ENDPOINT ESTIMATE FOR ROUGH MAXIMAL SINGULAR INTEGRALS

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ABSTRACT. We study the rough maximal singular integral

$$T_\Omega^\#(f)(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} |y|^{-n}\Omega(y/|y|)f(x-y)dy\right|,$$

where Ω is a function in $L^\infty(\mathbb{S}^{n-1})$ with vanishing integral. It is well known that the operator is bounded on L^p for $1<p<\infty$, but it is an open question if has to be of the weak type 1-1. We show that $T_\Omega^\#$ is bounded from $L(\log\log L)^{2+\varepsilon}$ to $L^{1,\infty}$ locally.

CONTENTS

1. INTRODUCTION

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [1]. The study of boundedness of rough singular integrals of convolution type has been an active area of research since the middle of the twentieth century. Calderón and Zygmund [2] first studied the rough singular integral

$$T_\Omega(f)(x) = \text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(y/|y|)}{|y|^n}f(x-y)\,dy$$

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where Ω is in $L\log L(\mathbb{S}^{n-1})$ with mean value zero and showed that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The weak type $(1,1)$ boundedness of T_Ω when $n = 2$ was established by Christ and Rubio de Francia [3] and independently by Hofmann [7]. The weak type $(1,1)$ property of T_Ω was proved by Seeger [8] in all dimensions and was later extended by Tao [10] to situations in which there is no Fourier transform structure. Several questions remain concerning the endpoint behavior of T_Ω, such as if the condition $\Omega \in L\log L(\mathbb{S}^{n-1})$ can be relaxed to $\Omega \in H^1(\mathbb{S}^{n-1})$, or merely $\Omega \in L^1(\mathbb{S}^{n-1})$ when Ω is an odd function.

The maximal counterpart of the rough singular integral,

$$T^{\#}_\Omega(f)(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} |y|^{-n}\Omega(y/|y|)f(x-y)dy\right|, \tag{1}$$

where Ω is in $L\log L(\mathbb{S}^{n-1})$ with mean value zero is bounded on L^p for $1 < p < \infty$, but it remains an open question if this operator is of the weak type $(1-1)$ even for a choice of $\Omega \in L^\infty(\mathbb{S}^{n-1})$. The standard Calderón and Zygmund method requires some Dini type smoothness on Ω, while the method of Christ, Rubio de Francia, Hofmann and Seeger fails due to the lack of orthogonality in the maximal function. This classical question was posed many times, for example recently in [6] as Problem 8.3.

While the full problem remains open, we provide an endpoint estimate, showing that for $\Omega \in L^\infty(\mathbb{S}^{n-1})$ the operator $T^{\#}_\Omega(f)(x)$ locally maps the space $L(\log\log L)^{2+\varepsilon}$ to $L^{1,\infty}$. Let us remind here that the space $L(\log\log L)^{2+\varepsilon}$ is the Orlicz space with the norm

$$\|f\|_{L(\log\log L)^{2+\varepsilon}} = \inf\{\lambda > 0 : \int |f/\lambda|\log(10+\log(10+|f/\lambda|))dx \le 1\}.$$

This is a better estimate than what follows from the standard extrapolation argument.

The main result of this paper is the following theorem.

Theorem 1. *Suppose $n \ge 2$, $\Omega \in L^\infty(\mathbb{S}^{n-1})$, $\varepsilon > 0$ and $T^{\#}_\Omega$ is as defined in* (1). *Then for a function $f \in L(\log\log L)^{2+\varepsilon}$, supported in the unit cube, we have*

$$\|T^{\#}_\Omega f\|_{L^{1,\infty}} \le C_\varepsilon \|f\|_{L(\log\log L)^{2+\varepsilon}}. \tag{2}$$

For the proof, we use the double dyadic decomposition of the kernel, which has been used previously by many authors, and carefully apply the method developed by Seeger at each level.

