+1)}{a_n}(1 - b_n) < \frac{h_n}{2}, \quad \text{and}$$

$$(4) \quad -r_n - \frac{4r_n(n+1)}{a_n}c_n > -\frac{h_{n-1}}{2}, \quad n \geq 2.$$

Now, for $n \in \mathbb{N}$ and $1 \leq j \leq n$, set $z_j^{(n)} := \varphi(t_j^{(n)}) = (t_j^{(n)}, g(t_j^{(n)}))$. The first inequality of (2) implies that the balls

$$B_j^{(n)} := B(z_j^{(n)}, r_n) \quad (n \in \mathbb{N}, 1 \leq j \leq n)$$

are pairwise disjoint subsets of D , and so we can define $f: D \rightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{n^2}(r_n - \|x - z_j^{(n)}\|) & \text{if } x \in B_j^{(n)}, \\ 0 & \text{if } x \in D \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^n B_j^{(n)}. \end{cases}$$

It is easy to see that f is Lipschitz with constant 1. Put $F(t) := f(\varphi(t))$, $t \in \mathbb{R}$, and notice that $F(t_j^{(n)} \pm r_n) = 0$, $F(t_j^{(n)}) = r_n/n^2$, and F is affine on $[t_j^{(n)} - r_n, t_j^{(n)}]$ and $[t_j^{(n)}, t_j^{(n)} + r_n]$ ($n \in \mathbb{N}$). Now (supposing that F'_+ exists on \mathbb{R}), we have

$$\begin{aligned} V(F'_+, [0, 1]) &\geq \sum_{n=1}^{\infty} V(F'_+, [b_n, c_n]) \geq \sum_{n=1}^{\infty} \sum_{j=1}^n |F'_+(t_j^{(n)} - r_n) - F'_+(t_j^{(n)})| \\ &\geq \sum_{n=1}^{\infty} n \cdot \frac{2(r_n/n^2)}{r_n} = \infty. \end{aligned}$$

So, [7, Theorem 2.3] implies that F is not DC. Consequently f is not DC either (see e.g. [7, Theorem 4.2 and Theorem 1.20]).

Third step: proof of (C).

In the sequel we will essentially use the following proposition which is a special case of [8, Theorem 4.1].

Proposition 2.1. *Let $D \subset \mathbb{R}^n$ be an open convex set and let f be a non-constant DC function on D which admits a Lipschitz control function γ . Let $\varphi: [\alpha, \beta] \rightarrow D$ be Lipschitz. Then*

$$K_\alpha^\beta(f \circ \varphi) \leq (\text{Lip} f + \text{Lip} \gamma) K_\alpha^\beta \varphi + 2 \text{Lip} \gamma \text{Lip} \varphi.$$

Here the notion of a *control function* is used, which is essential in the theory of DC mappings (see [7]). If f is a real DC function then γ is a control function for f iff both $\pm f + \gamma$ are convex. So,

(5) if $f = g - h$, with g, h convex, then $g + h$ is a control function for f .

Remark 2.2. Proposition 2.1 naturally motivates the following problem, which is a special case of [7, Problem 7].

Problem. Let $D \subset \mathbb{R}^n$ be an open convex set and let $f: D \rightarrow \mathbb{R}$ be a function. Suppose that there are $a \geq 0, b \geq 0$ such that

$$K_\alpha^\beta(f \circ \varphi) \leq a K_\alpha^\beta \varphi + b \text{Lip} \varphi$$

whenever $\varphi : [\alpha, \beta] \rightarrow D$ is Lipschitz. Is then f DC on D ?

