

# MAXIMAL OPERATORS ASSOCIATED WITH BILINEAR MULTIPLIERS OF LIMITED DECAY

LOUKAS GRAFAKOS, DANQING HE, AND PETR HONZÍK

ABSTRACT. Results analogous to those proved by Rubio de Francia [25] are obtained for a class of maximal functions formed by dilations of bilinear multiplier operators of limited decay. We focus our attention to  $L^2 \times L^2 \rightarrow L^1$  estimates. We discuss two applications: the boundedness of the bilinear maximal Bochner-Riesz operator and of the bilinear spherical maximal operator. For the latter we improve the known results in [1] by reducing the dimension restriction from  $n \geq 8$  to  $n \geq 4$ .

## 1. INTRODUCTION

Coifman and Meyer [5, 7, 6] initiated the study of bilinear singular integrals and set the cornerstone of a theory that has recently flourished in view of the breakthrough results in [21, 22] and of the foundational work in [19, 20]. The study of multipliers of limited decay in the bilinear setting, such as of Mihlin-Hörmander type, was initiated in [27] and pursued further in [9, 17, 18, 24] and other works. Many of these results have found weighted extensions in terms of the natural multilinear weights introduced in [23]. Meanwhile, the simple characterization of multipliers bounded on  $L^2$  does not have a bilinear analogue; see [2] and [15].

In this work we investigate the  $L^2 \times L^2 \rightarrow L^1$  boundedness of maximal operators related to bilinear multipliers with limited decay. Our source of inspiration is the following main theorem in Rubio de Francia [25]:

**Theorem** ([25, Theorem B]). *Let  $s$  be an integer with  $s > n/2$ , let  $a > 1/2$ , and suppose that  $m$  is a function of class  $C^{s+1}(\mathbb{R}^n)$  that satisfies*

$$|D^\alpha m(\xi)| \leq C|\xi|^{-a} \quad \text{for all } |\alpha| \leq s+1.$$

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Then,  $T_m^*(f) = \sup_{t>0} |(m(t\cdot)\widehat{f})^\vee|$  is bounded on  $L^p(\mathbb{R}^n)$  for

$$q_a = \frac{2n}{n+2a-1} < p < \frac{2n-2}{n-2a} = r_a$$

(with the understanding that  $q_a = 1$  if  $a > (n+1)/2$  and  $r_a = \infty$  if  $a \geq n/2$ ).

Here  $\widehat{f}$  is the Fourier transform of  $f$  given by  $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx$ .

In this paper we are concerned with maximal operators formed by dilations of bilinear multiplier operators of the form

$$T_m(f, g)(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|$$

for all Schwartz functions  $f$  and  $g$  on  $\mathbb{R}^n$ . Our main result is the following theorem, which presents a bilinear version of the aforementioned result of Rubio de Francia.

**Theorem 1.1.** *Let  $a > \frac{n}{2} + 1$ . Suppose that  $m(\xi, \eta) \in C^\infty(\mathbb{R}^{2n})$  satisfies*

$$|\partial^\beta m| \leq C_\beta |(\xi, \eta)|^{-a}$$

for all  $|\beta| \leq [\frac{n}{2}] + 2$ , where  $[\frac{n}{2}]$  is the integer part of  $\frac{n}{2}$ . Define

$$S_t(f, g)(x) = \int_{\mathbb{R}^{2n}} m(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

Then the bilinear maximal operator defined by

$$(1) \quad \mathcal{M}(f, g) = \sup_{t>0} |S_t(f, g)|$$

is bounded from  $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$ .

An important motivation for studying bilinear maximal operator of this type is the bilinear spherical maximal operator, in which case the multiplier  $m = \widehat{d\sigma}$  satisfies  $|m(\xi, \eta)| \leq C|(\xi, \eta)|^{-(n-1/2)}$ , where  $\sigma$  is surface measure on  $\mathbb{S}^{2n-1}$ . This operator was introduced in [10] and further studied in [1].

In studying linear and bilinear spherical maximal operators, we often decompose the multiplier  $m = \sum_{j=0}^{\infty} m_j$  with  $m_j = m\psi_j$  for smooth bumps  $\psi_j$  supported in annuli  $|(\xi, \eta)| \approx 2^j$ ,  $j \geq 1$  and  $\psi_0$  supported in a neighborhood of the origin. We recall the Sobolev space  $L_s^r$  of all functions  $g$  with  $\|(I - \Delta)^{s/2} g\|_{L^r} < \infty$ , where  $\Delta$  is the usual Laplacian and  $s > 0$ .

Motivated by Hörmander type conditions, we obtain Theorem 1.1 as a consequence of the following more general result, which is the main contribution of this paper.

**Theorem 1.2.** *Let  $\lambda > 1$ ,  $1 < r \leq 4$ ,  $s > \frac{2n}{r} + 1$ ,  $j \geq 1$ . Suppose that for each  $j \in \mathbb{N}$ ,  $M_j(\xi, \eta)$  is a multiplier supported in*

$$\{(\xi, \eta) \in \mathbb{R}^{2n} : 2^{j-1} \leq |(\xi, \eta)| \leq 2^{j+1}\},$$

*that satisfies*

$$(2) \quad \|M_j\|_{L^r_s(\mathbb{R}^{2n})} \leq A2^{-\lambda j}.$$

*Let*

$$S_t(f, g)(x) = \int_{\mathbb{R}^{2n}} \sum_{j \geq 0} M_j(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

*Then the maximal operator*

$$T(f, g) = \sup_{t > 0} |S_t(f, g)|$$

*is bounded from  $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$  with bound a constant multiple of  $A$ .*

We prove Theorem 1.2 in Section 3. Below we derive Theorem 1.1.

*Proof of Theorem 1.1 assuming Theorem 1.2.* We fix a smooth function  $\widehat{\varphi}$  supported in  $B(0, 2)$  whose value is 1 in the unit ball, and define

$$\widehat{\psi}(\cdot) = \widehat{\varphi}(2^{-1}\cdot) - \widehat{\varphi}(\cdot)$$

and

$$m_j(\xi, \eta) = m(\xi, \eta) \widehat{\psi}(2^{-j}(\xi, \eta))$$

for  $j \geq 1$ , and  $m_0 = m - \sum_{j \geq 1} m_j$ .

Then  $m_0$  is a compactly supported smooth function, so the corresponding bilinear maximal operator

$$T_{m_0}^*(f, g)(x) = \sup_{t > 0} \left| \int_{\mathbb{R}^{2n}} m_0(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|$$

is bounded by  $CM(f)M(g)$ , where  $M$  is the Hardy-Littlewood maximal function. So  $T_0$  is bounded from  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$  for all  $1 < p_1, p_2 < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ .

Let  $r = 4$ , then  $\|m_j\|_{L^4_s} \leq C2^{-ja}2^{jn/2}$ . Hence  $m_j$  satisfies conditions of Theorem 1.2 with the decay  $\lambda = a - \frac{n}{2} > 1$ . Theorem 1.2 then implies that the bilinear maximal operator (1) is bounded from  $L^2 \times L^2$  to  $L^1$ .  $\square$

As an application of Theorem 1.1, we improve the known results concerning the boundedness of the bilinear spherical maximal operator. It was shown in [1] that this operator is bounded from  $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$  for  $n \geq 8$ . Here we reduce the dimension restriction to  $n \geq 4$ .

**Theorem 1.3.** *Let  $m_\alpha(\xi, \eta) = \frac{J_{n+\alpha-1}(2\pi|\xi, \eta|)}{(|\xi, \eta|)^{n+\alpha-1}}$  for  $\alpha \in \mathbb{R}$ , then the bilinear maximal operator  $M_\alpha$  defined by*

$$M_\alpha(f, g) = \sup_{t>0} \left| \iint_{\mathbb{R}^{2n}} m_\alpha(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|$$

is bounded from  $L^2 \times L^2$  to  $L^1$  when  $n > 3 - 2\alpha$ .