2. Decomposition of the operator

Let us fix a function $\Omega \in L^\infty(\mathbb{S}^{n-1})$ with mean value zero. We fix a smooth function α in $\mathbb{R}^+$ such that $\alpha(t) = 1$ for $t \in (0,1]$, $0 < \alpha(t) < 1$

for $t \in (1, \frac{4}{3})$ and $\alpha(t) = 0$ for $t \geq \frac{4}{3}$. For $y \in \mathbb{R}^n$ and $j \in \mathbb{Z}$ we introduce the function

$$\beta_j(y) = \alpha(2^{-j}|y|) - \alpha(2^{-j+1}|y|).$$

We write $\beta = \beta_0$ and we note that this is a function supported in $\{|y| \in [1/2, 4/3]\}$.

Next, we fix a positive smooth function γ_0 supported in $B(0, 1/100)$ with $\int \gamma = 1$, we denote $\gamma_j(x) = 2^{jn}\gamma(2^j x)$ and we denote $\Delta_j f = (\gamma_{j+1} - \gamma_j) * f$.

Now, we decompose the kernel K as follows: we denote $K^i = \beta_i K$ and we set $K^i_j = \Delta_{j-i} K^i$ for $i, j \in \mathbb{Z}$. We see that

$$K = \sum_{i=-\infty}^{\infty} \left(\gamma_{-i} * K^i + \sum_{j=0}^{\infty} K^i_j \right).$$

Let us make the stadard reduction to the smooth dyadic truncations. We see that

$$T^{\#}_{\Omega}(f)(x) \leq CMf(x) + \sup_{k \in \mathbb{Z}} \left| f * \sum_{i>k} K^i \right| (x),$$

where M is the Hardy-Littlewood maximal operator.

Next, we write

$$\sup_{k \in \mathbb{Z}} \left| f * \sum_{i>k} K^i \right| (x) = \sup_{k \in \mathbb{Z}} \left| f * \sum_{i>k} \left(\gamma_{-i} * K^i + \sum_{j=0}^{\infty} K^i_j \right) \right| (x). \tag{3}$$

We define following maximal operators:

$$T^{\#}_{-1} f(x) = \sup_{k \in \mathbb{Z}} \left| \sum_{i>k} f * \left(\gamma_{-i} * K^i \right) \right| (x)$$

and for $m \geq 0$ and $i \in \mathbb{Z}$ we denote

$$H^i_m = \sum_{j=2^m-1}^{2^{m+1}-2} K^i_j$$

and we put

$$T^{\#}_m f(x) = \sup_{k \in \mathbb{Z}} \left| \sum_{i>k} f * H^i_m \right| (x).$$

Therefore, we have

$$T^{\#}_{\Omega}(f)(x) \leq CMf(x) + \sum_{m=-1}^{\infty} T^{\#}_m f(x). \tag{4}$$

Let us record two trivial estimates for convolutions of the functions γ_j, which will be usefull later.

Lemma 2. *Let us have $a,b,c \in \mathbb{Z}$. If $a > b$, then*

$$\int |\gamma_a * \gamma_b - \gamma_b| \leq C 2^{\delta(b-a)}$$

and the function $\gamma_a * \gamma_b - \gamma_b$ is supported in $B(0, 2^{-nb}/50)$. If $a < b < c$, then

$$\int |\gamma_a * (\gamma_c - \gamma_b)| \leq C 2^{\delta(a-b)}$$

and the function $\gamma_a * (\gamma_c - \gamma_b)$ is supported in $B(0, 2^{-na}/50)$. δ is a positive constant.

Both statements are clear from the suport properties of γ and the mean value theorem.

