On fragmented convex functions

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Abstract

Let X be a compact convex set and let $\operatorname{ext} X$ stand for the set of extreme points of X. Let $f\colon X \to \mathbb{R}$ be a bounded convex function with the point of continuity property. The first main result shows that $f \leq 0$ on X provided $f \leq 0$ on $\operatorname{ext} X$. As a byproduct of our method we generalize a result of Raja. Next we show that a resolvable convex semi-extremal nonempty set in X intersects $\operatorname{ext} X$. Finally we prove a Phelps maximum principle for abstract affine functions defined on a locally compact topological space.

1. *Introduction*

Let $\mathbb{F}$ stand for $\mathbb{R}$ or $\mathbb{C}$. Given a locally compact (Hausdorff) topological space X and $f\colon X \to \mathbb{F}$, the function f is said to have the *point of continuity property* if $f|_F$ has a point of continuity for any $F \subset X$ nonempty closed. The function f is called *fragmented* if for any $\epsilon > 0$ and nonempty closed set $F \subset X$ there exists a relatively open nonempty set $U \subset F$ such that $\operatorname{diam} f(U) < \epsilon$. A set $A \subset X$ is *resolvable* if the characteristic function χ_A is fragmented. By [**7**, Theorem 2.3], the following assertions for a function $f\colon X \to \mathbb{F}$ on a locally compact space X are equivalent:

(i) f is fragmented;
(ii) f has the point of continuity property;
(iii) for each $U \subset \mathbb{F}$ open the set $f^{-1}(U)$ is expressible as a countable union of resolvable sets.

(We remark that a locally compact space is hereditarily Baire, see [**4**, p. 196 and Theorem 3.9.6].) It easily follows from this characterization that fragmented functions form a Banach lattice and algebra, see [**9**, Theorem 5.10]. It also readily follows that any semicontinuous function $f\colon X \to \mathbb{R}$ is fragmented. Also, if X is a metrizable compact space, a set $F \subset X$ is resolvable if and only if F is both F_σ and G_δ, i.e., if χ_F is a classical Baire-one function.

Now we turn our attention to the case when X is a compact convex subset of a locally convex (Hausdorff) space. We recall that if H is a subset of X and $x \in H$, then x is an *extreme point* of H if whenever $x = \lambda y + (1-\lambda)z$ for some $y, z \in H$ and $\lambda \in (0,1)$, then $x = y = z$. We write $\operatorname{ext} H$ for the set of extreme points of H. It is well know that any semicontinuous convex function $f\colon X \to \mathbb{R}$ satisfies the maximum principle, i.e., $f \le 0$ on X provided $f \le 0$ on $\operatorname{ext} X$ (see e.g. [**9**, Section 3.9]). It is proved in [**3**] that a fragmented affine function satisfies the maximum principle. The first main result of this note is a proof of an analogous result for bounded convex fragmented functions (see Corollary 2·6). As a byproduct of our techniques we obtain an improvement of a theorem by Raja in [**13**] and also recover the Rainwater theorem.

As a subsequent result we verify that fragmented convex functions respect the Choquet ordering.

A set $F \subset X$ is called *extremal* if $\lambda x + (1-\lambda)y \in F$ for some $x, y \in X$ and $\lambda \in (0,1)$ implies $x, y \in F$. If $F \subset X$ is nonempty, resolvable and extremal, the function χ_F is convex and fragmented. Thus it follows from Corollary 2·6 that $F \cap \operatorname{ext} X \neq \emptyset$. In [**12**], it was showed that nonempty closed semi-extremal subsets of X intersect $\operatorname{ext} X$. (We recall that $F \subset X$ is *semi-extremal* if $X \setminus F$ is convex.) We prove a similar result by showing that any nonempty semi-extremal convex resolvable set intersects $\operatorname{ext} X$.

Finally we consider an abstract function space $\mathcal{H}$ on a locally compact space K with the Choquet boundary $\operatorname{Ch} K$ (see Section 5). We show that a completely $\mathcal{H}$-affine fragmented function f on K satisfies $\sup_{x \in K} |f(x)| = \sup_{x \in \operatorname{Ch} K} |f(x)|$, generalizing thus a theorem of Phelps, see [**5**, Theorem 2.3.8].

2. *Maximum principle*

First we recall several notions. Let K be a compact topological (Hausdorff) space. We write $\mathcal{M}(K, \mathbb{F})$ for the set of all (complete) Radon $\mathbb{F}$-valued measures on K, $\mathcal{M}^+(K)$ for the set of nonnegative Radon measures and $\mathcal{M}^1(K)$ for the set of probability Radon measures on K. We endow these sets with the weak* topology given by the duality $\mathcal{C}(K, \mathbb{F})^* = \mathcal{M}(K, \mathbb{F})$, where $\mathcal{C}(K, \mathbb{F})$ stands for the space of $\mathbb{F}$-valued continuous functions on K endowed with the sup-norm. A function $f : K \to \mathbb{F}$ is called *universally measurable* if f is measurable with respect to every measure $\mu \in \mathcal{M}(K, \mathbb{F})$.

Let X be a compact convex set in a locally convex (Hausdorff) space. We write $\mathfrak{A}(X, \mathbb{F})$ for the space of affine $\mathbb{F}$-valued continuous functions on X. For any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$ such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X, \mathbb{F})$, see [**1**, Proposition I.2.1]. We call $r(\mu)$ the *barycenter* of μ. Alternatively we say that the measure μ *represents* $r(\mu)$ and we write $\mathcal{M}_x$ for the set of all probability measures representing a point $x \in X$.

By the Choquet–Bishop–de-Leeuw representation theorem (see [**1**, Theorem I.4.8 and Proposition 1.4.6.]), for each $x \in X$ there exists a measure $\mu \in \mathcal{M}^1(X)$ carried by $\overline{\operatorname{ext} X}$ with $r(\mu) = x$.

