A REMARK ON FUNCTIONS CONTINUOUS ON ALL LINES

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ABSTRACT. We prove that each linearly continuous function $f$ on $\mathbb{R}^n$ (i.e., each function continuous on all lines) belongs to the first Baire class, which answers a problem formulated by K.C. Ciesielski and D. Miller (2016). The same result holds also for $f$ on an arbitrary Banach space $X$, if $f$ has moreover the Baire property. We also prove (extending a known finite-dimensional result) that such $f$ on a separable $X$ is continuous at all points outside a first category set which is also null in any usual sense.

1. Introduction

Separately continuous functions on $\mathbb{R}^n$ (i.e., functions continuous on all lines parallel to an coordinate axis) and also linearly continuous functions (i.e., functions continuous on all lines) were investigated in a number of articles, see the survey [1].

Recall here Lebesgue result of [4] which asserts that

\begin{equation}
\text{(1.1)} \quad \text{each separately continuous function on } \mathbb{R}^n \text{ belongs to the } (n-1)\text{-th Baire class.}
\end{equation}

We prove (see Theorem 3.5 below) that each linearly continuous function $f$ with the Baire property on a Banach space $X$ belongs to the first Baire class. Of course, if $X$ is infinite-dimensional, then there exists an (everywhere) discontinuous linear functional $f$ on $X$ (which is linearly continuous), which shows that, in Theorem 3.5, it is not possible to omit the assumption that $f$ has the Baire property. However, using Lebesgue result (1.1), we obtain that each linearly continuous function $f$ on $\mathbb{R}^n$ belongs to the first Baire class, which answers [1, Problem 2, p. 12].

The natural question how big can be the set $D(f)$ of all discontinuity points of a separately (resp. linearly) continuous function were considered in several works, see [1].

A complete characterization of sets $D(f)$ for separately continuous functions in $\mathbb{R}^n$ was given in [2] (and independently in [8]), cf. [1]. This characterization, in particular, shows that $D(f)$ is a first category set, but it can have positive Lebesgue measure (even its complement can be Lebesgue null).

Slobodnik proved in [8] that, for each linearly continuous $f$ on $\mathbb{R}^n$,

\begin{equation}
\text{(1.2)} \quad D(f) \text{ is contained in a countable union of Lipschitz hypersurfaces},
\end{equation}

in particular, the Hausdorff dimension of $D(f)$ is at most $(n-1)$ (and so $D(f)$ is Lebesgue null). We show that (1.2) holds also in each separable Banach space $X$ under the additional assumption that $f$ has the Baire property. Consequently $D(f)$ is null in any usual sense, in particular it is Aronszajn null and $\Gamma$-null.

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2. Preliminaries

In the following, by a Banach space we mean a real Banach space. If $X$ is a Banach space, we set $S_X := \{ x \in X : \|x\| = 1 \}$. The symbol $B(x, r)$ will denote the open ball with center $x$ and radius $r$. The oscillation of a function $f$ at a point $x$ will be denoted by $\text{osc}(f, x)$.

Let $X$ be a Banach space, $\emptyset \neq G \subset X$ an open set and $f : G \to \mathbb{R}$ a function. Then we say that $f$ is linearly continuous, if the restriction $f \upharpoonright_{L \cap G}$ is continuous for each line $L \subset X$ intersecting $G$.

We will essentially use the following well-known characterization of Baire class one functions (see e.g. [5, Theorem 2.12]).

**Lemma 2.1.** Let $X$ be a strong Baire metric space and $f : X \to \mathbb{R}$ a function. Then the following conditions are equivalent. h

i) $f$ is a Baire class one function.

ii) For every nonempty closed set $F \subset X$ and for every real numbers $\alpha < \beta$, the sets $\{ z \in F : f(z) \leq \alpha \}$ and $\{ z \in F : f(z) \geq \beta \}$ cannot be dense in $F$ simultaneously.

Recall that $X$ is called strong Baire if every closed subspace of $X$ is a Baire space. Thus each topologically complete metric space (and so each $G_\delta$ subspace of a complete space) is strong Baire.

We will use the classical Baire terminology concerning his category theory. So complements of first category sets (= meager sets) are called residual (= comeager) sets and sets of the second category are those which are not of the first category. We will need the following well-known fact which follows e.g. from [3, §10, (7) and (11)] (cf. the text below (11)).

**Lemma 2.2.** If $M$ is a second category subset of a metric space $X$, then there exists an open set $\emptyset \neq U \subset X$ such that $M \cap V$ is of the second category for each open $\emptyset \neq V \subset U$.

