ON DIFFERENTIABILITY OF SADDLE AND BICONVEX FUNCTIONS AND OPERATORS

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Abstract. We strengthen and generalize results of J.M. Borwein (1986) and of A. Ioffe and R.E. Lucchetti (2005) on Fréchet and Gâteaux differentiability of saddle and biconvex functions (and operators). For example, we prove that in many cases (also in some cases which were not considered before) these functions (and operators) are Fréchet differentiable except for a Γ-null, σ-lower porous set. Moreover, we prove these results for more general "partially convex (up or down)" functions and operators defined on the product of n Banach spaces.

1. Introduction

There exist many results concerning differentiability of continuous convex functions on Banach spaces (see e.g. the monographs [22] and [1]), and a number of results on differentiability of convex operators from Banach spaces to ordered normed linear spaces (see e.g. [2], [3], [14], [15], [29]).

Some of these results can be (partly) extended to results on differentiability of saddle (that is, convex-concave) and biconvex functions and operators.

Results which say that, in some situations, saddle or biconvex functions (and operators) are Fréchet or Gâteaux differentiable at all points outside a small set can be found in [13], [4] and [11].

Besides saddle and biconvex functions, J.M. Borwein [4] considered also more general functions f on $X_1 \times \cdots \times X_n$ which are "partially convex (up or down)" (in the sense that, for each i, either all partial functions of the form $f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$ are convex or all are concave). However, the main results of [4] are proved (using properties of partially monotone operators) only for saddle and biconvex functions (and operators). Rougly speaking, [4, Corollary 2,(a)] states the following:

(Bo1) *If $X_1 \times X_2$ is an Asplund space and $f \colon X_1 \times X_2 \to \mathbb{R}$ is a continuous saddle or biconvex function, then f is generically (i.e., on a dense G_δ set) Fréchet differentiable.*

The following generalization of (Bo1) is contained in [4, Theorem 2.3].

(Bo2) *The assertion of (Bo1) remains valid also in the case when $f \colon X_1 \times X_2 \to Y$ is a continuous saddle or biconvex operator and Y is an ordered Banach space whose positive cone Y_+ is well-based (see Definition 2.5 below).*

Further, [11, Theorem 3.1] (together with [4, Proposition 2.1]) gives the following "supergeneric" (i.e., strictly stronger than generic) result:

(IL) *If $X_1 \times X_2$ is a separable Asplund space and f is a real continuous saddle function on $X_1 \times X_2$, then f is Fréchet differentiable except for a σ-upper porous set.*

Note also that [37, Corollary 8.1] implies a generalization of (Bo1) in which $f\colon X_1 \times \cdots \times X_n \to \mathbb{R}$ is (e.g.) partially DC.

We strengten and generalize both (Bo1) and (Bo2) in the following directions (see Theorems 4.3, 4.4 and 4.5 below).

(i) (Bo1) and (Bo2) remain valid also for partially convex (up or down) functions and operators, respectively, of n variables,
(ii) and moreover the exceptional sets are even σ-*lower* porous.
(iii) Furthermore, our results cover also some cases on Fréchet differentiability of continuous saddle or biconvex operators $f\colon X = X_1 \times X_2 \to Y$, in which (Bo2) cannot be applied, namely when Y is a countably Daniell ordered Banach space and the space $\mathcal{L}(X, Y)$ of bounded linear operators is separable.

Note that (ii) gives also an improvement of (IL). Moreover, for saddle functions and operators we obtain even stronger results on Fréchet differentiability except for a cone small set (see our Theorems 5.4 and 5.7, respectively).

Further, [4, Corollary 2.2(b) and Theorem 2.3] contain also results on generic Gâteaux differentiability of saddle and biconvex functions (and operators). We generalize and strenghten most of these results.

We prove also results on Γ-almost everywhere Fréchet differentiability of partially convex (up or down) functions and operators. They are consequences of a deep result of [18] (see Theorem 2.20 below) and of the fact (see Theorem 3.6) that if a continuous partially convex (up or down) operator is Gâteaux differentiable at a point x, then it is strictly Gâteaux differentiable at x. Theorem 3.6 implies also an analogous result on strict (Fréchet) differentiability which can be of some independent interest.

2. Preliminaries

2.1. **Basic notation.** By a Banach space we mean a real Banach space, which can be also trivial (but ordered Banach spaces are assumed to be non-trivial). The open ball with center a and radius r in a metric space X will be denoted by $B(x, r)$ (or $B_X(x, r)$). By a K-Lipschitz mapping we mean a Lipschitz mapping with a (not necessary minimal) Lipschitz constant K. If X, Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. If $X_1, \dots, X_n$ are Banach spaces, we consider on $X := X_1 \times \cdots \times X_n$ the usual maximum norm. As usual, we frequently canonically identify X_i with a subset of X and, if Z is a Banach space, the operator space $\mathcal{L}(X, Z)$ will be canonically isomorphically identified with the Cartesian product $\prod_{i=1}^{n} \mathcal{L}(X_i, Z)$.

The directional and the one-sided directional derivative of a mapping f (between normed spaces) at x in the direction v are defined respectively by

$$f'(x, v) := \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad f'_+(x, v) := \lim_{t\to 0+} \frac{f(x + tv) - f(x)}{t}.$$

2.2. **Ordered Banach spaces.** By an *ordered Banach space* we mean a real Banach space Z with a given closed convex cone Z_+ which is pointed, i.e., $Z_+ \cap (-Z_+) = \{0\}$. In this case, Z is partially ordered by the relation

$$z_1 \le z_2 \ \Leftrightarrow \ z_2 - z_1 \in Z_+ \,.$$

The dual Z^* of an ordered Banach space Z is an ordered Banach space with the *dual positive cone*

$$Z^*_+ = \{z^* \in Z^* : \ z^*(z) \ge 0 \text{ for each } z \in Z_+\}\,.$$

Throughout the paper, all ordered Banach spaces are always assumed to be nontrivial.

Definition 2.1. Let Z be an ordered Banach space. We say that:

(a) Z is *normal* if there exists a constant $C > 0$ such that $\|x\| \le C\|y\|$ whenever $x, y \in Z$ and $0 \le x \le y$;

(b) Z is *countably Daniell* if every decreasing sequence in Z_+ converges.

Notice that every Banach lattice is clearly normal. Moreover, it is known (see [20, Proposition 1.a.8]) that: *a Banach lattice is countably Daniell if and only if it is order continuous if and only if it is σ-complete and σ-order continuous.*

In what follows, we collect some useful properties of ordered Banach spaces which are normal or countably Daniell. Most of these facts have been already used in our previous paper [29], where one can find references for the proofs.