Let us note that if $n = 2$ and a, b from the above problem exist, then condition (C) from the introduction holds with $C(D, f) = 2\pi a + b$. Indeed, using Proposition 2.1 with $\varphi(t) = r_K(t)$, $t \in [\alpha, \beta] = [0, T_K]$, then we obtain

$$K_\alpha^\beta \Phi \leq aK_\alpha^\beta r_K + b\text{Lip } r_K \leq 2\pi a + b$$

by Lemma 1.1 and 1-lipschitzness of r_K . So $K_\alpha^\beta \Phi < \infty$, and therefore $K_\alpha^\beta \Phi = V(\Phi'_+, [\alpha, \beta])$ (see, e.g., [8, Theorem 3.1]). Using (1) and the equality $\Phi'_-(\beta) = \lim_{t \rightarrow \beta^-} \Phi'_+(t)$ (see, e.g., [8, Proposition 3.4]), we obtain

$$V(\Phi', [0, T_K]) \leq V(\Phi'_+, [\alpha, \beta]) \leq 2\pi a + b.$$

So, if [4, Theorem 1] were correct, the above problem would have a positive answer for $n = 2$.

We continue with the following claim.

Claim 1. *Let $[u, v] \subset (-2, 2)$, and let $h : [u, v] \rightarrow (-2, 2)$ be a continuous function which is convex or concave. Then there exists at most one $n \in \mathbb{N}$ such that the graph of h intersects both $B_j^{(n)}$ and $B_k^{(n)}$ for some $1 \leq j < k \leq n$.*

Proof of Claim 1. Assume that the assertion of the claim is false. First, let h be convex. Then we can choose $1 \leq m < n$, $1 \leq j < k \leq n$, $1 \leq l \leq m$ and $x_1, x_2, x \in [u, v]$ such that

$$(6) \quad (x_1, h(x_1)) \in B_j^{(n)}, (x_2, h(x_2)) \in B_k^{(n)} \text{ and } (x, h(x)) \in B_l^{(m)}.$$

Clearly $b_n < x_1 < x_2 < c_n$ and $b_m < x < c_m$. Using convexity of h , we obtain

$$\begin{aligned} \frac{(g(c_n) - r_n) - h(x)}{x_1 - x} &\leq \frac{h(x_1) - h(x)}{x_1 - x} \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \\ &\leq \frac{(g(b_n) + r_n) - (g(b_n) - r_n)}{\frac{a_n}{2(n+1)}} = \frac{4(n+1)r_n}{a_n}, \end{aligned}$$

and hence

$$(7) \quad g(c_n) - r_n - h(x) \leq \frac{4(n+1)r_n}{a_n} (x_1 - x) \leq \frac{4(n+1)r_n}{a_n} c_n.$$

Notice that, in case $m < n - 1$, we have $g(c_{n-1}) = g(c_m) + \sum_{i=m}^{n-2} h_i > g(c_m) + \frac{h_m}{2}$. Consequently, we have

$$\begin{aligned} g(c_{n-1}) + \frac{h_{n-1}}{2} &\geq g(c_m) + \frac{h_m}{2} \geq h(x) - r_m + \frac{h_m}{2} \stackrel{\text{by (2)}}{>} h(x) \\ &\stackrel{\text{by (7)}}{\geq} g(c_n) - r_n - \frac{4(n+1)r_n}{a_n} c_n \stackrel{\text{by (4)}}{>} g(c_n) - \frac{h_{n-1}}{2} = g(c_{n-1}) + \frac{h_{n-1}}{2}, \end{aligned}$$

which is a contradiction.

Now, let h be concave. Then we choose $1 \leq n < m$, $1 \leq j < k \leq n$, $1 \leq l \leq m$ and $x_1, x_2, x \in [u, v]$ such that (6) holds.