For $x, y \in X$ we write $[x, y]$ for the segment $\{\lambda x + (1-\lambda)y;\ \lambda \in [0,1]\}$ and (x, y) for the set $\{\lambda x + (1-\lambda)y;\ \lambda \in (0,1)\}$. Further, we write $L(x, y)$ for the line given by x and y intersected with X, i.e.,

$$L(x, y) = \{x + t(y - x);\ t \in \mathbb{R}\} \cap X.$$

The following geometric lemma is a key ingredient for several results of this paper. In the proof we several times use the following simple geometric observation. Let $a_1, a_2, b_1, b_2, x \in X$ be distinct points satisfying $x \in (a_1, b_1) \cap (a_2, b_2)$. If $c \in [a_1, a_2]$, then the line segment $L(c, x)$ intersects $[b_1, b_2]$. If $c \in (a_1, a_2)$, then the line segment $L(c, x)$ intersects (b_1, b_2).

Lemma 2·1. *Let H be a semi-extremal subset of X. Then* $\operatorname{ext} X \cap H = \emptyset$ *if and only if* $\operatorname{ext} \overline{H} \cap H = \emptyset$.

Proof. It is obvious that if H contains an extreme point of X, then this point is extreme for $\overline{H}$.

We suppose that H contains no extreme point of X and we denote $F = X \setminus H$. We assume that there exists a point $x \in \operatorname{ext} \overline{H} \cap H$ and seek a contradiction. Since $x \in H$, it is not an extreme point of X, so there are distinct points $a, b \in X$ such that $x \in (a, b)$. Since $x \in \operatorname{ext} \overline{H}$, at least one of the points a and b does not belong to H, and since F is convex, a and b are not both in F. Thus we may assume that $a \in F$ and $b \in H$.

Now we denote

$$t = \max\{s \geq 0;\ b + s(b - a) \in X\},$$

and

$$c = b + t(b - a).$$

Then by the convexity of F, c belongs to H. We show that c is an extreme point of X which then yields the desired contradiction. Assuming the contrary, there exist distinct points $c_1, c_2 \in X$ such that $c \in (c_1, c_2)$. Then by the choice of c, the points c_1 and c_2 do not lie on the line segment $L(a, c)$, and thus the convex hull of the points a, c_1 and c_2 forms a non-degenerated triangle. Using the convexity of F, c_1 and c_2 are not both in F, thus we assume that $c_1 \in H$. Now we distinguish two cases.

Case 1. If $c_2 \notin F$, then let

$$u = \sup\{s \geq 0;\ a + s(c_2 - a) \in F\},$$

and

$$d_2 = a + u(c_2 - a). \tag{2.1}$$

Then such a defined point d_2 belongs to the line segment $[a, c_2]$. We show that d_2 is in the closure of H. If $d_2 = c_2 \in H$, then we are done. Assuming that $d_2 \neq c_2$, it follows from the definition of the point d_2 that the line segment (d_2, c_2) is a subset of H, see (2.1). This proves that d_2 belongs to the closure of H.

Using the extremality of the point x, the point of intersection of $L(c_1, x)$ and (a, c_2), which we denote by e_2, belongs to F, from which it follows that $d_2 \neq a$. We denote $L = (a, d_2)$ and we claim that $L \subset F$. Indeed, consider an arbitrary point $y \in L$. Then by the choice of d_2, there exists a point $z \in F \cap (y, d_2)$. Now, since $a \in F$, the convexity of F concludes that $y \in F$.

Now we claim that the line segment $L(d_2, x)$ intersects the line segment $[a, c_1]$. But this follows from the observation mentioned before the statement of the Lemma and the facts that the sets $L(c_2, x)$ and $L(e_2, x) = L(e_2, c_1)$ intersect the segment $[a, c_1]$, contain x in their interior and d_2 is contained in $[e_2, c_2]$. We denote this intersection point as d_1.

To finish the proof of this case it is enough to show that $d_1 \in \overline{H}$, which then yields a contradiction to the extremality of the point x. If $d_1 = c_1$, then we are done. Otherwise $d_1 \neq c_1$, from which it follows that $d_2 \neq e_2$. We claim that (d_1, c_1) is a subset of H. If not, then we pick a point $e_1 \in (d_1, c_1) \cap F$.

We know that the sets $L(c_1, e_2)$ and $L(d_1, d_2)$ are distinct and contain x in their interior. So we can use our observation to conclude that $L(e_1, x)$ intersects the set $(d_2, e_2) \subset L$. But L is a subset of F and $x \notin F$, which contradicts the convexity of F. Thus $(d_1, c_1) \subset H$, so d_1 belongs to the closure of H. This finishes the proof of the first case.

Case 2. Now we assume that $c_2 \in F$. By the convexity of F, the whole line segment

$[a, c_2]$ lies in F, and we define

$$u = \sup\{s \geq 0;\ c_2 + s(c_1 - c_2) \in F\},$$

and

$$d_2 = c_2 + u(c_1 - c_2). \tag{2.2}$$

Then, since $c \in H$, d_2 belongs to the line segment $[c_2, c]$. We denote $L = [a, c_2] \cup (c_2, d_2)$. Now the arguments are almost the same as in the previous case.

It holds that $d_2 \in \overline{H}$. Indeed, either $d_2 = c \in H$, and the other case is treated similarly as above. Moreover, we know that $[a, c_2] \subset F$ and using the same arguments as in the previous case it is easy to show that $(c_2, d_2) \subset F$, and so $L \subset F$.

Now, since the sets $L(c, x) = L(c, a)$ and $L(c_2, x)$ intersect the segment $[a, c_1]$, contain x in their interior and d_2 belongs to the closed line segment given by c_2 and c, using the observation it follows that $L(d_2, x)$ intersects $[a, c_1]$. We again denote this intersection point as d_1. As in the previous case we would show that this point d_1 belongs to the closure of H. This gives a contradiction to the extremality of the point x and finishes the proof.