Fact 2.2 (normal ordered Banach spaces)**.** *For every ordered Banach space Z which is normal, there exist real constants $M_Z, K_Z, P_Z > 0$ with the following properties.*

(a) For each $z^* \in Z^*$ there exist $z^*_1, z^*_2 \in Z^*_+$ such that $z^* = z^*_1 - z^*_2$ and $\|z^*_i\| \le M_Y\|z^*\|$ $(i = 1, 2)$ (see [29, Fact 1.2])*.*

(b) For each $z \in Z$ there exists $z^* \in Z^*_+$ such that $|z^*(z)| = \|z\|$ and $\|z^*\| \le K_Z$ (see [29, Lemma 1.3])*.*

(c) If $z, u, v \in Z$ are such that $u \le z \le v$ then $\|z\| \le P_Z(\|u\| + \|v\|)$ (see [29, Observation 1.4 and Fact 1.2])*.*

Fact 2.3 (countably Daniell ordered Banach spaces)**.** *Let Z be an ordered Banach space.*

(a) *If Z is countably Daniell then Z is normal* (see [29, Fact 1.5]).
(b) *For reflexive Z, also the vice-versa in (a) holds true* (see [29, Fact 1.5]).

Fact 2.4. *Every separable ordered Banach space admits a strictly positive functional, that is, a functional $z^* \in Z^*_+$ such that $z^*(z) > 0$ for each $z \in Z_+ \setminus \{0\}$* (see [2, Propositions 2.7 and 2.8]).

Definition 2.5. We say that a convex subset B of an ordered Banach space Z is a base for Z_+ if, for each $z \in Z_+ \setminus \{0\}$, there exists a unique $\lambda > 0$ such that $\lambda z \in B$. Following [12, p. 120], we say that Z_+ is *well-based* if it has a bounded base B such that $0 \notin \overline{B}$.

Fact 2.6 (well-based positive cones)**.** *Let Z be an ordered Banach space such that Z_+ is well-based. Then:*
(a) *Z is countably Daniell, and hence normal* (see [2, Proposition 3.6(c)])*;*
(b) *there exists $z^* \in Z^*_+$ such that $z^*(z) \geq \|z\|$ for each $z \in Z_+$* (see [12, 3.8.12]). (Such functional z^* is clearly strictly positive; it is sometimes called a strongly positive functional.)

2.3. Convex, saddle, partially convex operators, and generalized monotone mappings.

Definition 2.7. Let X be a normed space, $A \subset X$ an open convex set and Z an ordered Banach space. A mapping $f \colon A \to Z$ is called a *convex operator* if

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

whenever $x_1, x_2 \in A$, and λ_1, λ_2 are positive reals with $\lambda_1 + \lambda_2 = 1$.
Moreover, we say that f is a *concave operator* if $(-f)$ is a convex operator.

If $z^* \in Z^*_+$ and f is a Z-valued convex operator on A, then $z^* \circ f$ is clearly a convex function on A. Moreover, in the same way as for real-valued convex functions it is easy to see that, for each $x \in A$ and each $v \in X$,

$$\text{the function } \ t \mapsto \frac{f(x+tv) - f(x)}{t} \ \text{ is nondecreasing} \tag{1}$$

on the set $\{t \in \mathbb{R} \setminus \{0\} : x + tv \in A\}$. This easily implies the following fact (cf. [2, Proposition 3.7]).

Fact 2.8. *Let X,Z, A, f be as in Definition 2.7. Let $a \in A$ and $v \in X$.*
(a) *If $f'_+(a,v)$ exists then $f'_+(a,v) \leq (f(a+tv) - f(a))/t$ for each $t > 0$ with $a + tv \in A$.*
(b) *If Z is countably Daniell then $f'_+(a,v)$ always exists.*

Definition 2.9. Let X, Y be normed spaces, $A \subset X$ and $B \subset Y$ open convex sets and Z an ordered Banach space. A mapping $f \colon A \times B \to Z$ is called a *convex-concave* (or *saddle*) *operator* if

- $f(\cdot, b)$ is a convex operator (on A) for each $b \in B$, and
- $f(a, \cdot)$ is a concave operator (on B) for each $a \in A$.

In an analogous way as in Definition 2.9, one can define a *convex-convex* (or *biconvex*) *operator*. The next notion, introduced by J.M. Borwein [4], is a natural common generalization of both saddle and biconvex operators.

Definition 2.10. Let $X_1, \dots, X_n$ be normed spaces, $A_i \subset X_i$ $(i = 1, \dots, n)$ open convex sets, and Z an ordered Banach space. A mapping

$$f \colon A_1 \times \cdots \times A_n \to Z$$

is called a *partially convex (up or down) operator* if for each $i \in \{1, \dots, n\}$ the corresponding partial mapping

$$f(a_1, \dots, a_{i-1}, \cdot, a_{i+1}, \dots, a_n) \colon A_i \to Z$$

either is a convex operator whenever $a_j \in A_j$ $(j \in \{1, \dots, n\} \setminus \{i\})$ are fixed, or is a concave operator whenever $a_j \in A_j$ $(j \in \{1, \dots, n\} \setminus \{i\})$ are fixed.

Notice that Fact 2.8 immediately gives the following

Observation 2.11. *Let X_i, A_i $(i = 1, \dots, n)$, Z be as in Definition 2.10. Let Z be countably Daniell and $f \colon \prod_{i=1}^n A_i \to Z$ a partially convex (up or down) operator. Then for each $a \in \prod_{i=1}^n A_i$ and for each $k \in \{1, \dots, n\}$, f admits all one-sided "partial directional derivatives"*

$$f'_+(a, v) \quad \text{where } v = (0, \dots, 0, v_k, 0 \dots, 0),\ v_k \in X_k.$$

Definition 2.12 (Kirov [14])**.** Let X be a normed space, Z an ordered Banach space and $T \colon X \rightrightarrows \mathcal{L}(X, Z)$ a multivalued mapping. We will say that T is a *generalized monotone mapping* if

$$(A_1 - A_2)(x_1 - x_2) \geq 0 \quad \text{whenever} \quad x_i \in X,\ A_i \in T(x_i),\ i = 1, 2.$$

We set $D(T) := \{x \in X : T(x) \neq \emptyset\}$.

The following definition of the subdifferential of a continuous convex operator is a direct generalization of the classical notion of subdifferential in Convex Analysis.

Definition 2.13 ([27])**.** Let X be a normed space, $A \subset X$ an open convex set, Z an ordered Banach space and $f \colon A \to Z$ a continuous convex operator. For every $x \in A$ we define the *subdifferential* of f at x as the (possibly empty) set

$$\partial f(x) = \{M \in \mathcal{L}(X, Z) :\ f(y) \geq f(x) + M(y - x) \text{ for each } y \in A\}.$$

From formal reasons, we also define $\partial f(x) = \emptyset$ whenever $x \in X \setminus A$.

Fact 2.14 ([14, p. 264]). *Let X, A, Z, f be as in Definition 2.13. Then the subdifferential mapping $\partial f\colon X \rightrightarrows \mathcal{L}(X,Z)$ is a generalized monotone mapping.* (This is well known for $Z = \mathbb{R}$; the general case is proved in the same way.)

The following lemma is essentially well known, at least in some special cases (see [14, pp. 266–267]). For the sake of completeness, we give a simple proof.