In particular, for  $\alpha = 0$ , the bilinear spherical maximal operator

$$(3) \quad \mathcal{M}_0(f, g) = \sup_{t>0} \left| \int_{\mathbb{S}^{2n-1}} f(x - t\theta) g(x - t\phi) d\sigma(\theta, \phi) \right|$$

is bounded from  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  to  $L^1(\mathbb{R})$  when  $n \geq 4$ .

*Proof.* The function  $m_\alpha$  satisfies the conditions of Theorem 1.1 with  $a = n + \alpha - \frac{1}{2}$ . Hence when  $n > 3 - 2\alpha$ , we obtain the  $L^2 \times L^2 \rightarrow L^1$  boundedness of  $M_\alpha$ . Now recall that the  $(2n - 1)$ -dimensional surface measure  $\sigma$  satisfies  $\widehat{d\sigma} = m_0$ , i.e.,  $m_\alpha$  with  $\alpha = 0$ . Hence the bilinear spherical maximal operator (3) is bounded from  $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$  when  $n \geq 4$ .  $\square$

It was pointed out in [1] that  $L^2 \times L^2 \rightarrow L^1$  boundedness fails in dimension  $n = 1$ . As of this writing, we are uncertain about the behavior of this operator in dimensions  $n = 2, 3$ .

We discuss another application of Theorem 1.2 concerning the bilinear maximal Bochner-Riesz means in Section 4.

## 2. WAVELET DECOMPOSITION

We use the wavelet decomposition of multipliers as in [14]. So we need to introduce the tensor type wavelets due to [8], and the exact form we use here can be found in [28].

**Lemma 2.1** ([28, Section 1.7.3]). *For any fixed  $k \in \mathbb{N}$  there exist real compactly supported functions  $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$ , which satisfy that  $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$  and  $\int_{\mathbb{R}} x^\alpha \psi_M(x) dx = 0$  for  $0 \leq \alpha \leq k$ , such that, if  $\Psi^G$  is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for  $G = (G_1, \dots, G_{2n})$  in the set

$$\mathcal{I} := \left\{ (G_1, \dots, G_{2n}) : G_i \in \{F, M\} \right\},$$

then the family of functions

$$\bigcup_{\vec{\mu} \in \mathbb{Z}^{2n}} \left[ \left\{ \Psi^{(F, \dots, F)}(\vec{x} - \vec{\mu}) \right\} \cup \bigcup_{\gamma=0}^{\infty} \left\{ 2^{\gamma n} \Psi^G(2^\gamma \vec{x} - \vec{\mu}) : G \in \mathcal{I} \setminus \{(F, \dots, F)\} \right\} \right]$$

forms an orthonormal basis of  $L^2(\mathbb{R}^{2n})$ , where  $\vec{x} = (x_1, \dots, x_{2n})$ .

For simplicity, we use often below  $\omega(\xi, \eta) = \omega_{k,l}(\xi, \eta)$  to denote the wavelet  $2^{\gamma m} \Psi^G(2^\gamma(\xi, \eta) - (k, l))$  when the dilation factor  $\gamma$  is fixed. Moreover we may write  $\omega_{k,l}(\xi, \eta) = \omega_{1,k}(\xi) \omega_{2,l}(\eta)$ , where

$$(4) \quad \omega_{1,k}(\xi) = 2^{\gamma m/2} \psi_{G_1}(2^\gamma \xi_1 - k_1) \cdots \psi_{G_n}(2^\gamma \xi_n - k_n)$$

and  $\omega_{2,l}(\eta)$  is defined in an obvious similar way. For a good function  $m$ , we denote by  $a_{k,l}$  as the inner product  $\langle m, \omega_{k,l} \rangle$  of  $m$  and  $\omega_{k,l}$ .

Let  $F_{r,q}^s(\mathbb{R}^{2n})$  and  $f_{r,q}^s$  be the Triebel-Lizorkin spaces of functions and sequences, respectively; see [12, Sections 2.2 and 2.3]. To characterize general function spaces, we need the following lemma.

**Lemma 2.2** ([28, Theorem 1.64]). *Let  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$ , and for  $\gamma \in \mathbb{N}$  and  $\bar{\mu} \in \mathbb{N}^{2n}$  let  $\chi_{\gamma\bar{\mu}}$  be the characteristic function of the cube  $Q_{\gamma\bar{\mu}}$  centered at  $2^{-\gamma}\bar{\mu}$  with length  $2^{1-\gamma}$ . For a sequence  $\eta = \{\eta_{\bar{\mu}}^{\gamma,G}\}$  define the norm*

$$\|\eta\|_{f_{r,q}^s} = \left\| \left( \sum_{\gamma,G,\bar{\mu}} 2^{\gamma s q} |\eta_{\bar{\mu}}^{\gamma,G} \chi_{\gamma\bar{\mu}}(\cdot)|^q \right)^{1/q} \right\|_{L^r(\mathbb{R}^{2n})}.$$

*Let  $\mathbb{N} \ni k > \max\{s, \frac{4n}{\min(r,q)} + n - s\}$ . Let  $\Psi_{\bar{\mu}}^{\gamma,G}$  be the  $2n$ -dimensional Daubechies wavelet with smoothness  $k$  as in Lemma 2.1. Let  $m \in \mathcal{S}'(\mathbb{R}^{2n})$ . Then  $m \in F_{r,q}^s(\mathbb{R}^{2n})$  if and only if it can be represented as*

$$m = \sum_{\gamma,G,\bar{\mu}} \eta_{\bar{\mu}}^{\gamma,G} 2^{-\gamma m} \Psi_{\bar{\mu}}^{\gamma,G}$$

*with  $\|\eta\|_{f_{r,q}^s} < \infty$  with unconditional convergence in  $\mathcal{S}'(\mathbb{R}^n)$ . Furthermore this representation is unique,*

$$\eta_{\bar{\mu}}^{\gamma,G} = 2^{\gamma m} \langle m, \Psi_{\bar{\mu}}^{\gamma,G} \rangle,$$

*and*

$$I : m \rightarrow \{2^{\gamma m} \langle m, \Psi_{\bar{\mu}}^{\gamma,G} \rangle\}$$

*is an isomorphism from  $F_{r,q}^s(\mathbb{R}^{2n})$  onto  $f_{r,q}^s$ .*

We now return the multipliers considering their wavelet decompositions. Before doing so, we make some comments. The functions  $\psi_F$  and  $\psi_M$  have compact supports, and all elements in a fixed level, i.e., of the same dilation factor  $\gamma$ , in the basis come from translations of finitely many products, so their supports have finite overlaps. Consequently we can classify the elements in the basis into finitely many classes so that all elements in the same level in each class have distant supports, which means that if  $\omega$  and  $\omega'$  are in the same class with the same dilation parameter  $\gamma$ , then  $5 \text{ supp } \omega \cap 5 \text{ supp } \omega' = \emptyset$ , where  $5 \text{ supp } \omega = B(c_0, 5d)$  with  $c_0$  inside the

support of  $\omega$  and  $d$  the diameter of the support of  $\omega$ . So, from now on, we will assume that the supports of  $\omega$ 's related to a given dilation factor  $\gamma$  are far disjoint.

For the multiplier  $M_j$  in Theorem 1.2, we have a wavelet decomposition using Lemma 2.2, i.e.

$$(5) \quad M_j = \sum a_\omega \omega,$$

where the summation is over all  $\omega = \Psi_{\bar{\mu}}^{\gamma, G}$  in the orthonormal basis described in Lemma 2.1, the order of cancellations of  $\psi_M$  is  $M = 4n + 6$ , and  $a_\omega = \langle M_j, \omega \rangle$ .