3. The estimates of Seeger

We recall here the estimates A. Seeger uses in his article [8]. Let $\{H_j\}$ be a family of functions with

$$\mathrm{supp} H_j \subset \{x : 2^{j-2} \leq |x| \leq 2^{j+2}\}.$$

Suppose that for each $N = 0, \ldots, n+1$ we have estimates

$$\sup_{0 \leq l \leq N} \sup_j r^{n+1} \left| \left(\frac{\partial}{\partial r} \right)^l H_j(\theta r) \right| \leq \mathcal{M}_N \tag{5}$$

uniform in $\theta \in S^{n-1}$ and $r > 0$. Then for each positive integer $s > 3$ and a parameter $\kappa \in (0,1)$ there is a splitting

$$H_j = \Gamma_j^s + (H_j - \Gamma_j^s)$$

such that the following two estimates are valid.

Lemma 3. *Let $\mathcal{Q}$ be a collection of cubes Q with disjoint interiors. Define $L(Q) = m$ if $2^{m-1} < \mathrm{sidelength}(Q) \leq 2^m$ and let $\mathcal{Q}_m = \{Q \in \mathcal{Q} : L(Q) = m\}$. For each Q let f_Q be an integrable function supported in Q satisfying*

$$\int |f_Q(x)| dx \leq \alpha |Q|.$$

Let $F_m = \sum_{Q \in \mathcal{Q}_m} f_Q$. Then for $s > 3$

$$\|\sum_j \Gamma_j^s * F_{j-s}\|_2^2 \leq C \mathcal{M}_0^2 2^{-s(1-\kappa)} \alpha \sum_Q \|f_Q\|_1.$$

Lemma 4. *Let Q be a cube of sidelength 2^{j-s} and let b_Q be integrable and supported in Q; moreover, suppose that $\int b_Q = 0$. Then for $N = n+1$ and $0 \leq \varepsilon \leq 1$*

$$\|(H_j - \Gamma_j^s) * b_Q\|_1 \leq C_N \left[\mathcal{M}_0 2^{-s\varepsilon} + \mathcal{M}_N 2^{s(n+(\varepsilon-\kappa)N)} \right] \|b_Q\|_1$$

where C_N does not depend on j or Q.

In [8], these are Lemma 2.1. and 2.2.

4. Auxiliary estimates

We complement the estimates of the previous section with the following exponential estimate. We assume here that ν is a function supported in some annulus $\varepsilon \le |x| < \varepsilon^{-1}$ and is smooth and positive. Then we put $\nu_j(x) = 2^{-nj}\nu(2^{-j}x)$.

Lemma 5. *Suppose we have the collection of cubes and the functions from Lemma 3. Then for any $s \ge 0$ and $\lambda \ge 1$ we have*

$$|\{|\sum_j \nu_j * |F_{j-s}|| > \lambda\alpha\}| \le Ce^{-c\lambda}\sum_{\mathcal{Q}}|Q|.$$

Proof. We first show that $\sum_j \nu_j * F_{j-s}$ is in BMO with norm $C\alpha$. Let us fix a cube R. Clearly $|\nabla\nu_j * |F_{j-s}|| \le C\alpha 2^{-j}$, and by summation we get the statement for $\sum_{j\ge L(R)}\nu_j * |F_{j-s}|$. For $j < L(R)$ we get

$$\|(\nu_j * |F_{j-s}|)\chi_R\|_1 \le C\|F_{j-s}\chi_{3R}\|_1.$$

We have $\sum_{j<L(R)}\|F_{j-s}\chi_{3R}\|_1 \le C2^{nL(R)}\alpha$, and so $\sum_{j<L(R)}\nu_j * |F_{j-s}|$ is also in BMO with norm $C\alpha$.