Lemma 2.15. *Let X, A, Z, f be as in Definition 2.13. Let Z be countably Daniell and let f be Gâteaux differentiable at a point $a \in A$ with derivative $f'_G(a)$ ($\in \mathcal{L}(X,Z)$). Then $\partial f(a) = \{f'_G(a)\}$.*

Proof. By Fact 2.8, if $x \in A$ then $f(x) - f(a) = f(a + 1(x-a)) - f(a) \geq f'_+(a, x-a) = f'_G(a)(x-a)$. It follows that $f'_G(a) \in \partial f(a)$. Now assume that $\partial f(a)$ contains also some $L \neq f'_G(a)$. Fix $u \in X$ such that $Lu \neq f'_G(a)u$. By Facts 2.2(b) and 2.3, there exists $z^* \in Z^*_+$ such that $(z^* \circ L)u \neq (z^* \circ f'_G(a))u$. The continuous convex real-valued function $z^* \circ f$ is clearly Gâteaux differentiable at a, and hence its subdifferential at a reduces to the singleton $\{z^* \circ f'_G(a)\}$ (see e.g. [22, Proposition 1.8]). On the other hand, the fact that $L \in \partial f(a)$ easily implies that $z^* \circ L \in \partial(z^* \circ f)(a)$, which is a contradiction. □

Corollary 2.16. *Let X, A, Z, f be as in Definition 2.13, and $a \in A$. Then $\partial f(a)$ is nonempty in any of the following two cases:*
(a) $Z = \mathbb{R}$;
(b) Z is countably Daniell and f is Gâteaux differentiable at a.

Proof. The real case is well known (see e.g. [22, p. 6]). The second case follows immediately from Lemma 2.15. □

2.4. **Some known results we will need.**

We will recall several known results on differentiability of Lipschitz functions (and singlevaluedness and continuity of monotone opeartors) which we will apply. In these theorems, and in our new results on differentiability of partially convex operators, several systems of "small sets" are used. Since we will not work in our proofs with definitions of these systems (with the only exception of σ-lower porous sets), we will recall only some of them.

Definition 2.17. Let $M \subset X$, $x \in X$ and $R > 0$. Then we define $\gamma(x,R,M)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$ (where $B(z,0) := \emptyset$).

We will say that M is *upper porous at x* (resp. *lower porous at x*), if

$$\limsup_{R\to 0+} \frac{\gamma(x,R,M)}{R} > 0 \quad (\text{resp.} \liminf_{R\to 0+} \frac{\gamma(x,R,M)}{R} > 0). \tag{2}$$

We say that M is *upper porous* (resp. *lower porous*) if M is upper porous (resp. lower porous) at each point $y \in M$. We say that M is *σ-upper porous (resp. σ-lower porous)* if it is a countable union of upper porous (resp. lower porous) sets.

It is easy to see that each σ-lower porous set is σ-upper porous and each σ-upper porous set is a first category (=meager) set.

In results on Fréchet differentiability of saddle operators, we will use also the following notion of an *angle small* set (introduced in [24]). Obviously, each angle small set is σ-lower porous.

Definition 2.18. A set M in a Banach space X is called:

(a) α-angle porous (where $\alpha > 0$) if for every $x \in M$ and $\varepsilon > 0$ there exist $z \in X$ and $f \in X^*$ such that $\|z - x\| < \varepsilon$ and
$$M \cap \{w \in X : \ f(w - z) > \alpha \ \|f\| \|w - z\|\} = \emptyset\,;$$

(b) *angle-small* if for each $\alpha > 0$ it can be expressed as a countable union of α-angle porous sets.

We will use also the following notion of σ-directional porosity which is more restrictive than that of σ-upper porosity (but is, in infinite-dimensional spaces, incomparable with σ-lower porosity).

Definition 2.19. Let X be a Banach space. We say that $A \subset X$ is directionally porous at a point $x \in X$ if there exists $c > 0$, $u \in X$ with $\|u\| = 1$ and a sequence $\lambda_n \to 0$ of positive real numbers such that $B(x + \lambda_n u, c\lambda_n) \cap A = \emptyset$. The notions of directionally porous sets and of *σ-directionally porous* sets are defined in the standard way (cf. Definition 2.17).

In several results, we use the notion of a Γ-*null set*. This interesting and important notion which in a sophisticated way "combines category and measure" was defined in [18, Definition 2.1]. We will not use the definition of Γ-null sets, but only the following deep result [18, Theorem 3.10].

Theorem 2.20 ([18])**.** *Suppose that Y is a Banach space, G an open subset of a separable Banach space X, S a norm separable subspace of $\mathcal{L}(X, Y)$, and $f \colon G \to Y$ a Lipschitz mapping. Then f is Fréchet differentiable at Γ-almost every point $x \in G$ at which it is both regular (see below) and Gâteaux differentiable with $f'_G(x) \in S$.*

Here "*f is regular at x*" means (see [18, Definition 3.1]) that, for every $v \in X$ for which $f'(x, v)$ exists,

$$\lim_{t \to 0} \frac{f(x + tu + tv) - f(x + tu)}{t} = f'(x, v) \quad \text{uniformly for } \|u\| \le 1. \tag{3}$$

Obviously,

(4) if f is strictly Gâteaux differentiable at x, then f is regular at x

(for the definition of strict Gâteaux differentiability see Definition 2.32).

Note that Γ-null sets need not be of the first category and vice versa (and that Γ-null sets in $\mathbb{R}^n$ coincide with Lebesgue null sets).

Note (see, e.g., [36, p. 515, 3.1]) that if X is a separable Banach space, then

(5) *each σ-directionally porous set in X is Aronszajn null and also Γ-null.*

For definitions of the following types of small sets (which will be used in the sequel) we refer the reader to [36].

- The *cone-small sets* ([36, Definition 4.1]) are a natural generalization of angle-small sets (considered in separable spaces) to non-separable Banach spaces. In separable spaces, these two notions coincide.
- The notion of a *Lipschitz hypersurface* and a more restrictive notion of a *DC (or d.c.) hypersurface* ([36, Definition 4.3]) were used in a number of results in separable Banach spaces. Roughly speaking, a Lipschitz hypersurface M in X is a "graph of a Lipschitz function f defined on a closed hyperplane of X" and, if f is a difference of two convex Lipschitz functions, M is called a *DC* hypersurface. A set which can be covered by countably many Lipschitz (resp. DC) hypersurfaces is called a *sparse set* (resp. *DC sparse set*). It is easy to see that every sparse set is σ-directionally porous.
- A natural generalization of sparse sets (considered in a separable X) to non-separable Banach spaces are *σ-cone-supported sets* ([36, Definition 4.4]). In separable spaces, sparse sets and σ-cone-supported sets coincide.

Note that both cone-small sets and σ-cone-supported sets are σ-lower porous and thus they are first category sets.

In our proofs concerning Gâteaux differentiability we will use the following three theorems.

The following theorem is an easy consequence of [25, Theorem 5].

Theorem 2.21 ([25])**.** *If f is a locally Lipschitz mapping from an open subset G of a separable Banach space X to a Banach space Y, then the following implication holds at each point $x \in G$ except for a σ-directionally porous set:*

if the one-sided directional derivative $f'_+(x,u)$ exists for all vectors u from a set $U_x \subset X$ whose linear span is dense in X, then f is Gâteaux differentiable at x.

We will use also the following results on single-valuedness of monotone operators.