In a metric space $(X, \rho)$, the system of all sets with the Baire property is the smallest $\sigma$-algebra containing all open sets and all first category sets. We will say that a mapping $f : (X, \rho_1) \to (Y, \rho_2)$ has the Baire property if $f$ is measurable with respect to the $\sigma$-algebra of all sets with the Baire property. In other words, $f$ has the Baire property, if and only if $f^{-1}(B)$ has the Baire property for all Borel sets $B \subset Y$ (see [3, §32]). We will need the following fact (see e.g. [3, §32, II]).

**Lemma 2.3.** If $Y$ is separable, then $f$ has the Baire property, if and only there exists a residual set $R$ in $X$ such that the restriction $f \upharpoonright_R$ is continuous.

Let $X$ be a Banach space, $x \in X$, $v \in S_X$ and $\delta > 0$. Then we define the open cone $C(x, v, \delta)$ as the set of all $y \neq x$ for which $\|v - \frac{y-x}{\|y-x\|}\| < \delta$.

The following easy inequality is well known (see e.g. [6, Lemma 5.1]):

\[
(2.1) \quad \text{if } v, w \in X \setminus \{0\}, \text{ then } \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{2}{\|v\|} \|v - w\|.
\]

We will need the following special case of [7, Lemma 2.4]. It can be proved by the Kuratowski-Ulam theorem (as is noted in [7]), but the proof given in [7] is more direct.

**Lemma 2.4.** Let $U$ be an open subset of a Banach space $X$. Let $M \subset U$ be a set residual in $U$ and $z \in U$. Then there exists a line $L \subset X$ such that $z$ is a point of accumulation of $M \cap L$.  


3. Baire class one

**Lemma 3.1.** Let $X$ be a Banach space, $\emptyset \neq G \subset X$ an open set and let $f : G \to \mathbb{R}$ be a linearly continuous function having the Baire property. Then for each $\eta > 0$ there exist $u \in S_X$, $\delta > 0$ and $p \in \mathbb{N}$ such that

$$|f(y) - f(x)| \leq \eta \text{ whenever } y \in C(x,u,\delta) \cap B(x,1/p).$$

**Proof.** Let $x \in X$ and $\eta > 0$ be given; we can and will suppose that $x = 0$. For each $k \in \mathbb{N}$, set

$$S_k := \{v \in S_X : |f(x + tv) - f(x)| \leq \eta \text{ for each } 0 < t < 1/k\}.$$ 

Since $S_X$ is clearly covered by all sets $S_k$, by the Baire theorem (in $S_X$) we can choose $p \in \mathbb{N}$ such that $S_p$ is a second category set (in $S_X$). So Lemma 2.2 implies that we can find $u \in S_X$ and $\delta > 0$ such that $S_p \cap V$ is of the second category in $S_X$ whenever $\emptyset \neq V \subset S_X \cap B(u,\delta)$ is an open subset in $S_X$. Set

$$U := C(0,u,\delta) \cap B(0,1/p) \text{ and } M := \{y \in U : |f(y) - f(x)| \leq \eta\}.$$ 

Then (3.1) is equivalent to the equality $M = U$.

We will first prove that $M$ is residual in $U$. To this end consider the product metric space

$$U^* := (S_X \cap B(u,\delta)) \times (0,1/p)$$

and the mapping

$$\varphi : U^* \to U, \quad \varphi((v,t)) := tv.$$ 

Then $\varphi$ is clearly a homeomorphism (with $\varphi^{-1}(z) = (z/\|z\|,\|z\|)$ for $z \in U$). Since $f$ has the Baire property, we obtain that $M$ has the Baire property in $G$ (and consequently also in $U$). Therefore $M^* := \varphi^{-1}(M)$ has the Baire property in $U^*$. Consequently (cf. e.g. [3, § 11, IV, Corollary 2]), to prove that $M^*$ is residual in $U^*$, it is sufficient to prove that $M^* \cap (V \times G)$ is of the second category in $U^*$ whenever $\emptyset \neq V \subset S_X \cap B(u,\delta)$ is an open subset of $S_X$ and $\emptyset \neq G \subset (0,1/p)$ is open. To prove this last statement, observe that the definition of $S_p$ implies that

$$(S_p \cap V) \times G \subset M^* \cap (V \times G).$$

Further, since $S_p \cap V$ is of the second category in $S_X \cap B(u,\delta)$ and $G$ is of the second category in $(0,1/p)$, we obtain (see e.g. [3, § 22, V, Corollary 1b]) that $M^* \cap (V \times G)$ is of the second category in $U^*$.