For the proof of the maximum principle we need the following several lemmas.

LEMMA 2·2. *Let $x \in \operatorname{ext} X$. Then for every neighbourhood U of x and $\varepsilon > 0$ there exists a neighbourhood V of x such that for every $y \in V$ and $\mu \in \mathcal{M}_y$ it holds that $\mu(U) > 1 - \epsilon$.*

Proof. Assuming the contrary, there is an open neighbourhood U of x and $\varepsilon > 0$ such that for each neighbourhood V of x there exists $y_V \in V$ and $\mu_V \in \mathcal{M}_{y_V}$ satisfying $\mu_V(U) \leq 1 - \varepsilon$.

We denote the system of all neighbourhoods of x by $\mathcal{V}_x$ and we consider the family of measures

$$\{\mu_V;\ V \in \mathcal{V}_x\}$$

as a net in $\mathcal{M}^1(X)$ with the ordering given by $\mu_V \leq \mu_W$ if $W \subset V$. Since $\mathcal{M}^1(X)$ is weak* compact, we may assume that the net weak* converges to some measure $\nu \in \mathcal{M}^1(X)$. Then it is clear that the net $\{y_V;\ V \in \mathcal{V}_x\}$ with the natural ordering converges to x. Thus for every $a \in \mathfrak{A}(X, \mathbb{F})$ it holds that

$$\lim_{V \in \mathcal{V}_x} \mu_V(a) = \lim_{V \in \mathcal{V}_x} a(y_V) = a(x).$$

By the uniqueness of the limit of a net it follows that

$$\nu(a) = a(x), \quad a \in \mathfrak{A}(X, \mathbb{F}),$$

in other words, ν is a representing measure for x. Since $x \in \operatorname{ext} X$, $\nu = \varepsilon_x$, the Dirac measure centered at the point x.

Now we find a function $g \in \mathcal{C}(X, \mathbb{R})$ satisfying $0 \leq g \leq 1$, $g(x) = 1$ and $g = 0$ on $X \setminus U$. Then we have that

$$\lim_{V \in \mathcal{V}_x} \mu_V(g) = \varepsilon_x(g) = g(x) = 1.$$

On the other hand, for each $V \in \mathcal{V}_x$ it holds that

$$|\mu_V(g)| = \left| \int_{X \setminus U} g d\mu_V + \int_U g d\mu_V \right| \leq 0 + \int_U |g| d\mu_V \leq \mu_V(U) \leq 1 - \varepsilon,$$

which gives the desired contradiction.

LEMMA 2·3. *Let $f\colon X \to \mathbb{R}$ be a function with the point of continuity property. Then the set of points of continuity of f with respect to $\overline{\operatorname{ext} X}$ is a dense G_δ set in $\operatorname{ext} X$.*

Proof. We consider the restriction of f to the set $\overline{\operatorname{ext} X}$, which we denote again by f. Let C_f stand for the set of points of continuity of f. Then C_f is a G_δ set, which is moreover dense, see [**7**, Theorem 2.3]. We write $C_f = \bigcap_{n=1}^{\infty} G_n$, where G_n are open sets in $\overline{\operatorname{ext} X}$. Then

$$C_f \cap \operatorname{ext} X = \bigcap_{n=1}^{\infty} (G_n \cap \operatorname{ext} X)$$

is a G_δ set in $\operatorname{ext} X$. It remains to show that it is dense in $\operatorname{ext} X$. To see this, it is enough to realize that since $\operatorname{ext} X$ is dense in $\overline{\operatorname{ext} X}$ and each G_n is open and dense, $G_n \cap \operatorname{ext} X$ is an open dense set in $\operatorname{ext} X$. Since $\operatorname{ext} X$ is a Baire space (see [**1**, Theorem I.5.13.]), the assertion follows.

LEMMA 2·4. *Let $x \in \operatorname{ext} X$ and f be a bounded fragmented convex function on X which is upper semicontinuous with respect to $\overline{\operatorname{ext} X}$ at the point x. Then f is upper semicontinuous at the point x with respect to X.*

Proof. Let $L \geq 0$ be a constant satisfying $|f| \leq L$.

By [**9**, Theorem 10.75.], it holds that

$$\mu(f) \geq f(r(\mu)), \quad \mu \in \mathcal{M}^1(X).$$

Let $\varepsilon > 0$ be given.

Using the upper semicontinuity of f with respect to $\overline{\operatorname{ext} X}$ we find a neighbourhood $U \subset X$ of x such that

$$f(y) < f(x) + \varepsilon, \quad y \in U \cap \overline{\operatorname{ext} X}.$$

By Lemma 2·2 there exists a neighbourhood V of x such that for each $y \in V$ and $\mu \in \mathcal{M}_y$ it holds that $\mu(U) > 1 - \varepsilon$.

We choose arbitrary $y \in V$ and pick a measure $\mu_y \in \mathcal{M}_y$ carried by $\overline{\operatorname{ext} X}$. Now we have

$$\begin{aligned} f(y) \leq \mu_y(f) = \int_{\overline{\operatorname{ext} X}} f d\mu_y = \int_{\overline{\operatorname{ext} X} \cap U} f d\mu_y + \int_{\overline{\operatorname{ext} X} \setminus U} f d\mu_y \leq \\ \leq f(x) + \varepsilon + \varepsilon L = f(x) + \varepsilon(1 + L). \end{aligned}$$

This finishes the proof.

In [**13**], Raja proved that for a bounded lower semicontinuous convex function f on a compact convex set X the set of points of continuity of f is dense in $\operatorname{ext} X$. This can be reformulated in the way that the set of points of upper semicontinuity is dense in $\operatorname{ext} X$ for bounded lower semicontinuous convex functions. Since every semicontinuous function is fragmented, Lemmas 2·3 and 2·4 generalize this result.