Similarly as above, concavity of h and our construction imply that

$$\begin{aligned} \frac{4(n+1)r_n}{a_n} &= \frac{(g(b_n) + r_n) - (g(b_n) - r_n)}{\frac{a_n}{2(n+1)}} \geq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \\ &\geq \frac{h(x) - h(x_2)}{x - x_2} \geq \frac{h(x) - g(b_n) - r_n}{x - x_2}, \end{aligned}$$

and hence

$$(8) \quad h(x) - g(b_n) - r_n \leq \frac{4(n+1)r_n}{a_n} (x - x_2) \leq \frac{4(n+1)r_n}{a_n} (1 - b_n).$$

Notice that $g(b_m) = g(b_n) + \sum_{i=n}^{m-1} h_i \geq g(b_n) + \frac{h_n}{2} + \frac{h_{m-1}}{2}$. Thus we have

$$\begin{aligned} g(b_n) + \frac{h_n}{2} &\leq g(b_m) - \frac{h_{m-1}}{2} \leq h(x) + r_m - \frac{h_{m-1}}{2} \stackrel{\text{by (2)}}{<} h(x) \\ &\stackrel{\text{by (8)}}{\leq} g(b_n) + r_n + \frac{4(n+1)r_n}{a_n} (1 - b_n) \stackrel{\text{by (3)}}{<} g(b_n) + \frac{h_n}{2}, \end{aligned}$$

which is a contradiction. \square

Claim 2. f is differentiable at $\varphi(1) = (1, 1)$ with $f'(\varphi(1)) = 0$.

Proof. Recall that $f \equiv 0$ outside the balls $B_j^{(n)}$ ($1 \leq j \leq n < \infty$), and $f(\varphi(1)) = 0$. Let $P_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the canonical projection on the first coordinate. For $x \in B_j^{(n)}$, we have $\|x - \varphi(1)\| \geq 1 - P_1(x) \geq 1 - (t_j^{(n)} + r_n) > 1 - c_n > b_{n+1} - c_n = a_n$, and hence (by (2))

$$0 \leq \frac{f(x)}{\|x - \varphi(1)\|} \leq \frac{f(z_j^{(n)})}{a_n} = \frac{r_n}{n^2 a_n} < \frac{1}{4(n+1)n^2}.$$

Since $\varphi(1) \notin B_j^{(n)}$ ($1 \leq j \leq n < \infty$), it is easy to see that $\lim_{x \rightarrow \varphi(1)} \frac{f(x)}{\|x - \varphi(1)\|} = 0$. We are done. \square

Convention. In what follows, given a mapping $F: [0, \ell] \rightarrow \mathbb{R}^d$ such that $F(\ell) = F(0)$, if necessary we consider F extended to \mathbb{R} as a (uniquely determined) ℓ -periodic function. So, for instance, if the right derivative $F'_+(t)$ exists at each $t \in [0, \ell)$ then the variation $V(F'_+, [0, \ell])$ is calculated with $F'_+(\ell) := F'_+(0)$.

It is easy to see that this ‘‘circular variation’’ $V(F'_+, [0, \ell])$ does not depend on shifts of the parameter. More precisely, if F is extended periodically, $d \in \mathbb{R}$ and $G(t) := F(t + d)$, then $V(G'_+, [0, \ell]) = V(F'_+, [0, \ell])$.

We are going to prove that *our (Lipschitz non-DC) function f satisfies the following, a bit stronger variant of property (C):*

(C') *There exists a constant $c > 0$ such that for each two-dimensional compact convex set $K \subset D$ and for each arc-length parametrization $r: [0, \ell] \rightarrow \partial K$ of ∂K as a simple closed curve, and for $\Phi := f \circ r$, we have (in the sense of our Convention):*

- (a) *both one-sided derivatives $\Phi'_+(t)$ and $\Phi'_-(t)$ exist finite at each $t \in [0, \ell]$;*
 (b) *$V(\Phi'_+, [0, \ell]) \leq c$ and $V(\Phi'_-, [0, \ell]) \leq c$.*

To prove this, let $K \subset D$ and $r: [0, T] \rightarrow \partial K$ be as in (C'). We can clearly assume that the parametrization r is counterclockwise. The boundary ∂K is the union of four parts:

$$\partial K = G_1 \cup G_2 \cup G_3 \cup G_4,$$

where:

- G_1 and G_3 are the graphs of a continuous convex and a continuous concave function, respectively, both defined on an interval $[\alpha, \beta] \subset (-2, 2)$;
- G_2, G_4 are two (possibly degenerate) vertical line segments.