Thus we have proved that $M^*$ is residual in $U^*$ and consequently $M$ is residual in $U$. Now consider an arbitrary $z \in U$. By Lemma 2.4 there exists a line $L \subset X$ and points $z_n \in M \cap L \cap U$ with $z_n \to z$. Since the restriction of $f$ to $L \cap U$ is continuous, we obtain $f(z_n) \to f(z)$, and consequently $z \in M$. So $M = U$, which implies (3.1). \qed

**Lemma 3.2.** Let $X$ be a Banach space, $u \in S_X$, $0 < \delta \leq 1$ and $0 < \xi < \delta/2$. Then, for each $x,y \in X$ with $\|x - y\| < \delta\xi/4$, we have

(i) $z := y + (\xi/2)u \in C(x,u,\delta) \cap B(x,\delta)$ and

(ii) $C(x,u,\delta) \cap B(x,\delta) \cap C(y,u,\delta) \cap B(y,\delta) \neq \emptyset$.

**Proof.** Since

$$\|z - x\| \leq \|z - y\| + \|y - x\| \leq \frac{\xi}{2} + \frac{\delta\xi}{4} \leq \frac{\delta}{4} + \frac{\delta}{4} < \delta,$$
we have \( z \in B(x, \delta) \). Since
\[
\|z - x\| \geq \|z - y\| - \|y - x\| \geq \frac{\xi}{2} - \frac{\xi}{4} > 0,
\]
we can apply (2.1) to \( u := (\xi/2)u = z - y \) and \( w := z - x \neq 0 \). Because \( \|w - v\| = \|y - x\| < \delta/4 \), the inequality (2.1) gives
\[
\left\| \frac{u - w}{\|w\|} \right\| = \left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| < \frac{2}{\xi/2} \cdot \frac{\delta/4}{\xi/2} = \delta.
\]
Consequently \( z \in C(x, u, \delta) \) and so (i) follows. Since \( z \in C(y, u, \delta) \cap B(y, \delta) \), (i) implies (ii).

The following result is not labeled as a theorem, since it will be generalized to all Banach spaces.

**Proposition 3.3.** Let \( X \) be a separable Banach space, \( \emptyset \neq G \subset X \) an open set and let \( f : G \to \mathbb{R} \) be a linearly continuous function having the Baire property. Then \( f \) belongs to the first Baire class.

**Proof.** We can suppose \( \dim X > 1 \). Suppose to the contrary that \( f \) is not in the first Baire class. Then by Lemma 2.1 there exists a set \( \emptyset \neq F \subset G \) closed in \( G \) and reals \( \alpha < \beta \) such that the both sets
\[
A := \{ z \in F : f(z) \leq \alpha \} \quad \text{and} \quad B := \{ z \in F : f(z) \geq \beta \}
\]
are dense in \( F \). Set \( \eta := (1/7)(\beta - \alpha) \). Now choose a dense sequence \( (u_n)_n \in S_X \) and, for each \( n \in \mathbb{N} \), set
\[
P_n := \{ x \in F : |f(y) - f(x)| \leq \eta \quad \text{whenever} \quad y \in C(x, u_n, 1/n) \cap B(x, 1/n) \}.
\]
Lemma 3.1 implies that \( F = \bigcup_{n=1}^{\infty} P_n \). Indeed, for each \( x \in F \) we can choose \( u \in S_X \), \( \delta > 0 \) and \( p \in \mathbb{N} \) for which (3.1) holds. Further choose \( n > p \) such that \( 1/n < \delta/2 \) and \( \|u_n - u\| < \delta/2 \). Then clearly
\[
C(x, u_n, 1/n) \cap B(x, 1/n) \subset C(x, u, \delta) \cap B(x, 1/p)
\]
and consequently \( x \in P_n \) by (3.1).

Since \( F \) is closed in \( G \), the Baire theorem in \( F \) holds and thus there exists \( k \in \mathbb{N} \) such that \( P_k \) is not nowhere dense in \( F \). Therefore there exist \( c \in F \) and \( 0 < r < 1/(32k^2) \) such that \( P_k \) is dense in \( B(c, r) \cap F \).

Now choose \( y \in A \cap B(c, r) \) and \( y^* \in B \cap B(c, r) \). Since \( f \) is linearly continuous, we can choose \( 0 < \xi < 1/(2k) \) such that
\[
(3.2) \quad f(z) \leq \alpha + \eta \quad \text{for} \quad z := y + (\xi/2)u_k.
\]
Further choose \( x \in P_k \cap B(c, r) \) with \( \|y - x\| < \xi/(4k) \). Applying Lemma 3.2 (i) with \( u := u_k \) and \( \delta := 1/k \) we obtain that \( z \in C(x, u_k, 1/k) \cap B(x, 1/k) \), and consequently \( |f(z) - f(x)| \leq \eta \) since \( x \in P_k \). So (3.2) gives \( f(x) \leq \alpha + 2\eta \).