Now we are ready to prove the main theorem of this section.

THEOREM 2·5. *Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of convex functions on X with the point of continuity property.*

a) If $\{f_n\}_{n=1}^{\infty}$ is monotone and $\lim\limits_{n\to\infty} f_n \le 0$ on $\operatorname{ext} X$, then $\lim\limits_{n\to\infty} f_n \le 0$ on X.

b) If $\{f_n\}_{n=1}^{\infty}$ is nonincreasing and $\lim\limits_{n\to\infty} f_n < 0$ on $\operatorname{ext} X$, then $\lim\limits_{n\to\infty} f_n < 0$ on X.

Proof. We denote $f = \lim_{n\to\infty} f_n$. Then f is a convex function on X. We first prove the assertion $a)$ under the assumption that $\{f_n\}_{n=1}^{\infty}$ is a nonincreasing sequence.

Assuming the contrary, there exists $\eta > 0$ such that the set

$$H = \{x \in X;\ f(x) \ge \eta\}$$

is nonempty. Then H is a semi-extremal subset of X which does not intersect extreme points of X. By the monotonicity, for each n it holds that $f_n(x) \ge \eta$ on H. We denote $E = \overline{co}(H)$. By the Milman theorem it holds that $\operatorname{ext} E \subset \overline{H}$. Since E is a compact convex set, by Lemma 2·3, for each n, the set of points of continuity of f_n with respect to $\overline{\operatorname{ext} E}$ is a dense G_δ set in $\operatorname{ext} E$. Using the fact that $\operatorname{ext} E$ is a Baire space, there exists a point $x \in \operatorname{ext} E \subset \overline{H}$ such that all functions f_n are continuous at x with respect to $\overline{\operatorname{ext} E}$. Moreover, the functions f_n are at the point x upper semicontinuous with respect to E by Lemma 2·4. Since $x \in \overline{H}$, it follows that $f_n(x) \ge \eta$ for each n, hence $f(x) \ge \eta$. Thus $x \in H$, and hence $x \in H \cap \operatorname{ext} E$. In particular, $x \in H \cap \operatorname{ext} \overline{H}$. But this contradicts Lemma 2·1.

Now we assume that the sequence $\{f_n\}_{n=1}^{\infty}$ is nondecreasing. Let $n \in \mathbb{N}$ be arbitrary. Then $f_n \le f \le 0$ on $\operatorname{ext} X$. Applying the already proved result on a constant sequence we obtain that $f_n \le 0$ on X. Thus $f = \lim_{n\to\infty} f_n \le 0$ on X also. This finishes the proof of $a)$.

For the proof of $b)$ we proceed similarly as in $a)$. We assume that the set

$$H = \{x \in X;\ f(x) \ge 0\}$$

is nonempty and seek a contradiction. We denote $E = \overline{co}(H)$ and as above, we find a point $x \in \operatorname{ext} E$ which is a point of upper semicontinuity of all the functions f_n with respect to E. Since $f_n \ge 0$ on H for each n and $x \in \operatorname{ext} E \subset \overline{H}$, we obtain that $f_n(x) \ge 0$ for each n, and so $f(x) \ge 0$. Thus $x \in H \cap \operatorname{ext} E \subset H \cap \operatorname{ext} \overline{H}$. But the set $H \cap \operatorname{ext} \overline{H}$ is empty by Lemma 2·1, which is a contradiction. This finishes the proof.

As an immediate corollary of the preceding theorem we obtain the following maximum principles for convex functions having the point of continuity property.

COROLLARY 2·6. *Let $f : X \to \mathbb{R}$ be a bounded convex function with the point of continuity property.*

a)If $f \le 0$ on $\operatorname{ext} X$, then $f \le 0$ on X.

b)If $f < 0$ on $\operatorname{ext} X$, then $f < 0$ on X.

The classical Rainwater theorem (see [**11**, Section 5]) states that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of affine continuous functions on X which converges to 0 on $\operatorname{ext} X$, then it actually converges to 0 on X. Our Theorem 2·5 is in a sense not optimal since we need the monotonicity of the sequence of convex functions. Nevertheless, the Rainwater theorem follows from Theorem 2·5.

COROLLARY 2·7 (Rainwater). *Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of functions in $\mathfrak{A}(X,\mathbb{R})$. If $\lim\limits_{n\to\infty} f_n = 0$ on $\operatorname{ext} X$, then $\lim\limits_{n\to\infty} f_n = 0$ on X.*

Proof. For $n \in \mathbb{N}$ we denote

$$g_n(x) = \sup\{f_k(x);\ k \geq n\}, \quad x \in X.$$

Then $\{g_n\}_{n=1}^{\infty}$ is a bounded monotone sequence of convex lower semicontinuous functions, in particular, all the functions are fragmented. Moreover,

$$\lim_{n\to\infty} g_n(x) = \limsup_{n\to\infty} f_n(x) \leq 0, \quad x \in \operatorname{ext} X,$$

and so by Theorem 2·5 we obtain that

$$\limsup_{n\to\infty} f_n = \lim_{n\to\infty} g_n \leq 0 \quad \text{ on } X.$$

By an application of the argument above to the sequence $\{-f_n\}_{n=1}^{\infty}$ we get that

$$-\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} (-f_n(x)) \leq 0, \quad x \in X,$$

hence

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \in X.$$

This finishes the proof.

It would be interesting to know whether we can omit the assumption of the monotonicity in Theorem 2·5. In particular, the following generalization of the Rainwater theorem seems to be unknown.

QUESTION 2·8. *Let X be a compact convex set and $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of fragmented real affine functions converging to* 0 *on* $\operatorname{ext} X$*. Does it follow that $f_n \to 0$ on X?*

Also we do not know whether the strict maximum principle (Theorem 2·5(b)) holds for nondecreasing sequences. Thus we pose the following question.