We can (and do) assume that r “starts with the convex part”, that is, for some $0 < \ell_1 \leq \ell_2 < \ell_3 \leq \ell$ we have

$$r([0, \ell_1]) = G_1, \quad r([\ell_1, \ell_2]) = G_2, \quad r([\ell_2, \ell_3]) = G_3, \quad r([\ell_3, \ell]) = G_4.$$

(i) *The “convex part” $G_1 = r([0, \ell_1])$.*

First suppose that there exists $t_0 \in (0, \ell_1]$ such that $r(t_0) = \varphi(1) \equiv (1, 1)$. Given $\varepsilon \in (0, t_0)$, the set

$$L_\varepsilon := \{(n, j) : n, j \in \mathbb{N}, j \leq n, r([0, t_0 - \varepsilon]) \cap B_j^{(n)} \neq \emptyset\}$$

is finite. Let $n_\varepsilon := \max\{n \in \mathbb{N} : (n, j) \in L_\varepsilon \text{ for some } j \leq n\}$, and apply Claim 1 to choose an $m \leq n_\varepsilon$ such that $(n, j) \in L_\varepsilon$ for at most one j whenever $n \leq n_\varepsilon$ and $n \neq m$. Let $P_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ be as in the proof of Claim 2. Since the projections

$$I_j^{(n)} := P_1(B_j^{(n)}), \quad (n, j) \in L_\varepsilon,$$

are finitely many pairwise disjoint compact subintervals of $(-2, 2)$, there exist open intervals $U_j^{(n)}$, $(n, j) \in L_\varepsilon$, such that $I_j^{(n)} \subset U_j^{(n)} \subset \overline{U_j^{(n)}} \subset (-2, 2)$, $(n, j) \in L_\varepsilon$, and the intervals $\overline{U_j^{(n)}}$, $(n, j) \in L_\varepsilon$, are pairwise disjoint.

Let $J_j^{(n)}$, $(n, j) \in L_\varepsilon$, be relatively open subintervals of $[0, t_0 - \varepsilon]$ such that

$$K_j^{(n)} := r^{-1}(B_j^{(n)}) \cap [0, t_0 - \varepsilon] \subset J_j^{(n)} \subset \overline{J_j^{(n)}} \subset r^{-1}(U_j^{(n)} \times (-2, 2))$$

for each $(n, j) \in L_\varepsilon$. (This is clearly possible since $P_1 \circ r : [0, \ell_1] \rightarrow [\alpha, \beta]$ is an increasing homeomorphism.) Notice that

$$f(x) = \max \left\{ \frac{1}{n^2} (r_n - \|x - z_j^{(n)}\|), 0 \right\}, \quad x \in U_j^{(n)} \times (-2, 2), \quad (n, j) \in L_\varepsilon.$$

It is easy to see that the formula

$$f(x) = \left(f(x) + \frac{1}{n^2} \|x - z_j^{(n)}\| \right) - \frac{1}{n^2} \|x - z_j^{(n)}\|, \quad x \in U_j^{(n)} \times (-2, 2),$$

is a representation of f as a difference of two continuous convex functions. It follows from (5) that f is controlled on each $U_j^{(n)} \times (-2, 2)$, $(n, j) \in L_\varepsilon$, by the function

$$\gamma(x) = f(x) + \frac{2}{n^2} \|x - z_j^{(n)}\| = \max \left\{ \frac{r_n}{n^2} + \frac{1}{n^2} \|x - z_j^{(n)}\|, \frac{2}{n^2} \|x - z_j^{(n)}\| \right\},$$

which is Lipschitz with constant $\frac{2}{n^2}$. Denote $\Phi := f \circ r$.