Concerning the size of  $a_\omega = a_{k,l}$ , we have the following estimate.

**Corollary 2.3.** *The coefficient  $a_\omega$  in (5) related to  $\omega$  with dilation  $\gamma$  is bounded by  $C2^{-j\lambda}2^{-(s+n-\frac{2n}{r})\gamma}$ .*

*Proof.* Since  $F_{r,2}^s(\mathbb{R}^{2n}) = L_r^s(\mathbb{R}^{2n})$ , we have

$$\left\| \left( \sum_{k,l} 2^{2\gamma s} |2^{\gamma m} a_{k,l} \chi_{Q_{\gamma,k,l}}|^2 \right)^{1/2} \right\|_{L^r} \leq C \|M_j\|_{L_r^s},$$

by Lemma 2.2, where  $Q_{\gamma,k,l}$  is the cube centered at  $2^{-\gamma}(k,l)$  with length  $2^{1-\gamma}$ . Take just one term on the left hand side, and notice that  $|Q_{\gamma,k,l}| \sim 2^{-2n\gamma}$ , then

$$|a_{k,l}| \leq CA2^{-j\lambda}2^{-(s+n)\gamma}2^{2n\gamma/r} \leq CA2^{-j\lambda}2^{-\gamma(s+n-\frac{2n}{r})}.$$

□

With the wavelet decompositions in hand, we are able to prove Theorem 1.2. The proof is inspired by [13] and the square function technique (see [26], [4], and [25]). We control

$$(6) \quad T_j = \sup_{t>0} \left| \iint_{\mathbb{R}^{2n}} M_j(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|$$

by two integrals with the diagonal and the off-diagonal parts. For the diagonal part we have just one term, which can be handled using product wavelets. For the off-diagonal parts we introduce two square operators with each one bounded by a product of the Hardy-Littlewood maximal function and a linear operator bounded on  $L^2(\mathbb{R}^n)$ .

We need to decompose  $M_j$  further. Take  $N$  to be a fixed large enough number so that  $N/10$  is greater than  $d$ , the diameters of the support of  $\omega$  with dilation factor  $\gamma = 0$ . We write  $\omega(\xi, \eta) = \omega_{\bar{\mu}}(\xi, \eta) = \omega_{1,k}(\xi) \omega_{2,l}(\eta)$ ,

where  $\vec{\mu} = (k, l)$  with  $k, l \in \mathbb{Z}^n$ , and denote the corresponding coefficient  $\langle \omega_{k,l}, M_j \rangle$  by  $a_{k,l}$ . We define

$$(7) \quad M_j^1 = \sum_{\gamma \geq 0} M_{j,\gamma}^1 = \sum_{\gamma} \sum_{|k| \geq N} \sum_{|l| \geq N} a_{k,l} \omega_{1,k} \omega_{2,l}$$

$$(8) \quad M_j^2 = \sum_{\gamma \geq 0} M_{j,\gamma}^2 = \sum_{\gamma} \sum_k \sum_{|l| \leq N} a_{k,l} \omega_{1,k} \omega_{2,l}$$

$$(9) \quad M_j^3 = \sum_{\gamma \geq 0} M_{j,\gamma}^3 = \sum_{\gamma} \sum_{|k| \leq N} \sum_{|l| \geq N} a_{k,l} \omega_{1,k} \omega_{2,l}.$$

Here  $M_j^1$  is the diagonal part such that the support of each level is away from both  $\xi$  and  $\eta$  axes,  $M_j^2$  is the off-diagonal part with each level's support near the  $\xi$  axis, and the support of each level of  $M_j^3$  is near the  $\eta$  axis.

*Remark 1.* This decomposition is more delicate than that in [1], and allows us to handle more singular operators. Actually for each fix  $\gamma$ , the supports of the wavelets in  $M_j^2$  related to  $\gamma$  are contained in  $\{(\xi, \eta) : |\eta| \leq Nd2^{-\gamma}\}$ , while the corresponding part in [1] is contained in  $\{(\xi, \eta) : |\eta| \leq 2^{j\epsilon}\}$ .

Corresponding to  $M_{j,\gamma}^i$ ,  $i = 1, 2, 3$ , we define

$$B_{j,\gamma,t}^i(f, g)(x) = \iint_{\mathbb{R}^{2n}} M_{j,\gamma,t}^i(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We can define  $B_{j,\gamma,t}$  in a similar way and  $B_{j,\gamma,t}(f, g)(x) = \sum_{i=1}^3 B_{j,\gamma,t}^i(f, g)(x)$ . Moreover we can define  $T_{j,\gamma}^i$  in the way similar to (6) so that  $T_j = \sum_{i=1}^3 \sum_{\gamma} T_{j,\gamma}^i$ .

### 3. PROOF OF THEOREM 1.2

For  $f, g \in \mathcal{S}(\mathbb{R}^n)$ , using the fundamental theorem of Calculus, we rewrite

$$\begin{aligned} & B_{j,\gamma,t}^1(f, g)(x) \\ &= \iint_{\mathbb{R}^{2n}} M_{j,\gamma,t}^1(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \\ &= \int_0^t \iint_{\mathbb{R}^{2n}} (s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \frac{ds}{s}, \end{aligned}$$

where the existence of  $\nabla M_{j,\gamma}^1$  is guaranteed by that all components in  $M_{j,\gamma}^1$  are contained in the same level.

Define the operator related to  $(s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta)$  as

$$\tilde{B}_{j,\gamma,s}^1(f, g)(x) = \iint_{\mathbb{R}^{2n}} (s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

Then we have the pointwise estimate

$$(10) \quad T_{j,\gamma}^1(f,g)(x) = \sup_{t>0} |B_{j,\gamma,t}^1(f,g)(x)| \leq \int_0^\infty |\widetilde{B}_{j,\gamma,t}^1(f,g)(x)| \frac{dt}{t}$$

We now turn to the study of the boundedness of  $\widetilde{B}_{j,\gamma,t}^1$ . The basic idea is the observation that when  $r \in (1,4)$ , [13, Remark 2] shows that whenever  $\sigma$  is supported in  $B(0,R)$  we have

$$\|T_\sigma(f,g)\|_{L^2 \times L^2 \rightarrow L^1} \leq C \|\sigma\|_{L^r_s}$$

with  $C$  independent of  $R$ .

To make this argument rigorous, in the case  $t = 1$ , we have the following estimate, whose proof can be found in the Appendix (Section 5).

**Proposition 3.1.** *Let  $E = \{\xi \in \mathbb{R}^n : C2^{-\gamma} \leq |\xi| \leq 2^j\}$ . Then we have*

$$(11) \quad \|\widetilde{B}_{j,\gamma,1}^1(f,g)\|_{L^1} \leq CAC(j,\gamma) \|\widehat{f}\chi_E\|_{L^2} \|\widehat{g}\chi_E\|_{L^2},$$

with

$$\begin{aligned} C(j,\gamma) &= n(j+\gamma)2^{-j(\lambda-1)}2^{\gamma(1+\frac{n}{2}-s)} && \text{when } r = 4 \\ C(j,\gamma) &= 2^{-j(\lambda-1)}2^{\gamma(1+\frac{2n}{r}-s)} && \text{when } 1 < r < 4. \end{aligned}$$

In both cases we have good decay in  $j$ .