Now we select a system $\mathcal{R}$ of maximal dyadic cubes R such that

$$\|(\sum_j \nu_j * |F_{j-s}|)\chi_R\|_1 \ge \alpha|R|.$$

Clearly our set is contained in $\cup\mathcal{R}$ and

$$\sum_{\mathcal{R}}|R| \le \|\sum_j \nu_j * |F_{j-s}|\|_1/\alpha \le C\sum_{\mathcal{Q}}|Q|.$$

For $\lambda < 2^{n+1}$ the statement is trivial, so let us suppose that $\lambda \ge 2^{n+1}$. For each R we apply the John-Nirenberg theorem at the level $2\alpha\lambda$. From the maximality we see that the average over R is at most $2^n\alpha$. We get

$$|R \cap \{((\sum_j \nu_j * |F_{j-s}|) - 2^n\alpha)^+ > 2\lambda\alpha\}| \le C|R|e^{-c\lambda}.$$

We sum over R to prove the lemma. □

With the help of the previous lemma, we can combine the two estimates of Seeger into a single L^2 bound.

Lemma 6. *Suppose that the functions H_j are as defined in section 3 and we have the collection of cubes and the functions from Lemma 3. Suppose moreover that for each Q $\int f_Q = 0$ and that*

$$|Q| \leq C\alpha^{-1} \int |f_Q(x)|dx.$$

Then for $s > 3$

$$\| \sum_j H_j * F_{j-s} \|_2^2 \leq C 2^{-s\delta} \alpha \sum_Q \|f_Q\|_1.$$

The $\delta > 0$ depends on the radial smoothness of H_j and dimension.

We note that this simplified approach is suitable since we have Ω bounded and full smoothness in the radial direction, while in the article of Seeger much more general situation is studied.

Proof. Combining the Lemma 4 and the Lemma 3 with a suitable choice of κ, we obtain that there is an $\varepsilon > 0$ such that $\sum_j H_j * F_{j-s} = a_1 + a_1$, where

$$\|a_1\|_1 \leq C 2^{-s\varepsilon} \sum_Q \|f_Q\|_1$$

while

$$\|a_2\|_2^2 \leq C 2^{-s\varepsilon} \alpha \sum_Q \|f_Q\|_1.$$

Now, we can choose ν so that ν_j dominates H_j. Therefore, we get the estimate from the Lemma 5 for $a_1 + a_2$. From the L^2 estimate we get

$$|\{|a_2| \geq \alpha\}| \leq \|a_2\|_2^2 / \alpha^2 \leq C 2^{-s\varepsilon} \alpha^{-1} \sum_Q \|f_Q\|_1.$$

Let us consider set $A = \{|a_1| \leq \alpha 2^{s\varepsilon/2}\}$. We see that

$$\int_A |a_1 + a_2|^2 \leq C(\|a_2\|_2^2 + \alpha 2^{s\varepsilon/2} \int |a_1|) \leq C 2^{-s\varepsilon/2} \alpha \sum_Q \|f_Q\|_1.$$

On the other hand

$$A^c \subset \{|a_1 + a_2| \geq \alpha 2^{s\varepsilon/2}/2\} \cup \{|a_2| \geq \alpha/2\}.$$

Now we apply the Lemma 5, we get

$$|A^c| \leq C(2^{-c2^{s\varepsilon/2}} + 2^{-s\varepsilon}) \alpha^{-1} \sum_Q \|f_Q\|_1 \leq C 2^{-s\varepsilon/2} \alpha^{-1} \sum_Q \|f_Q\|_1.$$

Let us put $\lambda = s\varepsilon/2$ and denote for $k \geq 0$

$$B_k = \{(k+1)\lambda\alpha > |a_1 + a_2| \geq k\lambda\alpha\}.$$

Clearly, we may assume that $\lambda > 0$, otherwise the statement of the Lemma is trivial. It follows from the Lemma 5 that for $k > 0$ we get

$$\|B_k\| \leq Ce^{-ck\lambda}\sum_Q |Q| \leq Ce^{-cks\varepsilon/2}\alpha^{-1}\sum_Q \|f_Q\|_1.$$