Theorem 2.22 ([32], [28])**.** *Let X be a separable Banach space and let $T\colon X \rightrightarrows X^*$ be a (multivalued) monotone operator with an arbitrary domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. Then there exists a sparse set $A \subset D(T)$ such that T is single-valued at each point of $D(T) \setminus A$.*

If $X = \mathbb{R}^2$, then A can be chosen to be a DC sparse set.

Proof. The first part is contained in [32]. The second part follows from [28] via the simple fact that, in $\mathbb{R}^2$, every "CFC-fragment" of dimension 1 is contained in a DC hypersurface. (To see this, it suffices to observe that if $\emptyset \neq M \subset \mathbb{R}$ then every Lipschitz real function of finite convexity on M admits a Lipschitz extension of finite convexity, and hence DC, to the whole $\mathbb{R}$.) $\square$

Theorem 2.23 ([10], [9])**.** *Let X be either a Gâteaux smooth Banach space or a subspace of an Asplund generated (i.e., a GSG) space. Let $T\colon X \rightrightarrows X^*$ be a locally bounded (multivalued) monotone operator with an arbitrary domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. Then there exists a σ-cone supported set $A \subset D(T)$ such that T is single-valued at each point of $D(T) \setminus A$.*

Remark 2.24.
- (i) Let us recall (see [8]) that an *Asplund generated space* (or a GSG space [10]) is a Banach space which contains a dense continuous linear image of an Asplund space. Note that all Asplund spaces and all weakly compactly generated (WCG) spaces are Asplund generated.
- (ii) Note that σ-cone supported sets are called "σ-cone porous" in [9].
- (iii) The cases when X is Asplund or X^* is strictly convex are contained already in [33].

The following theorem is a special case of [29, Theorem 3.3].

Theorem 2.25. *Let X be a Banach space, let Y be a normal ordered Banach space, and let $\mathcal{L}(X,Y)$ be separable. Suppose that $T\colon X \rightrightarrows \mathcal{L}(X,Y)$ is a (multivalued) generalized monotone mapping with an arbitrary domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. Then there exists an angle small set $M \subset D(T)$ such that T is single-valued and upper-semicontinuous at every $x \in D(T) \setminus M$.*

2.5. **Rich families and separable reduction.** Some our results in non-separable Banach spaces will be proved by the method of separable reduction. Namely, we will first prove the result in separable spaces and from it we will obtain the non-separable result using some known deep results which say that some notions are "separably determined in the sense of rich families". The following notion of a "rich family" was defined and used in [5] (see also [19, p. 37] and [6]).

Definition 2.26. Let X be a normed linear space. A family $\mathcal{F}$ of closed separable subspaces of X is called a *rich family* if:

- (R1) If $Y_i \in \mathcal{F}$ $(i \in \mathbb{N})$ and $Y_1 \subset Y_2 \subset \dots$, then $\overline{\bigcup\{Y_n : n \in \mathbb{N}\}} \in \mathcal{F}$.
- (R2) For each closed separable subspace Y_0 of X there exists $Y \in \mathcal{F}$ such that $Y_0 \subset Y$.

A basic (easy) fact (see [5, Proposition 1.1] or [19, Proposition 3.6.2]) concerning rich families is the following.

Lemma 2.27. *Let X be a normed linear space and let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be rich families of closed separable subspaces of X. Then $\mathcal{F} := \bigcap\{\mathcal{F}_n : n \in \mathbb{N}\}$ is also a rich family of closed separable subspaces of X.*

We will need also the following simple fact which is a reformulation of [37, Lemma 4.4].

Lemma 2.28. *Let $X_1, \dots, X_n$ be normed linear spaces and $X := X_1 \times \cdots \times X_n$. Let $\mathcal{F}_k$ be a rich family of closed separable subspaces of X_k, $1 \le k \le n$. Then*

$$\mathcal{F} := \{Y_1 \times \cdots \times Y_n : \ Y_k \in \mathcal{F}_k, \ 1 \le k \le n\}$$

is a rich family in X.

Much more difficult is the following result which says that Fréchet differentiability at a point is "separably determined in the sense of rich families".

Theorem 2.29 ([19, Theorem 3.6.10])**.** *Let X, Y be Banach spaces, $A \subset X$ an open set and $f\colon A \to Y$ a mapping. Then there exists a rich family $\mathcal{F}$ of closed separable subspaces of X such that for every $Y \in \mathcal{F}$, f is Fréchet differentiable (with respect to X) at every $x \in Y \cap A$, at which its restriction to $Y \cap A$ is Fréchet differentiable (with respect to Y).*

(In fact, [19, Theorem 3.6.10] is formulated for $A = X$, but if we apply this formally weaker theorem to any extension $\widetilde{f}$ of f to X, we obtain the assertion of Theorem 2.29.)

Even more deeper is the result on "separable determination of σ-lower porosity and cone smallness" which was first proved in [7] "in the sense of suitable models" and then transferred to the following result in [6].

Theorem 2.30 ([6, Corollary 5])**.** *Let X be an Asplund space and $A \subset X$ a Souslin set. Then there exists a rich family $\mathcal{F}$ of closed separable subspaces of X such that for every $Y \in \mathcal{F}$ we have*

$$\begin{aligned} A \text{ is } \sigma\text{-lower porous in } X \quad &\Longleftrightarrow \quad A \cap Y \text{ is } \sigma\text{-lower porous in } Y, \\ A \text{ is cone small in } X \quad &\Longleftrightarrow \quad A \cap Y \text{ is cone small in } Y. \end{aligned}$$

Recall that every Borel set in X is Souslin.

We will use also the following similar result on "separable determination of Γ-nullness".

Theorem 2.31 ([19, Corollary 5.6.2])**.** *Let X be a Banach space and $A \subset X$ a Borel set. Then A is Γ-null in X if and only if there is a rich family $\mathcal{F}$ of closed separable subspaces of X such that for every $Y \in \mathcal{F}$, the set $A \cap Y$ is Γ-null in Y.*

2.6. **Directional strict differentiability.** We will use some facts concerning strict differentiability of a mapping with respect to a subspace. They are (at least essentially) well-known (see, e.g. [21, p. 143, Exercise 5]) but we were not able to find a suitable reference; so we present short proofs in this subsection.

Definition 2.32. (a) Let X, Y be Banach spaces, $V \subset X$ a closed linear space, $G \subset X$ an open set, $a \in G$ and $f : G \to Y$ a mapping. We say that f is *strictly differentiable at a with respect to V* if there exists $L \in \mathcal{L}(V, Y)$ such that

$$\lim_{\substack{(x,y)\to(a,a)\\ 0\neq y-x\in V}} \frac{f(y)-f(x)-L(y-x)}{\|y-x\|} = 0. \tag{6}$$

In such a case, we shall write $L =: f'_{\mathrm{str},V}(a)$.

(b) If this holds for $V = X$ (for $V = \operatorname{span}\{v\}$ where $v \in X$), we say that f is *strictly differentiable* at a (*strictly differentiable at a in the direction v*, respectively). We also say that f is *strictly Gâteaux differentiable at a* if it is both Gâteaux differentiable at a and strictly differentiable at a in all directions $v \in X$.