**Corollary 3.2.** *For the diagonal part we have*

$$\|T_{j,\gamma}^1(f,g)\|_{L^1} \leq CAC(j,\gamma)(j+\gamma) \|f\|_{L^2} \|g\|_{L^2}.$$

*Proof.* From (10) we know that

$$\|T_{j,\gamma}^1(f,g)\|_{L^1} \leq \int_0^\infty \|\widetilde{B}_{j,\gamma,t}^1(f,g)\|_{L^1} \frac{dt}{t} = \int_0^\infty \|\widetilde{B}_{j,\gamma,1}^1(f_t,g_t)\|_{L^1} \frac{dt}{t},$$

where  $\widehat{f}_t(\xi) = t^{-n/2}\widehat{f}(\xi/t)$ , and  $\widehat{g}_t(\xi) = t^{-n/2}\widehat{g}(\xi/t)$ . Applying Proposition 3.1, the last integral is dominated by

$$\begin{aligned} & CA \int_0^\infty C(j,\gamma) \|\widehat{f}_t\chi_E\|_{L^2} \|\widehat{g}_t\chi_E\|_{L^2} \frac{dt}{t} \\ & \leq CC(j,\gamma) \left( \int_{\mathbb{R}^n} \int_0^\infty |\widehat{f}_t\chi_E|^2 \frac{dt}{t} d\xi \right)^{1/2} \left( \int_{\mathbb{R}^n} \int_0^\infty |\widehat{g}_t\chi_E|^2 \frac{dt}{t} d\xi \right)^{1/2}. \end{aligned}$$

The double integral involving  $f_t$  is bounded by  $\int |\widehat{f}(\xi)|^2 \int_{C2^{-\gamma}/|\xi|}^{2^j/|\xi|} \frac{dt}{t} d\xi$ , which is less than  $C(j+\gamma) \|f\|_{L^2}^2$ . Hence the last expression is controlled by  $CAC(j,\gamma)(j+\gamma) \|f\|_{L^2} \|g\|_{L^2}$ .  $\square$



We next deal with the off-diagonal parts. More specifically, we consider  $B_{j,\gamma,t}^2$ , since the analysis of  $B_{j,\gamma,t}^3$  is similar in view of symmetry. Recall that

$$B_{j,\gamma,t}^2 = \iint_{\mathbb{R}^{2n}} M_{j,\gamma}^2(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We denote  $(\xi, \eta) \cdot (\nabla M_{j,\gamma}^2)(\xi, \eta)$  by  $\widetilde{M}_{j,\gamma}^2(\xi, \eta)$ . Then similar to  $B_{j,t}^2(f, g)(x)$  we define

$$\widetilde{B}_{j,\gamma,t}^2(f, g)(x) = \iint_{\mathbb{R}^{2n}} \widetilde{M}_{j,\gamma}^2(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

With these notations, by the fundamental theorem of Calculus, we have

$$\begin{aligned} (B_{j,\gamma,t}^2(f, g)(x))^2 &= 2 \int_0^t B_{j,\gamma,s}^2(f, g)(x) s \frac{dB_{j,\gamma,s}^2(f, g)(x)}{ds} \frac{ds}{s} \\ &\leq 2 \int_0^\infty |B_{j,\gamma,s}^2(f, g)(x)| |\widetilde{B}_{j,\gamma,s}^2(f, g)(x)| \frac{ds}{s} \\ &\leq 2G_{j,\gamma}(f, g)(x) \widetilde{G}_{j,\gamma}(f, g)(x), \end{aligned}$$

where we set

$$\begin{aligned} G_{j,\gamma}(f, g)(x) &= \left( \int_0^\infty |B_{j,\gamma,s}^2(f, g)(x)|^2 \frac{ds}{s} \right)^{1/2} \\ \widetilde{G}_{j,\gamma}(f, g)(x) &= \left( \int_0^\infty |\widetilde{B}_{j,\gamma,s}^2(f, g)(x)|^2 \frac{ds}{s} \right)^{1/2}. \end{aligned}$$

These  $g$ -functions are bounded from  $L^2 \times L^2$  to  $L^1$  with good decay in  $j$ . Indeed, we have the following.

**Lemma 3.3.** *For any  $\varepsilon > 0$  there exists a constant  $C_\varepsilon$  independent of  $j$  such that for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$ ,*

$$\|G_{j,\gamma}(f, g)\|_{L^1} \leq C_\varepsilon A 2^{-j\lambda} 2^{(n+1)\gamma} \|f\|_{L^2} \|g\|_{L^2}$$

and

$$\|\widetilde{G}_{j,\gamma}(f, g)\|_{L^1} \leq C_\varepsilon A 2^{-j(\lambda-1)} 2^{(n+1)\gamma} \|f\|_{L^2} \|g\|_{L^2}.$$

The proof of this lemma is inspired by [14].

*Proof.* We will focus on  $\widetilde{G}_{j,\gamma}$  first. For  $\widetilde{G}_{j,\gamma}$  we need to consider two typical cases, the derivative falling on  $\xi$  and the derivative falling on  $\eta$ .

Let us consider the multiplier

$$\xi_1 \partial_{\xi_1} M_{j,\gamma}^2 = \sum_{k,l} a_{k,l} v_k(\xi) \omega_{2,l}(\eta)$$

with  $v_k(\xi) = \xi_1 \partial_{\xi_1} \omega_{1,k}(\xi)$ . Using (4) we observe that

$$|v_k(\xi)| \leq C 2^j 2^{\gamma m/2} 2^\gamma = C 2^j 2^{\gamma(n+2)/2}.$$

The  $g$ -function related to  $\xi_1 \partial_{\xi_1} M_{j,\gamma}^2$  is denoted by  $\tilde{G}_{j,\gamma}^1(f, g)$ . By the definition (8), for a fixed  $\gamma$  at most  $N$  of  $\omega_{2,l}$  are involved, so we can consider a single fixed  $l$ . Observe that

$$\begin{aligned} & \iint_{\mathbb{R}^{2n}} \sum_k a_{k,l} v_k(\xi) \omega_{2,l}(\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi, \eta)} d\xi d\eta \\ &= \|a\|_{\ell^\infty} 2^{\gamma(n+2)/2} 2^j \left( \int_{\mathbb{R}^n} \omega_{2,l}(\eta) \widehat{g}(\eta) e^{2\pi i x \cdot \eta} d\eta \right) \\ & \quad \int_{\mathbb{R}^n} \frac{\sum_k a_{k,l} v_k(\xi)}{\|a\|_{\ell^\infty} 2^{\gamma(n+2)/2} 2^j} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \end{aligned}$$

By  $|v_k| \leq C 2^{\gamma(n+2)/2} 2^j$ ,  $|a_{k,l}| \leq \|a\|_{\ell^\infty}$ , and the disjointness of the supports of  $v_k$ , we know that  $\sigma(\xi) := (\sum_k a_{k,l} v_k(\xi)) / (\|a\|_{\ell^\infty} 2^{\gamma(n+2)/2} 2^j)$  is a compactly supported bounded function. Hence the bilinear operator related to the multiplier  $\sum_k a_{k,l} v_k(\xi) \omega_{2,l}(\eta)$  is pointwise bounded by

$$(12) \quad C \|a\|_{\ell^\infty} 2^{\gamma(n+1)} 2^j M(g)(x) T_\sigma(f)(x),$$

where  $T_\sigma(f)$  satisfies that  $\|T_\sigma(f)\|_{L^2} \leq C \|\widehat{f} \chi_F\|_{L^2}$  with

$$F = \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}.$$

The operator  $\tilde{G}_{j,\gamma}^1$  is then bounded from  $L^2 \times L^2$  to  $L^1$ . Indeed we can estimate it by a standard dilation argument as follows. Setting  $\widehat{f}_s(\xi) = s^{-n/2} \widehat{f}(\xi/s)$ , and  $\widehat{g}_s(\xi) = s^{-n/2} \widehat{g}(\xi/s)$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \tilde{G}_{j,\gamma}^1(f, g)(x) dx \\ &= \int_{\mathbb{R}^n} \left[ \int_0^\infty \left| \iint_{\mathbb{R}^{2n}} \sum_{k,l} a_{k,l} v_k(s\xi) \omega_{2,l}(s\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi, \eta)} d\xi d\eta \right|^2 \frac{ds}{s} \right]^{\frac{1}{2}} dx \\ &= \int_{\mathbb{R}^n} \left[ \int_0^\infty \left| \iint_{\mathbb{R}^{2n}} \sum_{k,l} a_{k,l} v_k(\xi) \omega_{2,l}(\eta) \widehat{f}_s(\xi) \widehat{g}_s(\eta) e^{2\pi i \frac{x}{s} \cdot (\xi, \eta)} \frac{d\xi d\eta}{s^n} \right|^2 \frac{ds}{s} \right]^{\frac{1}{2}} dx. \end{aligned}$$