Proceeding quite analogously as above (working now with \( y^* \) and \( B \) instead of \( y \) and \( A \) we find \( x^* \in P_k \cap B(c, r) \) with \( f(x^*) \geq \beta - 2\eta \). Since \( 0 < r < 1/(32k^2) \), we have \( \|x - x^*\| < 1/(16k^2) \). So we can apply Lemma 3.2 (ii) with \( u := u_k \), \( \delta := 1/k \), \( \xi := 1/(4k) \), \( x \) and \( y := x^* \) to find a point
\[
b \in C(x, u_k, 1/k) \cap B(x, 1/k) \cap C(x^*, u_k, 1/k) \cap B(x^*, 1/k).
\]
Since \( x, x^* \in P_k \), we have \( |f(b) - f(x)| \leq \eta \), \( |f(b) - f(x^*)| \leq \eta \), and therefore \( \beta - 3\eta \leq f(b) \leq \alpha + 3\eta \). Consequently \( \beta - \alpha \leq 6\eta \) which contradicts the choice of \( \eta \). \( \square \)
Since each function from \((n - 1)\)-th Baire class has the Baire property, Lebesgue result (1.1) and Proposition 3.3 give the following main result of the present note which answers [1, Problem 2].

**Theorem 3.4.** Each linearly continuous function on \(\mathbb{R}^n\) belongs to the first Baire class.

Using easy “separable reduction” arguments, we obtain that the assumption of separability of \(X\) in Proposition 3.3 can be deleted.

**Theorem 3.5.** Let \(X\) be an arbitrary Banach space, \(\emptyset \neq G \subset X\) an open set and let \(f : G \to \mathbb{R}\) be a linearly continuous function having the Baire property. Then \(f\) belongs to the first Baire class.

**Proof.** Suppose to the contrary that \(f\) is not in the first Baire class. Then by Lemma 2.1 there exist a set \(\emptyset \neq F \subset G\) closed in \(G\) and reals \(\alpha < \beta\) such that the both sets

\[
A := \{ z \in F : f(z) \leq \alpha \} \quad \text{and} \quad B := \{ z \in F : f(z) \geq \beta \}
\]

are dense in \(F\).

Now we will define inductively a nondecreasing sequence \((M_n)_{n=1}^\infty\) of countable subsets of \(F\). We set \(M_1 := \{a\}\), where \(a \in F\) is an arbitrarily chosen point. If \(n > 1\) and a countable set \(M_{n-1}\) is defined, we choose for each point \(\mu \in M_{n-1}\) sequences \((a_k^\mu)_{k=1}^\infty\), \((b_k^\mu)_{k=1}^\infty\) converging to \(\mu\) with \(a_k^\mu \in A\) and \(b_k^\mu \in B\), \(k \in \mathbb{N}\). Then we set

\[
M_n := M_{n-1} \cup \bigcup_{\mu \in M_{n-1}} \bigcup_{k \in \mathbb{N}} \{a_k^\mu, b_k^\mu\}.
\]

Setting

\[
\tilde{F} := \bigcup_{n \in \mathbb{N}} M_n \cap G,
\]

we easily see that \(\tilde{F}\) is a separable subset of \(F\) which is closed in \(F\) and

\[
A \cap \tilde{F} \quad \text{and} \quad B \cap \tilde{F} \quad \text{are dense in} \quad \tilde{F}.
\]

Denote by \(X_1\) the closure of the linear span of \(\tilde{F}\). Then \(X_1\) is a closed separable subspace of \(X\). By Lemma 2.3 there exists a residual set \(R\) in \(G\) such that the restriction \(f \restriction_R\) is continuous. [11, Lemma 4.6] implies that there exists a separable closed subspace \(X_2\) of \(X\) such that \(X_2 \supset X_1\) and \(R \cap X_2\) is residual in \(X_2\). Consequently the function \(g := f \restriction_{X_2\cap G}\) has the Baire property. Since \(g\) is linearly continuous on \(X_2 \cap G\), Proposition 3.3 implies that \(g\) is in the first Baire class. But this contradicts Lemma 2.1, since \(X_2 \cap G\) is a strong Baire space (even a topologically complete space), \(\tilde{F}\) is closed in \(X_2 \cap G\) and (3.3) holds.

\[\square\]

**4. SET OF DISCONTINUITY POINTS**

In this short section we will show that Lemma 3.1 easily implies a Slobodnik’s result of [8] (Corollary 4.3 below) and its analogues in infinite-dimensional Banach spaces. First we recall some definitions and facts.