Remark 2.33. (a) It is easy to show that (6) holds if and only if

$$\frac{f(z_n+t_nu_n)-f(z_n)-t_nL(u_n)}{t_n} \to 0 \quad \text{whenever} \tag{7}$$
$$\{t_n\} \subset (0,\infty), t_n \to 0, \{z_n\} \subset X, z_n \to a, \text{ and } \{u_n\} \subset V \text{ is bounded.}$$

(b) It is easy to see that f is strictly differentiable at x in a direction v if and only if

$$\lim_{z\to x,\, t\to 0+} \frac{f(z+tv)-f(z)}{t} = f'(x,v).$$

The following easy fact is (at least essentially) well known, see [21, p. 143, Exercise 5]).

Proposition 2.34. *Let X, Y be Banach spaces, $V \subset X$ a closed linear space, $G \subset X$ an open set, $a \in G$ and $f : G \to Y$ a mapping. Suppose that V is the topological direct sum of closed subspaces $V_1, \dots, V_n$ and f is strictly differentiable at a with respect to V_i, $i = 1, \dots, n$. Then f is strictly differentiable at a with respect to V.*

Proof. Proceeding by induction, we see that it is sufficient to prove the assertion for $n = 2$.

So let $L_1 := f'_{str,V_1}(a)$, $L_2 := f'_{str,V_2}(a)$ and let $L \in \mathcal{L}(V, Y)$ be the unique common extension of L_1 and L_2. To prove (7), suppose that $\{t_n\} \subset (0, \infty)$, $t_n \to 0$, $\{z_n\} \subset X$, $z_n \to a$ and $\{u_n\} \subset V$ is bounded. Write $u_n = v_n + w_n$, where $v_n \in V_1$, $w_n \in V_2$; then $\{v_n\}$ and $\{w_n\}$ are bounded. Now, observing that $z_n^* := z_n + t_n v_n \to a$, using the definition of L_1, L_2 and two times (7), we obtain

$$\begin{aligned}&\frac{f(z_n+t_nu_n)-f(z_n)-t_nL(u_n)}{t_n}\\ &=\frac{f(z_n^*+t_nw_n)-f(z_n^*)-t_nL_2(w_n)}{t_n}+\frac{f(z_n+t_nv_n)-f(z_n)-t_nL_1(v_n)}{t_n}\to 0.\end{aligned}$$

□

Aplying the above proposition in the special case when $V_i = \operatorname{span}\{v_i\}$, we easily obtain

Corollary 2.35. *Let X, Y, G, a, f be as in Proposition 2.34. Let $v_1, \dots, v_n \in X$ be linearly independent vectors and let f be strictly differentiable at a in the direction v_i for every $i = 1, \dots, n$. Then f is strictly differentiable at a in the direction $v := v_1 + \cdots + v_n$ with $f'(a, v) = \sum_{i=1}^n f'(a, v_i)$.*

Consequently, we obtain the following (surely well-known) fact:

Corollary 2.36. *Let X, Y, G, a, f be as in Proposition 2.34. Suppose that f is locally Lipschitz on G and strictly differentiable at a in each direction $v \in X$. Then f is strictly Gâteaux differentiable at a.*

Proof. Corollary 2.35 implies that the mapping $g := v \mapsto f'(a, v)$ is linear on X. Since f is locally Lipschitz, g is continuous and so f is Gâteaux differentiable at a. □

3. Local Lipschitz continuity and differentiability properties at a point

We shall need the following more or less standard fact.

Fact 3.1. *Let X be a normed space and g a real-valued convex function on a ball $B(a, 2\delta) \subset X$. If $C > 0$ is such that $|g| \le C$ on $B(a, 2\delta)$, then g is Lipschitz on $B(a, \delta)$ with Lipschitz constant $2C/\delta$.* (See e.g. [22, Proposition 1.6] and its proof.)

The following corollary is a quantitative version (for ordered Banach spaces) of J.M. Borwein's [2, Corollary 2.4].

Corollary 3.2. *Let $B(a, 2\delta)$ be an open ball in a normed space X. Let Z be an ordered Banach space which is normal, $c > 0$ a constant. If $f\colon B(a, 2\delta) \to Z$ is a convex operator such that $\|f\| \le c$, then f is Lipschitz on the ball $B(a, \delta)$ with Lipschitz constant $2K_Z c/\delta$, where K_Z is a fixed constant from Fact 2.2(b).*

Proof. Fix arbitrary $x, x' \in B_X(a, \delta)$. By the choice of K_Z, there exists $z^* \in Z_+^*$ such that $\|z^*\| \leq K_Z$ and $\|f(x) - f(x')\| = \left|z^*(f(x) - f(x'))\right|$. The real-valued function $g := z^* \circ f$ is clearly convex and $|g| \leq K_Z c$ on $B_X(a, 2\delta)$. By Fact 3.1, g is L-Lipschitz with $L := 2K_Z c/\delta$. Consequently, $\|f(x) - f(x')\| = |g(x) - g(x')| \leq L\|x - x'\|$, and we are done. □

Theorem 3.3. *Let $X_1, \ldots, X_n$ be normed linear spaces, $A_i \subset X_i$ $(i = 1, \ldots, n)$ open convex sets, Z an ordered Banach space which is normal and*

$$f : A_1 \times \cdots \times A_n \to Z$$

a partially convex (up or down) operator. The following assertions are equivalent:

(i) f is locally (norm) bounded;
(ii) f is continuous;
(iii) f is locally Lipschitz.

Proof. Since the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious, it remains to prove that $(i) \Rightarrow (iii)$. Let $(a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$. There exist $\delta > 0$ and $c > 0$ such that $\|f\| \leq c$ on $\prod_{i=1}^n B_{X_i}(a_i, 2\delta) \subset \prod_{i=1}^n A_i$. Now, Corollary 3.2 easily implies that there exists a constant $L = L(c, \delta, Z) > 0$ such that for each $1 \leq i \leq n$ all the partial mappings

$$f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n), \qquad \text{with } x_j \in B_{X_j}(a_j, 2\delta),\ 1 \leq j \leq n,\ j \neq i,$$

are L-Lipschitz on $B_{X_i}(a_i, \delta)$. It is an easy exercise to show that then f is Lipschitz on $\prod_{i=1}^n B_{X_i}(a_i, \delta)$ (with Lipschitz constant nL). □

Remark 3.4. Note that if all X_i in Theorem 3.3 are finite-dimensional then (i) (and hence also (ii) and (iii)) holds. The proof can be easily deduced from [2, Corollary 2.4] or [27, Proposition 9] in a standard way.

As we have already observed in Observation 2.11, every partially convex operator with values in a countably Daniell ordered Banach space admits all "one-sided partial directional derivatives" at each point. It is however known that one-sided directional derivative may not exist for some "non-partial" directions even if the operator is continuous; indeed, [17, Example 9.1] provides a (continuous) biconvex real-valued function on $\mathbb{R}^2$ for which the one-sided directional derivative $f'_+((0,0),(1,1))$ does not exist.