Since there are only finitely many  $l$  in the sum above, we can use the pointwise estimate (12) to estimate the last displayed expression by

$$\begin{aligned} & C \|a\|_{\ell^\infty} 2^j 2^{(n+1)\gamma} \int_{\mathbb{R}^n} \left( \int_0^\infty \left| s^{-n/2} M(g)(x) T_\sigma(f_s)(s^{-1}x) \right|^2 \frac{ds}{s} \right)^{1/2} dx \\ & \leq C \|a\|_{\ell^\infty} 2^j 2^{(n+1)\gamma} \|M(g)\|_{L^2} \left( \int_0^\infty \int_{\mathbb{R}^n} s^{-n} |\widehat{f}(\xi/s)|^2 \chi_F(\xi) d\xi \frac{ds}{s} \right)^{\frac{1}{2}} \\ & \leq C \|a\|_{\ell^\infty} 2^j 2^{(n+1)\gamma} \|g\|_{L^2} \left( \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \int_{(2^{j-1})/|\xi|}^{(2^{j+1})/|\xi|} \frac{ds}{s} d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

The integral with respect to  $s$  is  $\log \frac{2^{j+1}}{2^{j-1}} \leq C$ . This, combined with the bound of  $\|a\|_{\ell^\infty} \leq CA2^{-j\lambda}2^{-(s+n-\frac{2n}{r})\gamma}$  obtained in Corollary 2.3, shows that the last displayed expression is smaller than

$$(13) \quad C\|a\|_{\ell^\infty}2^j2^{(n+1)\gamma}\|g\|_{L^2}\|f\|_{L^2} \leq CA2^{-j(\lambda-1)}2^{-\gamma(s-\frac{2n}{r}-1)}\|g\|_{L^2}\|f\|_{L^2}.$$

When the derivative falls on  $\eta$ , for example we have differentiation with respect to  $\eta_1$ , using the notation  $v_l(\eta) = \eta_1 \partial_{\eta_1} \omega_{2,l}(\eta)$  we have a similar representation

$$\begin{aligned} & \iint_{\mathbb{R}^{2n}} \sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi, \eta)} d\xi d\eta \\ &= \|a\|_{\ell^\infty} 2^{\gamma m/2} \left( \int_{\mathbb{R}^n} v_l(\eta) \widehat{g}(\eta) e^{2\pi i x \cdot \eta} d\eta \right) \int_{\mathbb{R}^n} \frac{\sum_k a_{k,l} \omega_{1,k}(\xi)}{\|a\|_{\ell^\infty} 2^{\gamma m/2}} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \end{aligned}$$

The integral in the parenthesis in the last line is dominated by  $2^{\gamma m/2} M(g)(x)$  as both  $\partial_1(\omega_{2,l})^\vee(x) e^{2\pi i x \cdot l}$  and  $(\omega_{2,l})^\vee(x) l_1 e^{2\pi i x \cdot l}$  are Schwartz functions, and the number of the second type of functions is finite because  $|l| \leq N$ . The bilinear operator related to the multiplier  $\sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta)$  is therefore bounded by

$$C\|a\|_{\ell^\infty} 2^{\gamma m} M(g)(x) T_{\sigma'}(f)(x),$$

where  $T_{\sigma'}$  satisfies the same property as  $T_\sigma$ . For the  $L^1$  norm of the  $g$ -function  $\widetilde{G}_{j,\gamma}^2$  related to the multiplier  $\sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta)$  we apply an argument similar to that used for  $\|\widetilde{G}_{j,\gamma}^1\|_{L^1}$ . We obtain

$$(14) \quad \|\widetilde{G}_{j,\gamma}^2\|_{L^1} \leq CA2^{-j\lambda}2^{-\gamma(s-\frac{2n}{r})}\|g\|_{L^2}\|f\|_{L^2}.$$

This estimate and (13) show that

$$\|\widetilde{G}_{j,\gamma}(f, g)\|_{L^1} \leq CA2^{-j(\lambda-1)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

For  $G_{j,\gamma}(f, g)$  an analogous, but simpler argument, applied to the standard representation  $\sum a_{k,l} \omega_{1,k}(\xi) \omega_{2,l}(\eta)$  yields

$$\|G_{j,\gamma}(f, g)\|_{L^1} \leq CA2^{-j\lambda}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

The additional decay of  $2^{-j}$  comes from the fact that in the multiplier of  $B_{j,\gamma,s}^2$  we miss the term  $(\xi, \eta)$ , which is controlled by  $2^j$ .  $\square$

**Corollary 3.4.** *For the off-diagonal part the estimate below holds:*

$$(15) \quad \|T_{j,\gamma}^2(f, g)\|_{L^1} \leq CA2^{-j(\lambda-1/2)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

*Proof.* By the calculation before Lemma 3.3 we have the pointwise control

$$T_{j,\gamma}^2(f,g)(x) \leq \sqrt{2}(G_{j,\gamma}(f,g)(x)\tilde{G}_{j,\gamma}(f,g)(x))^{1/2},$$

which, combined with Lemma 3.3, implies that

$$\begin{aligned} \|T_j^2(f,g)\|_{L^1} &\leq \|\sqrt{2}(G_{j,\gamma}(f,g)\tilde{G}_{j,\gamma}(f,g))^{1/2}\|_{L^1} \\ &\leq C(\|G_{j,\gamma}(f,g)\|_{L^1}\|\tilde{G}_{j,\gamma}(f,g)\|_{L^1})^{1/2} \\ &\leq CA\left(2^{-j\lambda}2^{-j(\lambda-1)}2^{-2\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}^2\|g\|_{L^2}^2\right)^{1/2} \\ &= CA2^{-j(\lambda-1/2)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}. \end{aligned}$$

In this case we have nice decay in  $j$  for  $T_{j,\gamma}^2$  since  $\lambda > 1 > 1/2$ .  $\square$

We collect the known results to finish the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We observe that

$$T(f,g)(x) \leq \sum_{j=0}^{\infty} \sum_{\gamma} |T_{j,\gamma}(f,g)(x)|.$$

It is straightforward to verify that

$$\sum_{\gamma} C(j,\gamma)(j+\gamma) \leq C_{\varepsilon}2^{-j(\lambda-1-\varepsilon)} \leq C_{\varepsilon}2^{-j(\lambda-1)/2},$$

if we choose  $\varepsilon$  small enough. So we obtain

$$(16) \quad \sum_j \sum_{\gamma} \|T_{j,\gamma}^1(f,g)\|_{L^1} \leq \sum_j CA2^{-j\frac{\lambda-1}{2}}\|f\|_{L^2}\|g\|_{L^2} \leq CA\|f\|_{L^2}\|g\|_{L^2}.$$

This concludes the argument of the diagonal part. A similar argument using (15) show the boundedness of the off-diagonal part. Hence we deduce the conclusion of Theorem 1.2.  $\square$

#### 4. APPLICATIONS TO BILINEAR MAXIMAL BOCHNER-RIESZ

Theorem 1.2 can also be used to study the boundedness of the maximal bilinear Bochner-Riesz means. These are the means

$$(17) \quad A_t^{\lambda}(f,g)(x) = \iint_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)(1-|t\xi|^2-|t\eta|^2)_+^{\lambda} e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta,$$

which coincide with  $B_{1/t}^{\lambda}(f \otimes g)(x,x)$  with  $B_{1/t}^{\lambda}$  the linear Bochner-Riesz operator on  $\mathbb{R}^{2n}$  and  $x \in \mathbb{R}^n$ . For test functions we should have  $A_t^{\lambda}(f,g) \rightarrow fg$  as  $t \rightarrow 0$  in the  $L^p$  or in the pointwise sense. Grafakos and Li [16] and Bernicot Grafakos, Song and Yan [3] have proved positive results for  $\lambda = 0$  and  $\lambda > 0$  respectively, concerning their  $L^p$  convergence.