We then get for $k > 0$

$$\int_{A^c\cap B_k} |a_1+a_2|^2 \leq C((k+1)\alpha s\varepsilon/2)^2|B_k| \leq C((k+1)s\varepsilon/2)^2 e^{-cks\varepsilon/2}\alpha\sum_Q \|f_Q\|_1$$

and for $k = 0$

$$\int_{A^c\cap B_0} |a_1+a_2|^2 \leq C(\alpha s\varepsilon/2)^2|A^c| \leq C(s\varepsilon/2)^2 2^{-s\varepsilon/2}\alpha\sum_Q \|f_Q\|_1.$$

We sum these estimates in k and get

$$\int_{A^c} |a_1+a_2|^2 \leq C(s\varepsilon)^2 2^{-s\varepsilon/2}\alpha\sum_Q \|f_Q\|_1$$

and the lemma follows.

□

We also establish a simple L^2 estimate.

Lemma 7. *Suppose we have the collection of cubes and the functions from Lemma 3 and assume also*

$$|Q| \leq C\alpha^{-1}\int |f_Q(x)|dx.$$

Then for any s

$$\|\sum_j \nu_j * |F_{j-s}|\|_2^2 \leq C\alpha\sum_Q \|f_Q\|_1.$$

Proof. Clearly

$$\|\sum_j \nu_j * |F_{j-s}|\|_1 \leq C\sum_Q \|f_Q\|_1.$$

We denote A the set where the sum is less or equal to α and B the set where it is greater than alpha. We have

$$\int_A \left(\sum_j \nu_j * |F_{j-s}|(x)\right)^2 dx \leq C\alpha\sum_Q \|f_Q\|_1.$$

On the set B, we apply the Lemma 5, we denote B_n the set where the sum is greater than $n\alpha$ which gives

$$\int_B \left(\sum_j \nu_j * |F_{j-s}|(x)\right)^2 dx \leq \sum_n n^2|B_n| \leq \sum_n n^2 Ce^{-cn}\sum_Q |Q| \leq C\alpha\sum_Q \|f_Q\|_1.$$

□

5. THE L^2 ESTIMATE

Lemma 8. *For a $f \in L^2$ we have*

$$\|T_m^{\#} f\|_2 \leq C 2^{-c2^m} \|f\|_2.$$

Proof. The classical Fourier estimates from [5] give us $|\hat{K}^i|(\xi) \leq C \min(|2^i\xi|^{-\varepsilon}, |2^i\xi|^{\varepsilon})$ for some $\varepsilon > 0$. The operator Δ_{j-i} provides essential localization to the annulus $|\xi| \approx 2^{j-i}$. Using standard reasoning one gets that for any index set I and $j \geq 0$

$$|\sum_{i \in I} \hat{K}_j^i| \leq C 2^{-\varepsilon j}.$$

By summing a geometric series, we also have

$$|\sum_{i \in I} \hat{H}_m^i| \leq C 2^{-\varepsilon 2^m}.$$

Let us now define operators

$$T_{m,l}^{\#} f(x) = \sup_{r \in \mathbb{Z}} \left| f * \left(\sum_{s \in \mathbb{Z}, s \geq r} H_m^{s2^{m+2}+l} \right) \right| (x).$$

for $0 \leq l \leq 2^{m+2} - 1$. We see that

$$T_m^{\#} f(x) \leq 2^{m+2} \sup_{0 \leq l \leq 2^{m+2}-1} T_{m,l}^{\#} f(x).$$

Now, we establish following inequality of Cotlar type:

$$T_{m,l}^{\#} f(x) \leq CM \left(f * \left(\sum_{s \in \mathbb{Z}} H_m^{s2^{m+2}+l} \right) \right)(x) + C 2^{-\varepsilon 2^m} M(f)(x), \tag{6}$$

where M is the usual Hardy-Littlewood function.