Let \(X\) be a Banach space. We say that \(A \subset X\) is a Lipschitz hypersurface if there exists a 1-dimensional linear space \(F \subset X\), its topological complement \(E\) and a Lipschitz mapping \(\varphi : E \to F\) such that \(A = \{ x + \varphi(x) : x \in E\}\).

Recall (see [10, 4C]) that if \(X\) is separable, then each \(M \subset X\) which can be covered by countably many Lipschitz hypersurfaces (note that such sets are sometimes called
“sparse”, see [10]) is not only a first category set but is also Aronszajn (≡ Gauss) null and \( \Gamma \)-null (in Lindenstrauss-Preiss sense).

A natural generalization of “sparse sets” to arbitrary (nonseparable) spaces are \( \sigma \)-cone supported sets. Their definition (see e.g. [10, Definition 4.4]) works with cones defined by a slightly different way than the cones \( C(x, v, \delta) \) in Preliminaries; namely with cones \( A(v, c) := \bigcup_{\lambda > 0} \lambda \cdot B(v, c) \), where \( ||v|| = 1 \) and \( 0 < c < 1 \). However, for such \( v \) and \( c \), obviously \( C(0, v, c) \subset A(v, c) \) and (2.1) easily implies \( A(v, c/2) \subset C(0, v, c) \). Consequently [10, Definition 4.4] can be equivalently rewritten as follows:

We say that a subset \( M \) of a Banach space \( X \) is cone supported if for each \( x \in M \) there exist \( v \in S_X \), \( \delta > 0 \) and \( r > 0 \) such that \( M \cap C(x, v, \delta) \cap B(x, r) = \emptyset \). A set is called \( \sigma \)-cone supported if it is a countable union of cone supported sets.

Recall that [9, Lemma 1] easily implies that if \( X \) is separable, then

\[
(4.1) \quad M \subset X \text{ is } \sigma \text{-cone supported if and only if it can be covered by countably many Lipschitz hypersurfaces.}
\]

**Theorem 4.1.** Let \( X \) be an arbitrary Banach space, \( \emptyset \neq G \subset X \) an open set and let \( f : G \to \mathbb{R} \) be a linearly continuous function having the Baire property. Then the set \( D(f) \) of all discontinuity points of \( f \) is \( \sigma \)-cone supported.

**Proof.** Denote \( D_n := \{ x \in G : \text{osc}(f, x) \geq 1/n \}, n \in \mathbb{N} \). Since \( D(f) = \bigcup_{n=1}^{\infty} D_n \), it is sufficient to prove that each \( D_n \) is a cone supported set. To this end fix an arbitrary \( n \in \mathbb{N} \) and consider an arbitrary point \( x \in D_n \). By Lemma 3.1 there exist \( v \in S_X \), \( \delta > 0 \) and \( r > 0 \) such that

\[
|f(y) - f(x)| \leq \frac{1}{3n} \quad \text{whenever } y \in C(x, v, \delta) \cap B(x, r).
\]

Consequently the oscillation of \( f \) on the open set \( C(x, v, \delta) \cap B(x, r) \) is at most \( 2/(3n) \) and therefore \( D_n \cap C(x, v, \delta) \cap B(x, r) = \emptyset \). So we have proved that \( D_n \) is cone supported. \( \square \)

Using (4.1), we obtain the following corollary.

**Corollary 4.2.** Let \( X \) be a separable Banach space, \( \emptyset \neq G \subset X \) an open set and let \( f : G \to \mathbb{R} \) be a linearly continuous function having the Baire property. Then the set \( D(f) \) of all discontinuity points of \( f \) can be covered by countably many Lipschitz hypersurfaces. In particular, \( D(f) \) is a first category set which is Aronszajn null and also \( \Gamma \)-null.

We obtain also the following result which was proved by S.G. Slobodnik in [8] by an essentially different way.

**Corollary 4.3.** Let \( \emptyset \neq G \subset \mathbb{R}^n \) be an open set and let \( f : G \to \mathbb{R} \) be a linearly continuous function. Then the set \( D(f) \) of all discontinuity points of \( f \) can be covered by countably many Lipschitz hypersurfaces.

**Proof.** If \( G = \mathbb{R}^n \), it is sufficient to use Theorem 4.1 together with (1.1). If \( G \) is an open interval we can use instead of (1.1) its generalization [3, \S 31, V, Theorem 2]. Using this special case, we easily obtain the general one, if we write \( G = \bigcup_{n \in \mathbb{N}} I_n \), where \( I_n \) are open intervals. \( \square \)

**References**


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