In this section, we are concerned with the pointwise convergence of the means (17), in particular with the boundedness of the maximal bilinear Bochner-Riesz operator, which of course implies the boundedness of the bilinear Bochner-Riesz operators in the same range.

The bilinear maximal Bochner-Riesz operator for  $\lambda > 0$  is defined as

$$(18) \quad T_*^\lambda(f, g)(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m^\lambda(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|,$$

where  $m^\lambda(\xi, \eta) = (1 - |\xi|^2 - |\eta|^2)_+^\lambda$ , which is equal to  $(1 - (|\xi|^2 + |\eta|^2))^\lambda$  when  $|(\xi, \eta)| \leq 1$  and 0 when  $|(\xi, \eta)| > 1$ .

Our main theorem concerning the boundedness of bilinear maximal Bochner-Riesz means is as follows:

**Theorem 4.1.** *When  $\lambda > \frac{2n+3}{4}$ , for  $T_*^\lambda$  in (18) we have that*

$$\|T_*^\lambda(f, g)\|_{L^1} \leq C \|f\|_{L^2} \|g\|_{L^2}.$$

We fix a nonnegative smooth function  $\varphi(s)$  supported in  $[-\frac{3}{4}, \frac{3}{4}]$  and a smooth function  $\psi$  supported in  $[\frac{1}{8}, \frac{5}{8}]$  such that  $\sum_{j=0}^{\infty} \psi_j(1-s) = 1$  for  $s \in [0, 1)$ , where  $\psi_j(s) = \psi(2^j s)$  for  $j \geq 1$  and  $\psi_0 = \varphi$ .

We decompose the multiplier  $m(\xi, \eta) = (1 - (|\xi|^2 + |\eta|^2))_+^\lambda$  smoothly as  $m = \sum_{j \geq 0} m_j$ , where

$$m_j(\xi, \eta) = m(\xi, \eta) \psi_j(|(\xi, \eta)|)$$

is supported in an annulus of the form

$$\{(\xi, \eta) \in \mathbb{R}^{2n} : 1 - 2^{-j} \leq |(\xi, \eta)| \leq 1 - 2^{-j-2}\}$$

for  $j \geq 1$  and  $m_0$  is supported in a ball of radius  $3/4$  centered at the origin. If

$$(19) \quad T_j(f, g)(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) \widehat{g}(\eta) m_j(t\xi, t\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|,$$

then

$$T_*^\lambda(f, g)(x) \leq \sum_{j=0}^{\infty} T_j(f, g)(x).$$

The following are straightforward facts about  $T_*^\lambda$  and  $T_j$ . Let  $\|T\|_{X \times Y \rightarrow Z}$  denote the norm of  $T$  from  $X \times Y$  to  $Z$ .

**Proposition 4.2.** *Assume  $1 < p_1, p_2 \leq \infty$ , and  $1/p = 1/p_1 + 1/p_2$ . Then for  $\lambda > n - 1/2$ , there exists a finite constant  $C = C(p_1, p_2)$  such that  $\|T_*^\lambda\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C$ . For any fixed  $j$ , there exists a finite constant  $C_j(p_1, p_2)$  such that  $\|T_j\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C_j(p_1, p_2)$ .*

*Proof.* Let us consider the kernel  $K(y, z) = m^\vee(y, z)$  of  $A_1^\lambda$  defined in (17), which satisfies that  $|K(y, z)| \leq C(1 + |y| + |z|)^{-(n+\lambda+1/2)}$  (see, for example, [11]), hence for  $\lambda > n - 1/2$ , we have

$$\begin{aligned} |A_t(f, g)(x)| &= \left| \int_{\mathbb{R}^{2n}} t^{-2n} K\left(\frac{x-y}{t}, \frac{x-z}{t}\right) f(y)g(z) dy dz \right| \\ &\leq C(\varphi_t * |f|)(x)(\varphi_t * |g|)(x) \\ &\leq CM(f)(x)M(g)(x), \end{aligned}$$

where  $M$  is the Hardy-Littlewood maximal function, and  $\varphi_t(y) = t^{-n}\varphi(y/t)$  with  $\varphi(y) = (1 + |y|)^{-(n+\lambda+1/2)/2}$ , which is integrable when  $\lambda > n - 1/2$ . Then  $T_*(f, g)(x) \leq CM(f)(x)M(g)(x)$ , which implies that  $\|T_*(f, g)\|_{L^p} \leq C(p_1, p_2)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$  for  $1 < p_1, p_2 \leq \infty$  with  $1/p = 1/p_1 + 1/p_2$  in view of the boundedness of the Hardy-Littlewood maximal function.

We observe that each  $m_j$  is smooth and compactly supported, hence for each  $j$  a similar argument yields  $\|T_j\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C_j(p_1, p_2) < \infty$ .  $\square$

With the aid of the preceding decomposition and the boundedness of  $T_j$ , the study of the boundedness of  $T_*$  is reduced to the decay of  $C_j$  in  $j$ .

We now go back to the multipliers and will apply Theorem 1.2. For this purpose we should study kinds of norms of  $m_j$ .

**Lemma 4.3.** *There exists a constant  $C$  such that*

$$\|m_j\|_{L^2} \leq C2^{-j(\lambda+1/2)},$$

and for any multiindex  $\alpha$ ,

$$(20) \quad \|\partial^\alpha m_j\|_{L^\infty} \leq C_\alpha 2^{-j(\lambda-|\alpha|)}.$$

*Proof.* A change of variables using polar coordinates implies that

$$\begin{aligned} \|m_j\|_{L^2} &= \left( \int_{\mathbb{R}^{2n}} |m_j(\xi, \eta)|^2 d\xi d\eta \right)^{1/2} \\ &\leq C \left( \int_{1-2^{-j}}^{1-2^{-j-2}} (1-r^2)^{2\lambda} r^{2n-1} dr \right)^{1/2} \\ &\leq C(2^{-2j\lambda} 2^{-j})^{1/2} \\ &= C2^{-j(\lambda+1/2)} \end{aligned}$$

To estimate the  $\alpha$ -th derivatives, we use the Leibniz's rule to write

$$\partial^\alpha m_j(\xi, \eta) = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1} \partial^{\alpha_1} m(\xi, \eta) \partial^{\alpha_2} \psi_j(|(\xi, \eta)|).$$

Noticing that  $|\partial^{\alpha_1} m(\xi, \eta)| \leq C2^{-j(\lambda-|\alpha_1|)}$  and  $\partial^{\alpha_2} \psi_j(|(\xi, \eta)|) \leq C2^{j|\alpha_2|}$ , we derive the bound  $\partial^\alpha m_j(\xi, \eta)$  by  $C2^{-j(\lambda-|\alpha|)}$ .  $\square$

The multiplier  $m_j$  is not supported in the annulus of radius  $2^j$  and one can verify that its Sobolev norm is not as good as would wish. Actually the norm increases as the number of derivatives is large. So a dilation is necessary to apply Theorem 1.2.