Lemma 3.5. *Let X, Y be normed spaces, $G \subset X$ and $H \subset Y$ open convex sets, Z an ordered Banach space which is normal, $f : G \times H \to Z$ a mapping, $(a, b) \in G \times H$. Suppose that:*

(a) f is Lipschitz on $G \times H$;
(b) for each $y \in H$, $f(\cdot, y)$ is a convex operator on G;

(c) $f(\cdot, b)$ is Fréchet differentiable at a (with derivative $D_1 f(a,b) \in \mathcal{L}(X,Z)$).

Then f is strictly differentiable at (a,b) with respect to X ($\equiv X \times \{0\}$) in the sense of Definition 2.32.

Proof. We can (and do) assume that $D_1 f(a,b) = 0$ (by considering the operator $\widetilde{f}(x,y) = f(x,y) - D_1 f(a,b)x$). Let us show that $f'_{str,X}(a,b) = 0$. We will use Remark 2.33(a). So fix sequences $\{t_n\} \subset (0,+\infty)$, $\{(x_n,y_n)\} \subset G \times H$ and a bounded sequence $\{u_n\} \subset X$, such that $t_n \to 0$, $x_n \to a$ and $y_n \to b$. Our aim is to show that

$$\Delta_n := \frac{f(x_n + t_n u_n, y_n) - f(x_n, y_n)}{t_n} \to 0.$$

Define $s_n := t_n + \sqrt{\|x_n - a\|} + \sqrt{\|y_n - b\|}$. For each sufficiently large n, Δ_n is well-defined and we can use convexity to obtain

$$\begin{aligned}\Delta_n \le \frac{f(x_n + s_n u_n, y_n) - f(x_n, y_n)}{s_n} &= \frac{f(a + s_n u_n, b) - f(a,b)}{s_n} \\ &+ \frac{f(x_n + s_n u_n, y_n) - f(a + s_n u_n, b)}{s_n} + \frac{f(a,b) - f(x_n,y_n)}{s_n} \\ &=: a_n + b_n + c_n\,,\end{aligned}$$

and also

$$\begin{aligned}\Delta_n \ge \frac{f(x_n, y_n) - f(x_n - s_n u_n, y_n)}{s_n} &= \frac{f(a,b) - f(a - s_n u_n, b)}{s_n} \\ &+ \frac{f(a - s_n u_n, b) - f(x_n - s_n u_n, y_n)}{s_n} + \frac{f(x_n,y_n) - f(a,b)}{s_n} \\ &=: \widetilde{a}_n + \widetilde{b}_n + \widetilde{c}_n\,.\end{aligned}$$

By our assumption (c), $\|a_n\| \to 0$ and $\|\widetilde{a}_n\| \to 0$. Moreover, (a) and definition of s_n easily imply that also the sequences $\{b_n\}, \{c_n\}, \{\widetilde{b}_n\}, \{\widetilde{c}_n\}$ tend to zero. Then $\|\Delta_n\| \to 0$ by Fact 2.2(c). □

Theorem 3.6. *Let X_i, A_i ($i = 1, \dots, n$) and Z be as in Theorem 3.3, and $a = (a_1, \dots, a_n) \in A := \prod_{i=1}^n A_i$. Let $f\colon A \to Z$ be a continuous partially convex (up or down) operator. For each $i \in \{1, \dots, n\}$ set*

$$g_i := f(a_1, \dots, a_{i-1}, \cdot, a_{i+1}, \dots, a_n).$$

(a) If g_i is Fréchet differentiable at a_i, $i \in \{1, \dots, n\}$, then f is strictly differentiable at a.

(b) If g_i is Gâteaux differentiable at a_i, $i \in \{1, \dots, n\}$, then f is strictly Gâteaux differentiable at a.

Proof. (a) The case $n = 1$ holds by Lemma 3.5 for $Y = H = \{0\}$. Let $n > 1$. By Theorem 3.3 we can (and do) assume that f is Lipschitz on A. For every $1 \leq j \leq n$ we set

$$G := A_j\,, \quad \text{and} \quad H := \prod_{\substack{i=1\\ i\neq j}}^{n} A_i\,,$$

and identify in the canonical way A with $G \times H$. Now, considering f as a mapping on $G \times H$ and applying Lemma 3.5 (to f or $-f$), we conclude that f is strictly differentiable at a with respect to X_j, where X_j is considered canonically immersed in X. Since, after such immersion, X becomes the topological direct sum of X_j, $1 \leq j \leq n$, we can apply Proposition 2.34 to conclude that f is strictly differentiable at a.

(b) By Theorem 3.3 and Corollary 2.36 it is sufficient to prove that f is strictly differentiable at a in each direction $v = (v_1, \ldots, v_n) \in X$. By Corollary 2.35, it suffices to show that f is strictly differentiable at a in the direction $\hat{v}_i := (0, \ldots, 0, v_i, 0, \ldots, 0)$, $1 \leq i \leq n$. For simplicity, let us prove this for $i = 1$; the other cases are completely analogous. By Theorem 3.3, we can clearly assume that f is Lipschitz on A. Let Y_1 be a closed subspace of X_1 such that X_1 is the topological direct sum $X_1 = \operatorname{span}\{v_1\} \oplus Y_1$. Without any loss of generality, we can suppose that $A_1 = G + C$, where G is an open convex set in $\operatorname{span}\{v_1\}$ and C is an open convex set in Y_1. Identify in the canonical way A with $G \times H$, where $H := C \times A_2 \times \cdots \times A_n \subset Y_1 \times X_2 \times \cdots \times X_n$. Now, considering f as a mapping on $G \times H$ and applying Lemma 3.5 (to f or $-f$), we conclude that f is strictly differentiable at a with respect to $\operatorname{span}\{\hat{v}_1\}$. □

Proposition 3.7 (reduction to the scalar case)**.** *Let X_i, A_i $(i = 1, \ldots, n)$ and Z be as in Theorem 3.3, let $f\colon \prod_{i=1}^n A_i \to Z$ be a continuous partially convex (up or down) operator. Let $a = (a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$ and $z^* \in Z_+^*$.*

(a) If z^* is strictly positive and $z^* \circ f$ is Gâteaux differentiable at a, then f is Gâteaux differentiable at a.

(b) If $z^* \in Z_+^*$ is as in Fact 2.6 (that is, "strongly positive") and if $z^* \circ f$ is Fréchet differentiable at a, then f is Fréchet differentiable at a.

Proof. First, let us consider the case $n = 1$; thus we can assume that f is a continuous convex operator. Then (a) holds by [29, Lemma 2.1]. To show (b) for $n = 1$, let z^* be a "strongly positive functional". By [31, Proposition 4.1] and its proof, f is a DC mapping with control function $z^* \circ f$. Then (b) follows by the fact (see [30, Proposition 3.9]) that a DC mapping is Fréchet differentiable at a whenever its control function is.

Now, let $n > 1$. The previous case then implies, in both cases (a) and (b), that f is "partially Gâteaux or Fréchet (respectively) differentiable" at a. The rest follows by Theorem 3.6. □

We shall also need the sufficient condition for Fréchet differentiability at a point from Corollary 3.9.