QUESTION 2·9. *Let X be a compact convex set and $\{f_n\}_{n=1}^{\infty}$ be a bounded nondecreasing sequence of fragmented convex functions. Assume that $\lim_{n\to\infty} f_n(x) < 0$ for $x \in \operatorname{ext} X$. Does it follow that $\lim_{n\to\infty} f_n(x) < 0$ for $x \in X$?*

There is a general class of functions for which it is known that the maximum principle holds. A function $f : X \to \mathbb{R}$ is called strongly universally measurable if for each $x \in X$ and $\varepsilon > 0$ there exist lower semicontinuous affine function g and upper semicontinuous affine function h such that $h \leq f \leq g$ on X and $g(x) - h(x) < \varepsilon$, see [**16**, p. 519], [**10**, p. 435], [**15**, p. 102] or [**9**, Definition 9.30 and Proposition 9.32]. We plan to investigate potential applicability of Theorem 2·6 to extending results on strongly universally measurable functions to the context of fragmented functions.

3. *Fragmented convex functions and the Choquet ordering*

We recall that for a pair $\mu, \nu \in \mathcal{M}^+(X)$, the *Choquet ordering* $\mu \prec \nu$ means that $\mu(k) \leq \nu(k)$ for each convex continuous function k on X. It is known that $\mu(k) \leq \nu(k)$ provided $\mu \prec \nu$ and k is either lower or upper semicontinuous convex function, see [**9**, Proposition 3.56].

The aim of this section is to further extend this inequality to fragmented convex functions.

THEOREM 3·1. *Let X be a compact convex set and $\mu, \nu \in \mathcal{M}^1(X)$ satisfy $\mu \prec \nu$. If f is a fragmented convex function on X, then f is lower bounded and $\mu(f) \leq \nu(f)$.*

Proof. Let f be a convex fragmented function on X. By [**9**, Theorem 10.75], f is lower bounded. For each $n \in \mathbb{N}$ we consider the function $f_n(x) = \min\{f(x), n\}$. Then $f_n \nearrow f$, and by the Monotone convergence theorem, $\mu(f_n) \nearrow \mu(f)$. We fix $n \in \mathbb{N}$.

Let

$$M = \{(\varepsilon_x, \lambda) \in \mathcal{M}^1(X) \times \mathcal{M}^1(X);\, \varepsilon_x \prec \lambda\}.$$

Then M is a compact subset of $\mathcal{M}^1(X) \times \mathcal{M}^1(X)$. By [**9**, Proposition 3.89] there exists a measure $\Lambda \in \mathcal{M}^1(M)$ such that the point (μ, ν) is the barycenter of Λ.

By [**9**, Proposition 3.90], for any pair (f_1, f_2) of bounded universally measurable functions on X we have the equality

$$\mu(f_1) + \nu(f_2) = \int_M (\lambda_1(f_1) + \lambda_2(f_2))\, \mathrm{d}\Lambda(\lambda_1, \lambda_2).$$

Since it follows from [**9**, Proposition A.118] that fragmented functions are universally measurable, we may apply this identity to the pairs $(f_n, 0)$ and $(0, f_n)$. Since for any $(\varepsilon_x, \lambda) \in M$ we have $\lambda \in \mathcal{M}_x$ (see e.g. [**9**, Proposition 3.20]), we obtain $f(x) \leq \lambda(f)$ (see [**9**, Theorem 10.75]). Thus we get

$$\begin{aligned}
\mu(f_n) =& \mu(f_n) + \nu(0) = \int_M \lambda_1(f_n)\, \mathrm{d}\Lambda(\lambda_1, \lambda_2) = \int_M \varepsilon_x(f_n)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) \\
&= \int_{\{(\varepsilon_x,\lambda)\in M;\, f(x)<n\}} \varepsilon_x(f)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) + \int_{\{(\varepsilon_x,\lambda)\in M;\, f(x)\geq n\}} \varepsilon_x(n)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) \\
&\leq \int_{\{(\varepsilon_x,\lambda)\in M;\, f(x)<n\}} \lambda(f)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) + \int_{\{(\varepsilon_x,\lambda)\in M;\, f(x)\geq n\}} \lambda(n)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) \\
&= \int_M \lambda(f_n)\, \mathrm{d}\Lambda(\varepsilon_x, \lambda) = \mu(0) + \nu(f_n) = \nu(f_n) \leq \nu(f).
\end{aligned}$$

By letting n goes to infinity we obtain $\mu(f) \leq \nu(f)$.

This concludes the proof.

We mention that it follows from the proof of the previous theorem that it is not particularly important that the considered function f is fragmented. All we need to know is that f is lower bounded, universally measurable and satisfies the subbarycentric formula.

4. *The intersection of convex semi-extremal sets with extreme points*

Let X be a compact convex set and H be a nonempty resolvable extremal set. Then its characteristic function is convex. Thus we know from Corollary 2·6 that H satisfies $H \cap \operatorname{ext} X \neq \emptyset$. It is clear that every extremal set is semi-extremal, so it is natural to ask whether it is also true that nonempty resolvable semi-extremal subset H of X intersects extreme points of X. The answer is positive provided that H is moreover convex as Theorem 4·2 shows.

LEMMA 4·1. *Let $H \subset X$ be a nonempty resolvable set. Then the relative interior of H in $\overline{H}$ is nonempty.*

Proof. Assuming that the statement does not hold, the set $\overline{H}\setminus H$ is a resolvable dense set in $\overline{H}$. It is known (see [**9**, Propositon A.117.(e)]) that dense resolvable sets are residual. Thus H and $\overline{H}\setminus H$ are two disjoint residual subsets of $\overline{H}$, which is a Baire space. This gives a contradiction and finishes the proof.