Let us define  $M_j(\xi, \eta) = m_j(2^{-j}\xi, 2^{-j}\eta)$ , which is supported in the annulus  $\{(\xi, \eta) \in \mathbb{R}^{2n} : 2^j - 1 \leq |(\xi, \eta)| \leq 2^j - 1/4\}$ , whose width is  $3/4$ . Based on Lemma 4.3, we have the following corollary.

**Corollary 4.4.** *The multipliers  $M_j(\xi, \eta) = m_j(2^{-j}\xi, 2^{-j}\eta)$  satisfy*

$$\|\partial^\alpha M_j\|_{L^\infty} \leq C2^{-j\lambda} \quad \text{for all multiindex } \alpha,$$

$$\nabla m_j(\xi, \eta) = 2^j(\nabla M_j)(2^j\xi, 2^j\eta),$$

and

$$\|M_j\|_{L^r_s} \leq C2^{-j\lambda} 2^{j(2n-1)/r},$$

where  $\|M_j\|_{L^r_s} = \|(I - \Delta)^{s/2} M_j\|_{L^r}$  is the Sobolev norm of  $M_j$ .

*Proof.* We have

$$|\partial^\alpha M_j| \leq 2^{-j|\alpha|} |(\partial^\alpha m_j)(2^{-j}\xi, 2^{-j}\eta)| \leq C2^{-j|\alpha| - j(\lambda - |\alpha|)} = C2^{-j\lambda},$$

using (20). The verification of the last identity is straightforward once we notice that  $M_j$  is supported in the annulus

$$\{(\xi, \eta) : 2^j - 4 \leq |(\xi, \eta)| \leq 2^j - 1\}$$

whose volume is about  $2^{j(2n-1)}$ .  $\square$

*Proof of Theorem 4.1.* It is easy to verify that  $T_j$  in (19) stays the same if we replace  $m_j$  by  $M_j$ . We apply Theorem 1.2 to  $M_j$  with  $r = 4$ , then it follows from this and Proposition 4.2 that  $T_*$  is bounded from  $L^2 \times L^2$  to  $L^1$  when  $\lambda > \frac{2n+3}{4}$ .  $\square$

Using complex interpolation between Theorem 4.1 and Proposition 19 we can obtain a larger range of boundedness, which we will not pursue here.

As a corollary of Theorem 4.1, we obtain the pointwise convergence, as  $t \rightarrow 0$ , of the operator  $A_t^\lambda(f, g)(x)$ , which we denote by  $A_t(f, g)(x)$  as well.

**Proposition 4.5.** *Suppose  $\lambda > \frac{2n+3}{4}$ , then for  $f \in L^2$  and  $g \in L^2$  we have*

$$(21) \quad \lim_{t \rightarrow 0} A_t(f, g)(x) \rightarrow f(x)g(x) \quad \text{a.e..}$$

The proof of this proposition is similar to the linear case, but we sketch it here for completeness.

*Proof.* It is easy to establish (21) when both  $f$  and  $g$  are Schwartz functions. To prove (21) for  $f \in L^2$  and  $g \in L^2$  it suffices to show that for any given  $\delta > 0$  the set  $E_{f,g}(\delta) = \{y \in \mathbb{R}^n : O_{f,g}(y) > \delta\}$  has measure 0, where

$$O_{f,g}(y) = \limsup_{\theta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} |A_\theta(f, g)(y) - A_\varepsilon(f, g)(y)|.$$

For any positive number  $\eta$  smaller than  $\|f\|_{L^2}, \|g\|_{L^2}$ , there exist Schwartz functions  $f_1 = f - a$  and  $g_1 = g - b$  such that both  $\|a\|_{L^2}$ , and  $\|b\|_{L^2}$  are bounded by  $\eta$ . We observe that

$$|E_{f,g}(\delta)| \leq |E_{f_1,g_1}(\delta/4)| + |E_{a,g_1}(\delta/4)| + |E_{f_1,b}(\delta/4)| + |E_{a,b}(\delta/4)|.$$

Notice that  $|E_{f_1,g_1}(\delta/4)| = 0$  since (21) is valid for  $f_1, g_1$ . To control the remaining three terms, we observe that, for instance,

$$\begin{aligned} |E_{a,g_1}(\delta/4)| &\leq |\{y : 2T_*(a, g_1)(y) > \delta/4\}| \\ &\leq C \frac{\|a\|_{L^2} \|g_1\|_{L^2}}{\delta} \\ &\leq C \frac{\eta \|g\|_{L^2}}{\delta}, \end{aligned}$$

where the last term goes to 0 as  $\eta \rightarrow 0$  since  $g$  and  $\delta$  are fixed.  $\square$

## 5. APPENDIX: PROOF OF PROPOSITION 3.1

The proof of this proposition is essentially contained in [13, Lemma 6], but for the sake of completeness we include it, ignoring some routine calculations that can be found in [13].

*Proof of Proposition 3.1.* Notice that in the support of  $\nabla M_{j,\gamma}^1(\xi, \eta)$ , we have  $\xi \in E$  and  $\eta \in E$ , hence we may always assume that  $\widehat{f} = \widehat{f}\chi_E$  and  $\widehat{g} = \widehat{g}\chi_E$ . In other words, it suffices to establish (11) without  $\chi_E$ .

It suffices to consider, for example, the typical term  $\xi_1 \partial_{\xi_1} M_{j,\gamma}^1(\xi, \eta)$ , which is  $\sum_k \sum_l a_{k,l} \partial_{\xi_1} \omega_{1,k}(\xi) \xi_1 \omega_{2,l}(\eta)$  for allowed  $k, l$  in  $M_{j,\gamma}^1$ . We rewrite this as

$$(22) \quad 2^j 2^\gamma \sum_k \sum_l b_{k,l} \tilde{\omega}_{1,k}(\xi) \tilde{\omega}_{2,l}(\eta),$$

where  $\tilde{\omega}_{1,k}(\xi) = 2^{-j} \partial_{\xi_1} \omega_{1,k}(\xi) \xi_1 / \|\partial_{\xi_1} \omega_{1,k}(\xi)\|_{L^r}$ ,  $\tilde{\omega}_{2,l} = \omega_{2,l} / \|\omega_{2,l}\|_{L^r}$ , and  $b_{k,l} = 2^{-\gamma} a_{k,l} \|\partial_{\xi_1} \omega_{1,k}(\xi)\|_{L^r} \|\omega_{2,l}\|_{L^r}$ .

We need some estimates of  $\tilde{\omega}_{1,k}$  which will be useful later. The function  $\partial_{\xi_1} \omega_{1,k}(\xi)$  is of the form  $2^\gamma 2^{m/2} \varphi(2^\gamma \xi)$  for a compactly supported smooth function  $\varphi$ , hence  $\|\partial_{\xi_1} \omega_{1,k}(\xi)\|_{L^r} \approx 2^{\gamma(1+\frac{n}{2}-\frac{n}{r})}$ . This implies that  $\|\tilde{\omega}_{1,k}\|_{L^\infty} \leq C 2^{\gamma m/r}$  since  $|\xi_1| \leq C 2^j$ .