To see this, fix r and take $t = r2^{m+2} + l - 2^{m+1}$. Let us remind that

$$H_m^i = (\gamma_{2^{m+1}-i-2} - \gamma_{2^m-i-1}) * K^i.$$

We fix $\tau = s2^{m+2} + l$, for $s \geq r$ and we obtain

$$\begin{aligned} &H_m^\tau - \gamma_{-t} * H_m^\tau \\ &= ((\gamma_{2^{m+1}-\tau-2} - \gamma_{2^m-\tau-1}) - \gamma_{-t} * (\gamma_{2^{m+1}-\tau} - \gamma_{2^m-\tau-1})) * K^\tau \end{aligned}$$

and by an application of the Lemma 2 we get

$$|H_m^\tau - \gamma_{-t} * H_m^\tau(x)| \leq C 2^{-c2^m} 2^{-c(s-r)2^m} 2^{-n\tau} \chi_{B(0,2^{\tau+2}) \setminus B(0,2^{\tau-2})}.$$

Therefore

$$|f * (H_m^\tau - \gamma_{-t} * H_m^\tau(x))| \leq C 2^{-c2^m} 2^{-c(s-r)2^m} Mf.$$

For $s < r$ and $0 \leq v \leq 2^m - 1$ we get

$$\gamma_{-t} * H_m^\tau = \gamma_{-t} * (\gamma_{2^{m+1}-\tau} - \gamma_{2^m-\tau-1}) * K^\tau$$

and by the Lemma 2 we get

$$|\gamma_{-t} * H_m^\tau| \le C2^{-c2^m} 2^{-c(r-s)2^m} 2^{-n\tau} \chi_{B(0,2^{\tau+2})\setminus B(0,2^{\tau-2})}.$$

Therefore

$$|\gamma_{-t} * H_m^\tau * f| \le C2^{-c2^m} 2^{-c(r-s)2^m} Mf.$$

Together, this means that

$$\left| \left(\gamma_{-t} * \left(\sum_{s\in\mathbb{Z}} H_m^{s2^{m+2}+l} \right) - \left(\sum_{s\in\mathbb{Z}, s>r} \sum_{v=0}^{2^m-1} H_m^{s2^{m+2}+l} \right) \right) * f \right| (x)$$
$$\le C2^{-c2^m} M|f|(x).$$

and we get (6) by adding and subtracting this term and passing to the supremum in t.

From (6) the lemma follows easily, because it gives the estimate

$$\|T^{\#}_{m,l} f\|_2 \le C2^{-c2^m} \|f\|_2$$

and

$$T^{\#}_m f(x) \le 2^{m+2} \sup_{l\in\mathbb{Z}, 0\le l\le 2^{m+2}} T^{\#}_{m,l} f(x).$$

□

6. The proof of the theorem 1

We start by observing that H_m^i from the definition of our operator satisfy the condition (5). Since the estimate is scaling invariant, it is enough to verify it for $i=0$. We have

$$H_m^0 = \sum_{j=2^m-1}^{2^{m+1}-2} K_j^0 = \sum_{j=2^m-1}^{2^{m+1}-2} \Delta_j K^0 = (\gamma_{2^{m+1}-1} - \gamma_{2^m-1}) * K^0.$$

Therefore, it is enough to show that (5) holds for any $\gamma_k * K^0$, $k \ge 0$. We write

$$\gamma_k * K^0(x) = \int \gamma_k(x-y) * K^0(y) dy.$$

Now we denote $x = \theta r$, $\theta \in S^{n-1}$ and $r > 0$, and pass to polar coordinates. We get

$$\gamma_k * K^0(\theta r) = C \int_{S^{n-1}} \int_{1/4}^{4} (r_1)^{n-1} K^0(\theta_1 r_1) \gamma_k(\theta r - \theta_1 r_1) dr_1 d\theta_1.$$

It is important to make use of the radial smoothness of K^0. Therefore, we make change of variables $r_2 = r_1/r$. We see that by the support properties of γ_k, the effective range of r_2 is $(1-2^{-k}, 1+2^{-k})$, while θ_1 ranges over a region $S_1 \subset S^{n-1}$ of diameter less than 2^{-k}. We write

$$\gamma_k * K^0(\theta r) = C \int_{S_1} \int_{1-2^{-k}}^{1+2^{-k}} r^{n-2} r_2^{n-1} K^0(\theta_1 r r_2) \gamma_k(r(\theta - \theta_1 r_2)) dr_2 d\theta_1.$$