THEOREM 4·2. *Let $H \subset X$ be a nonempty resolvable convex semi-extremal set. Then H contains an extreme point of X.*

Proof. We denote $F = X \setminus H$. We claim that the intersection of $\operatorname{ext}\overline{H}$ and H is nonempty. Otherwise it holds that $\operatorname{ext}\overline{H} \subset \overline{H}\setminus H \subset F$. From the Krein-Milman theorem we have

$$\overline{\operatorname{co}(\operatorname{ext}\overline{H}) \cap \overline{H}} = \overline{\operatorname{co}(\operatorname{ext}\overline{H})} = \overline{\operatorname{co}}(\operatorname{ext}\overline{H}) = \overline{H},$$

and so we obtain that

$$F = \operatorname{co}(F) \supset \operatorname{co}(\operatorname{ext}\overline{H})$$

is relatively dense in $\overline{H}$. On the other hand, Lemma 4·1 claims that the relative interior of H in $\overline{H}$ is nonempty. But this contradicts the density of its complement in $\overline{H}$. Thus $\operatorname{ext}\overline{H} \cap H \neq \emptyset$. Now we use Lemma 2·1 to conclude that $\operatorname{ext} X \cap H$ is nonempty.

Our method of proof of Theorem 4·2 only works if H is convex. On the other hand, Pryce proved in [**12**] that a nonempty closed semi-extremal subset H of X intersect extreme points of X, without the assumption on convexity of the set H. Thus it seems reasonable to hope that the answer to the following question is positive.

QUESTION 4·3. *Let $H \subset X$ be a nonempty resolvable semi-extremal set. Does H necessarily contain an extreme point of X?*

One might ask to what extent it is possible to generalize Theorem 4·2. The following example shows that there are faces of low Borel complexity which does not intersect the set of extreme points.

EXAMPLE 4·4. *There exists a metrizable compact convex set X and nonempty faces $F, G \subset X$ (i.e., convex extremal sets) such that F is of type F_σ, G is of type G_δ and*

$$F \cap \operatorname{ext} X = G \cap \operatorname{ext} X = \emptyset.$$

Proof. Let $X = \mathcal{M}^1([0,1])$. Further, let F be the face generated by the Lebesgue measure $\lambda \in X$ (i.e., F is the smallest face containing λ) and let G consist of all measures in X that are continuous on $[0,1]$ (we recall that a measure $\mu \in X$ is continuous if $\mu(\{x\}) = 0$ for each $x \in [0,1]$). By [**9**, Proposition 2.58] and [**9**, Proposition 2.94], F is an F_σ set, G is a G_δ set, both these sets are faces and their intersection with $\operatorname{ext} X = \{\varepsilon_x;\, x \in [0,1]\}$ is empty.

5. *Phelps theorem for abstract fragmented functions*

Let K be a locally compact space and $\mathcal{C}_0(K,\mathbb{F})$ stand for the space of all $\mathbb{F}$-valued continuous functions on K vanishing at infinity.

Let $\mathcal{H}$ be a subspace of $\mathcal{C}_0(K,\mathbb{F})$. Let $\phi\colon K \to B_{\mathcal{H}^*}$ be the evaluation mapping from K to the dual unit ball $B_{\mathcal{H}^*}$. By the *Choquet boundary* $\operatorname{Ch} K$ of $\mathcal{H}$ we mean the set of those points $x \in K$, for which $\phi(x)$ lies in $\operatorname{ext} B_{H^*}$. The classical result states that $\operatorname{Ch} K$ is a boundary for $\mathcal{H}$, which means that for any $h \in \mathcal{H}$ we have $\max_{x\in K} |f(x)| =$

$\max_{x\in \operatorname{Ch} K} |f(x)|$, see [**5**, Theorem 2.3.8]. The aim of this section is to generalize this result for fragmented "$\mathcal{H}$-affine" functions on K.

To explain this notion, assume for a while that K is even compact and $\mathcal{H}\subset\mathcal{C}(K,\mathbb{F})$ is a subspace. Let $f\colon K\to\mathbb{F}$ be a bounded universally measurable function. Then we may regard f as an element of $\mathcal{M}(K,\mathbb{F})^*$ via the formula $f(\mu)=\int_K f\,\mathrm{d}\mu$, $\mu\in\mathcal{M}(K,\mathbb{F})$. Let

$$\mathcal{H}^{\perp}=\{\mu\in\mathcal{M}(K,\mathbb{F});\ \mu(h)=0, h\in\mathcal{H}\}$$

be the annihilator of $\mathcal{H}$ in $\mathcal{M}(K,\mathbb{F})$. Then a bounded universally measurable function f belongs to $\mathcal{H}^{\perp\perp}$ provided $\mu(f)=0$ for each $\mu\in\mathcal{H}^{\perp}$. Such functions are called *completely $\mathcal{H}$-affine* in [**9**].

Assume now that K is only locally compact. Let $J=K\cup\{\alpha\}$ be the one-point compactification of K, where α is the point at infinity. If f is an $\mathbb{F}$-valued function on K, then let f_0 be its extension to J satisfying $f(\alpha)=0$. If $\mathcal{H}$ is a subspace of $\mathcal{C}_0(K,\mathbb{F})$, then we define the corresponding subspace of $\mathcal{C}(J,\mathbb{F})$ by

$$\mathcal{H}_0=\{g_0;\ g\in\mathcal{H}\}.$$

If we now consider the Choquet boundary $\operatorname{Ch} J$ of J with respect to the space $\mathcal{H}_0$, then by [**14**, Lemma 2.8] it holds that $\operatorname{Ch} K=\operatorname{Ch} J$.

Now we define

$$\mathcal{H}^{\perp\perp}=\{f \text{ bounded universally measurable on } K;\ f_0\in\mathcal{H}_0^{\perp\perp}\}.$$

For these functions we obtain the following maximum principle.