We have

$$\left\| \left( \sum_{k,l} 2^{\gamma s} |2^{\gamma m} a_{k,l} \chi_{Q_{\gamma,k,l}}|^2 \right)^{1/2} \right\|_{L^r} \leq C \|M_j\|_{L^r_s},$$

by Lemma 2.2, where  $Q_{\gamma,k,l}$  is the cube centered at  $2^{-\gamma}(k,l)$  with length  $2^{1-\gamma}$ . This leads to

$$\left\| \left( \sum_{k,l} 2^{\gamma s} |2^{-j} 2^{-\gamma} a_{k,l} \partial_{\xi_1} \omega_{1,k}(\xi) \xi_1 \omega_{2,l}(\eta)|^2 \right)^{1/2} \right\|_{L^r} \leq C \|M_j\|_{L^r_s}.$$

Recall that  $\|M_j\|_{L^r_s} \leq C 2^{-j\lambda}$ . Then using the disjointness of supports of  $\omega_{k,l}$  we obtain further that

$$B = \left( \sum |b_{k,l}|^r \right)^{1/r} \leq C 2^{-j\lambda} 2^{-s\gamma}.$$

Each  $\omega$  in level  $\gamma$  is of the form  $\omega = \omega_k \omega_l$  with  $\vec{\mu} = (k,l)$ , where  $k$  and  $l$  both range over index sets of cardinality at most  $C 2^{jn} 2^{\gamma m}$ . Moreover we denote by  $b_{kl}$  the coefficient  $b_\omega$ , and we define a bilinear multiplier

$$\zeta_\gamma = \sum_{k \in U_1} \tilde{\omega}_k \sum_{l \in U_2} b_{kl} \tilde{\omega}_l.$$

Let  $A$  be a number between  $\|b\|_\infty$  and  $B = \|b\|_r$ . Related to  $\tau \geq 0$  we define  $U_\tau = \{(k,l) : 2^{-\tau-1}A \leq |b_{k,l}| \leq 2^{-\tau}A\}$ . Denote by  $col_k = \{(k,l) \in U_\tau : k \text{ fixed}\}$ . Define

$$U_\tau^1 = \{(k,l) \in U_\tau : \#col_k \geq N_1\},$$

where  $N_1$  is a to be determined number. So  $U_\tau^1$  is a union of long columns. We denote by  $P_1 U_\tau^1 = \{k : \exists l \text{ s.t. } (k,l) \in U_\tau^1\}$ , the projection of  $U_\tau^1$  onto the  $k$ -axis. Then the number of columns is  $\#P_1 U_\tau^1 \leq B^r (2^{-\tau}A)^{-r} N_1^{-1} := N_2$ .

Let  $U_\tau^2$  be the complement of  $U_\tau^1$  in  $U_\tau$ . Associated to  $U_\tau^i$  we can define a bilinear multiplier  $\zeta_\tau^i = 2^j 2^\gamma \sum_{(k,l) \in U_\tau^i} b_{k,l} \tilde{\omega}_{k,l}$ , and a bilinear operator  $T_{\zeta_\tau^i}$ . A well-known argument (see, for instance, [14] or [13]) shows that

$$\|T_{\zeta_\tau^1}(f,g)\|_{L^1} \leq C 2^j 2^\gamma N_2^{1/2} 2^{2\gamma m/r} 2^{-\tau} A \|f\|_{L^2} \|g\|_{L^2}$$

and

$$\|T_{\zeta_\tau^2}(f,g)\|_{L^1} \leq C 2^j 2^\gamma N_1^{1/2} 2^{2\gamma m/r} 2^{-\tau} A \|f\|_{L^2} \|g\|_{L^2}.$$

Identifying  $N_1$  and  $N_2$ , and taking  $A = B$  in our situation, we obtain that  $N_1 = N_2 = C 2^{\tau r/2}$ , which implies that the  $\|T_{\zeta_\tau^i}\|_{L^2 \times L^2 \rightarrow L^1}$  is bounded by  $C 2^{-j(\lambda-1)} 2^{-\gamma(s-\frac{2n}{r}-1)} 2^{-\tau(1-\frac{r}{4})}$ .

Summing over  $\tau$ , we obtain the claimed bound for  $r < 4$ .

For the case  $r = 4$ , we may assume that  $\tau \leq \tau_m = 2(j+\gamma)n/4$  since  $N_2 = 2^{\tau r/2} \leq 2^{(j+\gamma)n}$  with  $r = 4$ . Actually we define

$$U_{\tau_m} = \{(k,l) : |b_{k,l}| \leq 2^{-\tau_m} A\}.$$

Then the previous argument gives the bound  $C(j + \gamma)n2^{-j(\lambda-1)}2^{-\gamma(s-\frac{n}{2}-1)}$  when  $r = 4$ .  $\square$

A lemma concerning the decay of the coefficients related to the orthonormal basis in Lemma 2.1 is given below.

**Lemma 5.1** ([14]). *Suppose  $\zeta(\xi, \eta)$  defined on  $\mathbb{R}^{2n}$  satisfies that there exists a constant  $C_M$  such that  $\|\partial^\alpha(\zeta(\xi, \eta))\|_{L^\infty} \leq C_M$  for each multiindex  $|\alpha| \leq M$ , where  $M$  is the number of vanishing moments of  $\psi_M$ . Then for any nonnegative integer  $\gamma \in \mathbb{N}_0 = \{n \in \mathbb{Z} : n \geq 0\}$  we have*

$$(23) \quad |\langle \Psi_{\vec{\mu}}^{\gamma, G}, \zeta \rangle| \leq CC_M 2^{-(M+n)\gamma}.$$

This lemma can be proved by applying Appendix B.2 in [12], and we delete the details which can be found in [14].

By this lemma we have a better decay in  $j$  for  $b_{k,l}$  compared with Corollary 2.3, namely  $|b_{k,l}| \leq C2^{-ja}2^{-\gamma(s+n)}$ , using  $|\partial^\beta m| \leq C|(\xi, \eta)|^{-a}$  in Theorem 1.1 if we assume  $s$  number of derivatives. It is natural to conjecture that this better decay in  $j$  can lower the restriction on  $a$ . This, unfortunately, is not true.

As we did before, setting  $N_1 = N_2$  implies that  $N_1 = 2^{\tau r/2}$ . An important observation is that  $|b_{k,l}| \ll B$ . Actually the smallest  $\tau$  such that  $2^{-\tau}B \sim \|b_{k,l}\|_{\ell^\infty} \leq C2^{-ja}2^{-\gamma(s+n)}$  is  $\tau_0 = \frac{2nj}{r} + n\gamma$ , which means that the summation in  $\tau$  starts from  $\tau_0$  other than 0.

Another observation is that  $N_2$  related to  $\tau_0$  is  $2^{\tau_0 r/2} \sim 2^{nj+n\gamma r/2}$ , which is smaller than  $2^{\tau_m r/2} \sim 2^{nj+n\gamma}$  when  $r > 2$ . So for  $r \in (2, 4)$ , we take  $N_2 = 2^{nj+n\gamma}$ . And the summation in  $\tau$  consists just one term  $\tau_0$ .

By the calculation in the proof of Proposition 3.1 the norm of  $T_{\zeta_\gamma} \sum_\tau T_{\zeta_\tau^i}$ , which consists of one term with  $\tau = \tau_0$ , is bounded by a constant multiple of  $2^{-j(a-n/2-1)}2^{-\gamma(s-n/2-1)}$ . This provides no new information except for a bound independent of  $r$ , which is natural since there is no  $r$  in the conditions of Theorem 1.1.

So we still have the restriction  $a > \frac{n}{2} + 1$ . It is also easy to verify that when  $r = 4$  the bound for  $T_{\zeta_\tau^i}$  does not change, so in this case we need  $a > \frac{n}{2} + 1$  as well.

*Remark 2.* We use mainly the case  $r = 4$  in applying Proposition 3.1, while a smaller  $A$ , which reduces the number of  $\tau$ 's involved, does not change the exponential decay in  $j$  at all.

*Remark 3.* Lemma 5.1 implies also a better decay of the off-diagonal part in  $j$ , namely  $2^{-j(a-1/2)}$ , which, however, is useless for us due to the restriction of the diagonal part.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211,  
USA

*E-mail address:* grafakosl@missouri.edu

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275,  
P. R. CHINA

*E-mail address:* hedanqing@mail.sysu.edu.cn

MFF UK, SOKOLOVSKA 83, PRAHA 7, CZECH REPUBLIC

*E-mail address:* honzik@gmail.com