Now, we differentiate l times in r. The radial derivatives of K^0 are bounded, r^{n-2} also has bounded derivatives, and the l-th derivative of γ_k is bounded by $2^{l(n+k)}$, which is counteracted by the fact that $|\theta - \theta_1 r_2| \leq 2^{-k}$ and that we integrate over region with measure 2^{-kn}. Therefore the bounds on the radial derivatives are independent on k.

Now, let us take a function f from the space $L(\log\log L)^{2+\varepsilon}$, supported in the unit cube Q. We may assume that f has unit norm. The operators M and $T^{\#}_{-1}$ are clearly of the weak type, so it is enough to consider $\sum_{m=0}^{\infty} T^{\#}_m(f)$. Let us fix $\alpha > 10$, for smaller α the theorem is obvious. We need to prove an estimate

$$|\{\sum_{m=0}^{\infty} T^{\#}_m(f) \geq \alpha\}| \leq C/\alpha.$$

We are going to apply stopping time argument to find maximal dyadic subcubes Q_j of Q, such that

$$|Q_j|^{-1} \int_{Q_j} |f|(x)(\log(10 + \log(10 + |f|(x))))^{2+\varepsilon} dx \geq \alpha. \tag{7}$$

We denote $E = \bigcup_j 5Q_j$, clearly $|E| \leq \frac{C}{\alpha}$.

Let us denote $g = f\chi_{Q \setminus \cup Q_j}$. Clearly $|g|(x) \leq \alpha$ almost everywhere on Q and therefore it has L^2 norm at most α. Each of the operators $T^{\#}_m$ is L^2 bounded as in Lemma 8, therefore $\sum_m T^{\#}_m$ is L^2 bounded and we may deduce the bound

$$|\{T^{\#} g(x) \geq \alpha/2\}| \leq C/\alpha.$$

We denote $b = f\chi_{\cup Q_j}$.

Next, we fix $m \geq 1$. We take $\lambda = 2^{c2^{m-2}}\alpha$, where c is the constant from the Lemma 8. We split the function $b = b^1_\lambda + b^2_\lambda$, where $|b^1_\lambda| \leq \lambda$ and $|b^2_\lambda| > \lambda$. We take a Calderon-Zygmund decomposition of the function b^2_λ at the level $Km^{-2-\varepsilon}\alpha$. We use the same dyadic grid as before, and for each Q_j, we have from the maximality

$$\int_{Q_j} |b^2_\lambda| \leq Cm^{-2-\varepsilon}\alpha|Q_j|.$$

Therefore we may choose K such that the Calderon-Zygmund cubes we obtain will be each contained in some Q_j. So, we get $b^2_\lambda = g^m + b^m$, $\|g^m\|_1 \leq \|b^2_\lambda\|_1 \leq Cm^{-2-\varepsilon}$, $\|g\|_\infty \leq Cm^{-2-\varepsilon}\alpha$ and $b^m = \sum_j b^m_{R_i}$, the $b^m_{R_i}$ are supported in dyadic cubes R_i with disjoint interiors, $\int b^m_{R_i} = 0$, $\|b^m_{R_i}\|_1 \le Cm^{-2-\varepsilon}\alpha|R_i|$, and

$$\sum_j |R_i| \le \frac{\|b^2_\lambda\|_1}{Km^{-2-\varepsilon}\alpha} \le C/\alpha$$

and also for each i there si a j such that $R_i \subset Q_j$.