Fix $(n, j) \in L_\varepsilon$. If $J_j^{(n)}$ is open in \mathbb{R} , put $\mathcal{J}_j^{(n)} := J_j^{(n)}$. Otherwise, since the length of $J_j^{(n)}$ is at most ℓ_1 , there exists an open (in \mathbb{R}) interval $\mathcal{J}_j^{(n)}$ of length smaller than ℓ , such that $J_j^{(n)} \subset \mathcal{J}_j^{(n)}$ and $r(\overline{\mathcal{J}_j^{(n)}}) \subset U_j^{(n)} \times (-2, 2)$. Then (we use Convention above for r and Φ) $K(r, \overline{\mathcal{J}_j^{(n)}}) \leq K_0^\ell r \leq 2\pi$ by Lemma 1.1, and applying Proposition 2.1 we obtain

(9)

$$K(\Phi, \overline{\mathcal{J}_j^{(n)}}) \leq (\text{Lip}(f) + \text{Lip}(\gamma)) K(r, \overline{\mathcal{J}_j^{(n)}}) + 2\text{Lip}(\gamma) \text{Lip}(r) \leq \frac{3}{n^2} 2\pi + \frac{4}{n^2}.$$

Hence by [8, Proposition 3.4, p. 328], both derivatives Φ'_\pm exist at each point of $\mathcal{J}_j^{(n)}$, and we have

$$(10) \quad V(\Phi'_\pm, J_j^{(n)}) \leq V(\Phi'_\pm, \mathcal{J}_j^{(n)}) = K(\Phi, \overline{\mathcal{J}_j^{(n)}}) \leq \frac{6\pi + 4}{n^2}.$$

Since $f \equiv 0$ outside all the balls $B_j^{(n)}$, $1 \leq j \leq n < \infty$, we have that $\Phi \equiv 0$ outside the compact sets $K_j^{(n)}$, $(n, j) \in L_\varepsilon$. Therefore both derivatives Φ'_\pm exist at each point of $(0, t_0 - \varepsilon)$, and

$$\Phi'_\pm(x) = 0, \quad x \in (0, t_0 - \varepsilon) \setminus \bigcup \{K_j^{(n)} : (n, j) \in L_\varepsilon\}.$$

Choose relatively open subintervals $M_j^{(n)}$, $(n, j) \in L_\varepsilon$, of $[0, t_0 - \varepsilon]$ such that $K_j^{(n)} \subset M_j^{(n)} \subset \overline{M_j^{(n)}} \subset J_j^{(n)}$. Consider the extreme points of all the intervals $M_j^{(n)}$, $(n, j) \in L_\varepsilon$, and index them in an increasing order to obtain the points

$$0 \leq u_1 < v_1 < u_2 < v_2 < \cdots < u_d < v_d \leq t_0 - \varepsilon,$$

where $d = \text{card } L_\varepsilon$. Then for each $(n, j) \in L_\varepsilon$ there is a unique $1 \leq k \leq d$ such that $\inf M_j^{(n)} = u_k$ and $\sup M_j^{(n)} = v_k$.

Observe that both Φ'_\pm are identically null: on $(0, u_1]$ (if $u_1 > 0$); on $[v_d, t_0 - \varepsilon)$ (if $v_d < t_0 - \varepsilon$); on all intervals $[v_{k-1}, u_k]$, $2 \leq k \leq d$; on a neighborhood of any of the points u_k, v_k ($1 \leq k \leq d$) that belongs to $(0, t_0 - \varepsilon)$. Now, we can use this observation together with the additivity of variation and (1) to write

$$\begin{aligned} V(\Phi'_\pm, (0, t_0 - \varepsilon)) &= V(\Phi'_\pm, (u_1, v_d)) \\ &= V(\Phi'_\pm, (u_1, v_2]) + \sum_{k=2}^{d-1} V(\Phi'_\pm, [u_k, v_k]) + V(\Phi'_\pm, [u_d, v_d)) \\ &= \sum_{k=1}^d V(\Phi'_\pm, (u_k, v_k)) \leq \sum_{(n,j) \in L_\varepsilon} V(\Phi'_\pm, J_j^{(n)}). \end{aligned}$$