Theorem 5·1. *Let K be a locally compact space and $\mathcal{H}\subset\mathcal{C}_0(K,\mathbb{F})$ be a subspace. Let $f\colon K\to\mathbb{F}$ be a bounded fragmented function in $\mathcal{H}^{\perp\perp}$. Then*

$$\sup_{x\in\operatorname{Ch} K}|f(x)|=\sup_{x\in K}|f(x)|.$$

Before embarking on the proof of Theorem 5·1 we need a couple of lemmas allowing us to consider a function $f\in\mathcal{H}^{\perp\perp}$ as a function on $B_{\mathcal{H}^*}$. It turns out that we can use methods from [**14**].

The key ingredient of the proof is a characterization of fragmented functions mentioned in the introduction via resolvable sets. So let us mention again that a function $f\colon K\to\mathbb{F}$ on a compact space K is fragmented if and only if it is $\Sigma_1(\mathrm{Hs})$-measurable, i.e., the inverse image $f^{-1}(U)$ belongs to $\Sigma_1(\mathrm{Hs}(K))$ for any open set $U\subset\mathbb{F}$ (here $\Sigma_1(\mathrm{Hs}(K))$ denotes the family of all countable unions of resolvable sets in K). For the proof of this result see [**7**, Theorem 2.3] or [**9**, Theorem A.121].

Lemma 5·2. *Let $L_1,\dots,L_n$ be compact convex sets in a locally convex space and $L=\operatorname{co}(L_1\cup\cdots\cup L_n)$. Let $f\colon L\to\mathbb{F}$ be an affine function such that $f|_{L_i}$ is fragmented for each $i=1,\dots,n$. Then f is fragmented on L.*

Proof. Let

$$\Delta=\left\{\lambda\in[0,\infty)^n;\ \sum_{i=1}^n\lambda_i=1\right\}$$

and

$$H=L_1\times\cdots\times L_n\times\Delta.$$

Let further $g\colon H \to \mathbb{F}$ be defined as

$$g(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i f(x_i), \quad (x_1, \ldots, x_n, \lambda) \in H.$$

By the proof of [**9**, Theorem 5.10], g is $\Sigma_1(\mathrm{Hs})$-measurable on H. We consider a continuous surjection $\varphi\colon H \to L$ defined as

$$\varphi(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} \lambda_i x_i, \quad (x_1, \ldots, x_n, \lambda) \in H.$$

Since f is affine on L, we obtain $f \circ \varphi = g$. By [**6**, Theorem 4], f is $\Sigma_1(\mathrm{Hs})$-measurable on L, and thus fragmented on L.

LEMMA 5·3. *Let K be a compact space and $f\colon K \to \mathbb{F}$ be a bounded fragmented function. Then $\widehat{f}\colon \mathcal{M}(K, \mathbb{F}) \to \mathbb{F}$ defined as*

$$\widehat{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}(K, \mathbb{F}),$$

is fragmented on any ball $rB_{\mathcal{M}(K,\mathbb{F})}$, $r > 0$.

Proof. We provide a proof for the case of complex measures, the easier case of real measures would be done similarly.

Assume first that f is real. By [**8**, Lemma 3.3(a)], $\widehat{f}$ is fragmented on $\mathcal{M}^1(K)$. Let $L_1 = 2\mathcal{M}^1(K)$, $L_2 = -2\mathcal{M}^1(K)$, $L_3 = 2i\mathcal{M}^1(K)$ and $L_4 = -2i\mathcal{M}^1(K)$. Since L_i is affinely homeomorphic to $\mathcal{M}^1(K)$ and $\widehat{f}$ is linear, $\widehat{f}$ is fragmented on each L_i, $i = 1, \ldots, 4$. By the decomposition of a complex measure we obtain

$$B_{\mathcal{M}(K,\mathbb{F})} \subset L = \mathrm{co}\,(L_1 \cup L_2 \cup L_3 \cup L_4)\,.$$

By linearity it is enough to prove that $\widehat{f}$ is fragmented on L. But this follows from Lemma 5·2.

If $f\colon K \to \mathbb{C}$, we have $f = f_1 + if_2$, where f_1, f_2 are real bounded fragmented functions. Then the function

$$\widehat{f}(\mu) = \mu(f) = \mu(f_1) + i\mu(f_2) = \widehat{f_1}(\mu) + i\widehat{f_2}(\mu), \quad \mu \in B_{\mathcal{M}(K,\mathbb{F})},$$

is fragmented as well. (Indeed, since the functions $\mu \mapsto \widehat{f_1}$ and $\mu \mapsto i\widehat{f_2}(\mu)$ are fragmented, their sum is easily seen to be fragmented as well, see e.g. the proof of [**9**, Theorem 5.10].)

We write $S_{\mathbb{F}}$ for the set $\{\lambda \in \mathbb{F};\ |\lambda| = 1\}$.

LEMMA 5·4. *Let $\mathcal{H}$ be a subspace of $\mathcal{C}(K, \mathbb{F})$ for some compact space K. Then*

$$\mathrm{ext}\, B_{\mathcal{H}^*} \subset S_{\mathbb{F}} \cdot \phi(\mathrm{Ch}\, K).$$

Proof. See [**14**, Lemma 2.1].

Now we can prove Theorem 5·1.