We see that

$$\|T^\#_m(g^m + b^1_\lambda)\|_2^2 \le C2^{-c2^m}\|g_m + b^1_\lambda\|_2^2 \le C2^{-c2^m}\lambda\alpha \le C'2^{-c2^m}\alpha.$$

Next, we seek to estimate $T^\#_m(b^m)$ outside the set E. We organize the cubes R_j into groups D_l such that the cubes in D_l have sidelength 2^l. Using the idea of Christ, we write for x not in E

$$T^\#_m b^m(x) = \sup_{k\in\mathbb{Z}}\left|\sum_{i>k} b^m * H^i_m\right|(x) \le \sum_{s>0}\sup_k \Big|\sum_i \sum_{Q\in D_{i-s}} b^m_Q * H^i_m\Big|(x).$$

Let us fix $S \ge 3$. For each $s \le S$ we may use the Lemma 7 to obtain the estimate

$$\|\sup_k \Big|\sum_{i>k}\sum_{Q\in D_{i-s}} b^m_Q * H^i_m\Big|\|_2^2 \le C\alpha\sum_Q \|b^m_Q\|_1 \le C\alpha m^{-2-\varepsilon}.$$

For $s > S$ we first observe that H^i_m satisfy the condition (5). We want to use the Lemma 6 and then apply a technique similar to the proof of the Lemma 8. From Lemma 6 we get that there is a ε such that for any index set I

$$\|\sum_{i\in I}\sum_{Q\in D_{i-s}} b^m_Q * H^i_m\|_2^2 \le C2^{-s\varepsilon}m^{-2-\varepsilon}\alpha\sum_Q \|b^m_Q\|_1.$$

We denote $I_r = \cup_{i\in\mathbb{Z}}(r + is2^m)$. We see that there are $s2^m$ such sets. We denote

$$\mathcal{M}_r(x) = \sup_k\left|\sum_{i>k,\, i\in I_r}\sum_{Q\in D_{i-s}} b^m_Q * H^i_m\right|(x).$$

We have

$$\|\sup_k \Big|\sum_{i>k}\sum_{Q\in D_{i-s}} b^m_Q * H^i_m\Big|\|_2^2 \le \sum_{r=1}^{s2^m}\|\mathcal{M}_r\|.$$

Using the same argument as proof of (6) we get

$$\mathcal{M}_r(x) \le CM\Big(\sum_{i\in I_r}\sum_{Q\in D_{i-s}} b^m_Q * H^i_m\Big) + C2^{-cs}\sum_{i\in I}\sum_{Q\in D_{i-s}} |b^m_Q| * \nu_i,$$

where ν is the function from Lemma 7. This gives

$$\|\mathcal{M}_r\|_2^2 \le C2^{-\gamma s}\alpha\sum_Q \|b^m_Q\|_1$$

for some fixed positive γ. Therefore

$$\|\sup_k|\sum_{i>k}\sum_{Q\in D_{i-s}} b_Q^m * H_m^i|\|_2^2 \le Cs2^m 2^{-\gamma s}\alpha\sum_Q \|b_Q^m\|_1.$$

Now we can choose $S = Km$ for some large K and sum the convergent series in s, to get an estimate

$$\begin{aligned}\|\sum_{s>S}\sup_k|\sum_{i>k}\sum_{Q\in D_{i-s}} b_Q^m * H_m^i|\|_2^2 &\le Cm\alpha\sum_Q \|b_Q^m\|_1\\ &\le C\alpha m^{-1-\varepsilon}.\end{aligned}$$

Collecting the previous estimates, we obtain

$$\|T_m^{\#}(f)\chi_{E^c}\|_2^2 \le Cm^{-1-\varepsilon}\alpha.$$

and summing in m gives

$$\|\sum_{m=0}^{\infty} T_m^{\#}(f)\chi_{E^c}\|_2^2 \le C\alpha,$$

therefore

$$\left|\left\{\sum_{m=0}^{\infty} T_m^{\#}(f)(x) > \alpha\right\}\cap E^c\right| \le C/\alpha,$$

which finishes the proof.

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