Thus, by (10) and the properties of L_ε (see the text after its definition), we obtain

$$\begin{aligned} V(\Phi'_\pm, (0, t_0 - \varepsilon)) &\leq \sum_{n=1}^{n_\varepsilon} \frac{6\pi + 4}{n^2} + m \frac{6\pi + 4}{m^2} \\ &\leq (6\pi + 4) \left(1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \right) =: M. \end{aligned}$$

Consequently, $V(\Phi'_\pm, (0, t_0)) \leq M$ since $\varepsilon \in (0, t_0)$ was arbitrary.

We claim that $V(\Phi'_\pm, (0, \ell_1)) \leq M + 1$. This is obvious if $t_0 = \ell_1$. If $t_0 \neq \ell_1$, we have $0 < t_0 < \ell_1$, $\Phi'(t_0) = 0$ (by Lemma 1.1 and Claim 2), and $\Phi \equiv 0$ on $[t_0, \ell_1)$. Since Φ is Lipschitz with constant 1, we can use (1) to get

$$V(\Phi'_\pm, (0, \ell_1)) = V(\Phi'_\pm, (0, t_0)) \leq V(\Phi'_\pm, (0, t_0)) + 1 \leq M + 1,$$

and we are done.

In case that t_0 does not exist, G_1 intersects at most finitely many balls $B_j^{(n)}$, and we can get the same estimate $V(\Phi'_\pm, (0, \ell_1)) \leq M + 1$ directly (as above with ℓ_1 in place of $t_0 - \varepsilon$).

(ii) The “concave part” $G_3 = r([\ell_2, \ell_3])$.

This part can be treated in the very same way to obtain $V(\Phi'_\pm, (\ell_2, \ell_3)) \leq M + 1$.

(iii) The two vertical segments $G_2 = r([\ell_1, \ell_2])$ and $G_4 = r([\ell_3, \ell])$.

By our construction, each of the two vertical segments intersects at most one of the balls $B_j^{(n)}$, $1 \leq j \leq n < \infty$. As in (9), for such n we have $K(\Phi, [\ell_1, \ell_2]) \leq \frac{6\pi+4}{n^2} \leq M + 1$. As above, it follows that $V(\Phi'_\pm, (\ell_1, \ell_2)) \leq M + 1$ provided $\ell_1 < \ell_2$; and in the same way $V(\Phi'_\pm, (\ell_3, \ell)) \leq M + 1$ provided $\ell_3 < \ell$.

(iv) Conclusion of the proof of (C').

For simplicity denote $\ell_0 := 0$ and $\ell_4 := \ell$. Recall that Φ is Lipschitz with

constant 1. Now, combining (1) with [8, Propositions 3.4, pp. 328–329], we obtain for $i = 1, 2, 3, 4$ that the derivatives $\Phi'_\pm(\ell_i)$ exist, and

$$V(\Phi'_+, [\ell_{i-1}, \ell_i]) \leq V(\Phi'_+, (\ell_{i-1}, \ell_i)) + 2 \leq M + 3.$$

Thus

$$V(\Phi'_+, [0, \ell]) = \sum_{i=1}^4 V(\Phi'_+, [\ell_{i-1}, \ell_i]) \leq 4(M + 3)$$

and, symmetrically, $V(\Phi'_-, [0, \ell]) \leq 4(M + 3)$. The proof is complete.

Acknowledgement. The research of the first author was partially supported by the MIUR, Italy, and by the GNAMPA - INdAM. The research of the second author was partially supported by the grant P201/15-08218S from the Grant Agency of Czech Republic.

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