Proof of Theorem 5·1 First we assume that K is compact. Let $\pi\colon B_{\mathcal{M}(K,\mathbb{F})} \to B_{\mathcal{H}^*}$ be the restriction mapping. We denote

$$\widehat{f}\colon B_{\mathcal{M}(K,\mathbb{F})} \to \mathbb{F},$$
$$\mu \mapsto \mu(f).$$

By Lemma 5·3, $\widehat{f}$ is a fragmented function on $B_{\mathcal{M}(K,\mathbb{F})}$. Since $f \in \mathcal{H}^{\perp\perp}$, there exists a unique function $a\colon B_{\mathcal{H}^*} \to \mathbb{F}$ satisfying $a \circ \pi = \widehat{f}$. By [**6**, Theorem 4], the function a is fragmented on $B_{\mathcal{H}^*}$. Since a is obviously affine, the function $|a|$ is convex and fragmented. By Corollary 2·6,

$$\begin{aligned} \sup_{s \in \operatorname{ext} B_{\mathcal{H}^*}} |a(s)| &= \sup_{s \in B_{\mathcal{H}^*}} |a(s)| = \sup_{\mu \in B_{\mathcal{M}(K,\mathbb{F})}} |a(\pi(\mu))| \\ &= \sup_{\mu \in B_{\mathcal{M}(K,\mathbb{F})}} |\mu(f)| \geq \sup_{x \in K} |f(x)| . \end{aligned}$$

By Lemma 5·4 we have

$$\operatorname{ext} B_{\mathcal{H}^*} \subset S_{\mathbb{F}} \cdot \phi(\operatorname{Ch} K),$$

and hence we obtain

$$\sup_{x \in K} |f(x)| \leq \sup_{s \in \operatorname{ext} B_{\mathcal{H}^*}} |a(s)| \leq \sup_{x \in \operatorname{Ch} K} |a(\phi(x))| = \sup_{x \in \operatorname{Ch} K} |f(x)| .$$

This concludes the proof in the case when K is compact.

If K is locally compact, then we consider its one-point compactification $J = K \cup \{\alpha\}$ and the subspace $\mathcal{H}_0 \subset \mathcal{C}(J, \mathbb{F})$ corresponding to $\mathcal{H}$. If f is a fragmented function in $\mathcal{H}^{\perp\perp}$, then it easily follows that its extension f_0 is a fragmented function in $\mathcal{H}_0^{\perp\perp}$. Thus we have

$$\sup_{x \in \operatorname{Ch} K} |f(x)| = \sup_{x \in \operatorname{Ch} J} |f(x)| = \sup_{x \in \operatorname{Ch} J} |f_0(x)| = \sup_{x \in J} |f_0(x)| = \sup_{x \in K} |f(x)|,$$

which finishes the proof.

Assume now that $\mathcal{H} \subset \mathcal{C}(K, \mathbb{R})$ is a subspace containing constants and separating points of a compact space K. (Such a space is called a *function space* in [**9**].) For each $x \in K$ let $\mathcal{M}_x(\mathcal{H})$ denote the set

$$\mathcal{M}_x(\mathcal{H}) = \{\mu \in \mathcal{M}^1(K);\ \mu(h) = h(x), h \in \mathcal{H}\}.$$

Then $\operatorname{Ch} K$ defined above coincides with the set $\{x \in K;\ \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$, see [**9**, Definition 3.4 and Proposition 4.26(d)].

Further, a bounded function $f\colon K \to \mathbb{R}$ is termed $\mathcal{H}$*-convex* if $f(x) \leq \mu(f)$, $x \in K$, $\mu \in \mathcal{M}_x(\mathcal{H})$.

The function f is called $\mathcal{H}$*-affine* if both f and $-f$ are $\mathcal{H}$-convex.

Let

$$\mathcal{A}^c(\mathcal{H}) = \{f \in \mathcal{C}(K, \mathbb{R});\ f \text{ is } \mathcal{H}\text{-affine}\}.$$

Then $\mathcal{H} \subset \mathcal{A}^c(\mathcal{H})$, $\mathcal{A}^c(\mathcal{H})$ is a function space and the Choquet boundary of $\mathcal{H}$ coincides with the Choquet boundary of $\mathcal{A}^c(\mathcal{H})$.

We mention that in the case when K is a compact convex set X and $H = \mathfrak{A}(X, \mathbb{R})$ the space $\mathcal{A}^c(\mathcal{H})$ coincides with $\mathcal{H}$ and the points of the Choquet boundary of X are exactly the extreme points of X.

The function space $\mathcal{H}$ is called *simplicial* if the set

$$\mathbf{S}(\mathcal{A}^c(\mathcal{H})) = \{s \in (\mathcal{A}^c(\mathcal{H}))^*;\ s(1) = \|s\| = 1\}$$

endowed with the weak* topology is a Choquet simplex, see [**9**, Theorem 6.54] or [**2**].

For a simplicial space $\mathcal{H}$ we have the following improvement of Theorem 5·1.

COROLLARY 5·5. *Let K be a compact space and $H \subset \mathcal{C}(K,\mathbb{R})$ be a simplicial subspace containing constants and separating points of K.*

Let $f\colon K \to \mathbb{R}$ be a bounded $\mathcal{H}$-affine fragmented function on K. Then

$$\sup_{x\in \operatorname{Ch} K} |f(x)| = \sup_{x\in K} |f(x)| .$$

Proof. The assertion follows from Theorem 5·1 once we verify that any $\mathcal{H}$-affine function is in $(\mathcal{A}^c(\mathcal{H}))^{\perp\perp}$. Since $\mathcal{H}$ is simplicial, this follows from [**9**, Corollary 6.12].

The theory of function spaces very often imitates results from the convexity theory, nevertheless, not always is the transfer straightforward. As an example of this phenomenon we offer a problem on fragmented $\mathcal{H}$-convex functions. Of course, by Corollary 2·6 we know its positive answer for the case $\mathcal{H} = \mathfrak{A}(X,\mathbb{R})$ on some compact convex set X.

QUESTION 5·6. *Let $\mathcal{H} \subset \mathcal{C}(K,\mathbb{R})$ be a function space and $f\colon K \to \mathbb{R}$ be a bounded fragmented $\mathcal{H}$-convex function. Assume that $f \leq 0$ on $\operatorname{Ch} K$. Does it follow that $f \leq 0$ on K?*

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