Lemma 3.8. *Let X, Y be normed spaces, $A \subset X$ and $B \subset Y$ open convex sets, Z an ordered Banach space which is normal, and $f\colon A \times B \to Z$ a Lipschitz mapping such that:*

(a) $f(\cdot, y)$ is convex on A for every $y \in B$;

(b) there exists a dense set $D \subset A \times B$ such that for each $(x, y) \in D$, $f(\cdot, y)$ is Gâteaux differentiable at x with derivative $D_1 f(x, y)$.

If the mapping $D \ni (x, y) \mapsto D_1 f(x, y)$ is continuous at a point $(a, b) \in D$, then $f(\cdot, b)$ is Fréchet differentiable at a.

Proof. By translation if necessary, we can (and do) suppose that $(a, b) = (0, 0)$. Assume that f is L-Lipschitz. Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\|D_1 f(x, y) - D_1 f(0, 0)\| < \varepsilon \quad \text{whenever } (x, y) \in D,\ \|x\| + \|y\| < \delta. \tag{8}$$

Now, let $x \in A$ be such that $0 < \|x\| < \delta/2$. Since $(x, 0) \in A \times B$, there exists $(u, v) \in X \times Y$ such that $(x + u, v) \in D$ and $\|u\| + \|v\| < \min\{\delta/2, \varepsilon\|x\|/L\}$. By Fact 2.8(a) (applied to $f(\cdot, 0)$), we have

$$\Delta(x) := f(x, 0) - f(0, 0) - D_1 f(0, 0)x \geq 0.$$

On the other hand,

$$\begin{aligned}\Delta(x) &= \big[f(x + u, v) - f(u, v) - D_1 f(0, 0)x\big] + \big[f(x, 0) - f(x + u, v)\big] \\ &\quad + \big[f(u, v) - f(0, 0)\big] =: a(x) + b(x) + c(x).\end{aligned}$$

Notice that $D_1 f(x + u, v) \in \partial[f(\cdot, v)](x + u)$ by Lemma 2.15; hence we have $f(x + u, v) - f(u, v) - D_1 f(x + u, v)x \leq 0$ and so $a(x) \leq \big[D_1 f(x + u, v) - D_1 f(0, 0)\big]x =: \widetilde{a}(x)$. Since $\|x + u\| + \|v\| \leq \|x\| + \|u\| + \|v\| < \delta$, we obtain $\|\widetilde{a}(x)\| < \varepsilon\|x\|$ by (8). Moreover, $\|b(x)\| \leq L(\|u\| + \|v\|) < \varepsilon\|x\|$, and similarly $\|c(x)\| < \varepsilon\|x\|$. Since $0 \leq \Delta(x) \leq \widetilde{a}(x) + b(x) + c(x)$, we conclude that $\|\Delta(x)\| \leq 3C\varepsilon\|x\|$ where C is a fixed "normality constant" of Z from Definition 2.1. This concludes the proof. □

Corollary 3.9. *For each $i \in \{1, \dots, n\}$, let X_i be a normed space and $A_i \subset X_i$ an open convex set. Denote $A := \prod_{i=1}^n A_i$. Let Z be an ordered Banach space which is normal and $f\colon A \to Z$ a continuous partially convex (up or down) operator. If f is Gâteaux differentiable at the points of a dense set $D \subset A$ and this Gâteaux derivative is continuous at a point $a \in D$, then f is Fréchet differentiable at a.*

Proof. By Theorem 3.3, we can (and do) assume that f is Lipschitz on A. An easy application of Lemma 3.8 shows that f is partially Fréchet differentiable at a with respect to every X_i, that is, for each $1 \leq i \leq n$, the partial operator

$f(a_1, \dots, a_{i-1}, \cdot, a_{i+1}, \dots, a_n)$ is Fréchet differentiable at a_i. By Theorem 3.6, f is Fréchet differentiable at a. □

4. Differentiability of general partially convex operators except for a small set

Theorem 4.1. *Let $X = X_1 \times \dots \times X_n$ where $X_1, \dots, X_n$ are separable Banach spaces, and $A = A_1 \times \dots \times A_n$ where $A_i \subset X_i$ $(i = 1, \dots, n)$ are nonempty open convex sets. Let $(Z, \leq)$ be a countably Daniell ordered Banach space. Let $f : A \to Z$ be a continuous partially convex (up or down) operator. Then f is Gâteaux differentiable at all points of A except for a σ-directionally porous set. In particular, f is Gâteaux differentiable at all points of A except for a set which is Γ-null and Aronszajn null (and so Haar null).*

Proof. Let $U := \{(0, \dots, 0, v_k, 0, \dots, 0) : 1 \leq k \leq n, v_k \in X_k\}$. Then $f'_+(x, u)$ exists for all $x \in A$ and $u \in U$ by Fact 2.8, and the linear span of U equals to X. Further, f is locally Lipschitz by Theorem 3.3.

Therefore f is Gâteaux differentiable at all points of A except for a σ-directionally porous set by Theorem 2.21. Using (5), we complete the proof. □

In what follows, we shall sometimes write briefly "*F-differentiable*" instead of "Fréchet differentiable".

Lemma 4.2. *Let Y_1, Y_2 be Banach spaces, $G \subset Y_1$ and $H \subset Y_2$ nonempty open convex sets, and let Z be a countably Daniell ordered Banach space. Suppose that the space $\mathcal{L}(Y_2, Z)$ is separable. Let $f : G \times H \to Z$ be a Lipschitz operator such that $f(x, .)$ is a convex operator on H for each $x \in G$. Then the set*

$$Q := \{(x, y) \in G \times H : \ f(x, .) \text{ is not Fréchet differentiable at } y\}$$

is σ-lower porous.

Proof. Let $S := \{(x, y) \in G \times H : \ \partial f(x, .)(y) \neq \emptyset\}$.

(a) In the first step we will prove that $Q \cap S$ is σ-lower porous.

Let f be Lipschitz with constant $L > 1$ and let $K := K_Z \geq 1$ be a fixed constant from Fact 2.2(b). For any $(x, y) \in Q \cap S$ we choose a $p_{xy} \in \partial f(x, .)(y)$ and find a natural number n_{xy} such that

$$\limsup_{h \to 0} \frac{\|f(x, y + h) - f(x, y) - p_{xy}(h)\|}{\|h\|} > \frac{1}{n_{xy}}. \tag{9}$$

Put $Q_n := \{(x, y) \in Q \cap S : \ n_{xy} = n\}$. Since $\mathcal{L}(Y_2, Z)$ is separable, we can choose for each n a sequence $(Q_{n,k})_{k=1}^{\infty}$ such that $Q_n = \bigcup_{k=1}^{\infty} Q_{n,k}$ and

$$\|p_{xy} - p_{uv}\| < \tfrac{1}{6nLK} \quad \text{whenever} \quad (x, y) \in Q_{n,k}, (u, v) \in Q_{n,k}. \tag{10}$$

Obviously $Q \cap S = \bigcup_{n,k=1}^{\infty} Q_{n,k}$ and therefore it is sufficient to show that each of the sets $Q_{n,k}$ is lower porous. To this end, choose arbitrary n, k and $(x,y) \in Q_{n,k}$. Now it is sufficient to prove (see Definition 2.17) that

$$\gamma((x,y), Q_{n,k}, R) \geq \tfrac{R}{6nLK} \quad \text{for all sufficiently small} \quad R > 0. \tag{11}$$

Without any loss of generality we can suppose that $(x,y) = (0,0)$. Considering the operator $\widetilde{f}(u,v) := f(u,v) - p_{xy}(v)$ instead of f, we see that we can also suppose that $p_{xy} = p_{00} = 0$. Now consider an arbitrary $R > 0$ such that $B((x,y),R) \subset G \times H$, find $h \in Y_2$ with $\|h\| < R/2$ such that

$$\frac{\|f(0,h) - f(0,0) - p_{00}(h)\|}{\|h\|} = \frac{\|f(0,h) - f(0,0)\|}{\|h\|} > \frac{1}{n} \tag{12}$$

and put $e := \frac{h}{\|h\|}$. To prove (11), it is sufficient to prove that

$$B((0, \tfrac{R}{2}e), \tfrac{R}{6nLK}) \cap Q_{n,k} = \emptyset. \tag{13}$$

So suppose to the contrary that some $(u,v) \in B((0, \frac{R}{2}e), \frac{R}{6nLK}) \cap Q_{n,k}$ is given. By (10), we have

$$\|p_{uv}\| \leq \frac{1}{6nLK}. \tag{14}$$

By the choice of K we can choose $z^* \in Z_+^*$ such that $\|z^*\| \leq K$ and

$$z^*\big(f(0,h) - f(0,0)\big) = \|f(0,h) - f(0,0)\|. \tag{15}$$

Now consider the real function $g := z^* \circ f$. Then $g(x,.)$ is convex on H for each $x \in G$, and (12) with (15) imply

$$\frac{g(0,h) - g(0,0)}{\|h\|} > \frac{1}{n}. \tag{16}$$

Obviously,

$$g \text{ is Lipschitz with constant } KL \tag{17}$$

and using the definition of z^* and (14), we easily obtain

$$\widetilde{p}_{uv} := z^* \circ p_{uv} \in \partial g(u,.)(v) \quad \text{and} \quad \|\widetilde{p}_{uv}\| \leq K \cdot \tfrac{1}{6nLK} = \tfrac{1}{6nL}. \tag{18}$$

Since $g(0,.)$ is convex and $\|h\| < R/2$, we obtain (using also (16))

$$\frac{g(0, \frac{R}{2}e) - g(0,0)}{R/2} \geq \frac{g(0,h) - g(0,0)}{\|h\|} > \frac{1}{n}$$

and consequently

$$g(0, \tfrac{R}{2}e) - g(0,0) > \tfrac{R}{2n}. \tag{19}$$

Using (18), we obtain

$$g(u, v - \tfrac{R}{2}e) - g(u,v) \geq \widetilde{p}_{uv}(-\tfrac{R}{2}e) \geq -\tfrac{1}{6nL} \cdot \tfrac{R}{2} \geq -\tfrac{R}{12\,n}. \tag{20}$$

By (17) and the choice of (u,v) we obtain

$$|g(u,v) - g(0, \tfrac{R}{2}e)| \leq KL \cdot \tfrac{R}{6nLK} = \tfrac{R}{6n} \tag{21}$$

and

$$|g(0,0) - g(u, v - \tfrac{R}{2}e)| \leq KL \cdot \tfrac{R}{6nLK} = \tfrac{R}{6n}. \tag{22}$$

Using (20), (21) and (22), we obtain

$$g(0,0) - g(0, \tfrac{R}{2}e) \geq -\tfrac{R}{12n} - \tfrac{R}{6n} - \tfrac{R}{6n} = -\tfrac{5}{12}\tfrac{R}{n},$$

which contradicts (19); so (13) and (11) follow.

(b) To finish the proof, it is sufficient to show that $(G \times H) \setminus S$ is σ-lower porous. First notice that our assumptions imply that both Y_2^* and Z are separable. Using Fact 2.4, choose a strictly positive functional $z^* \in Z_+^*$, and set $\widetilde{f} := z^* \circ f$. Then $\widetilde{S} := \{(x,y) \in G \times H :\ \partial\widetilde{f}(x,.)(y) \neq \emptyset\} = G \times H$ by Corollary 2.16(a). So, by the first step (a) of the present proof,

$$\widetilde{Q} := \{(x,y) \in G \times H :\ \widetilde{f}(x,.) \text{ is not Fréchet differentiable at } y\}$$

is σ-lower porous. Consequently Proposition 3.7(a) (with $n = 1$) implies that

$$T := \{(x,y) \in G \times H :\ f(x,.) \text{ is not Gâteaux differentiable at } y\}$$

is σ-lower porous. Since $(G\times H)\setminus S \subset T$ by Corollary 2.16(b), we are done. □

Theorem 4.3. *Let $X = X_1 \times \cdots \times X_n$, where $X_1, \dots, X_n$ are Banach spaces, and $A = A_1 \times \cdots \times A_n$, where $A_i \subset X_i$ $(i = 1, \dots, n)$ are nonempty open convex sets. Let Z be a countably Daniell ordered Banach space and suppose that the space $\mathcal{L}(X, Z)$ is separable. Let $f\colon A \to Z$ be a continuous partially convex (up or down) operator. Then f is Fréchet differentiable except for a σ-lower porous Γ-null set.*

Proof. Since $Z \neq \{0\}$, we easily obtain that X^* and consequently also all X_i^* are separable. We can suppose that f is Lipschitz, since f is locally Lipschitz (by Theorem 3.3), X is separable and both σ-lower porous sets and Γ-null sets form a σ-ideal.

Now denote $Y_1 := X_1 \times \cdots \times X_{n-1}$, $Y_2 := X_n$, $G := A_1 \times \cdots \times A_{n-1}$ and $H := A_n$. Then either f (if f is convex in the n-th coordinate) or $-f$ (if f is concave in the n-th coordinate) considered as a mapping $Y_1 \times Y_2 \to Z$ fulfills the assumptions of Lemma 4.2 and consequently this lemma implies that the set

$$Q_n := \big\{(x_1, \dots, x_n) \in A :\ f(x_1, \dots, x_{n-1}, \cdot) \text{ is not F-differentiable at } x_n\big\}$$

is σ-lower porous. Quite analogously we obtain that, for each $1 \leq i \leq n$,

$$Q_i := \big\{(x_1, \dots, x_n) \in A :\ f(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n) \text{ is not F-differentiable at } x_i\big\}$$

is σ-lower porous. Now, Theorem 3.6 implies that f is Fréchet differentiable on A outside the σ-lower porous set $Q_1 \cup \cdots \cup Q_n$.

Now let $D \subset A$ be the set of all Gâteaux differentiability points of f. By Theorem 4.1, $A \setminus D$ is a Γ-null set. Further, f is strictly Gâteaux differentiable at all points of D by Theorem 3.6. Consequently (4) and Theorem 2.20 (applied with $S := \mathcal{L}(X, Y)$) imply that f is Fréchet differentiable at Γ-almost every point of D. So f is Fréchet differentiable at all points of A except for a Γ-null set. $\square$

In the real-valued case we obtain the following "nonseparable" result.

Theorem 4.4. *Let $X = X_1 \times \cdots \times X_n$ where $X_1, \ldots, X_n$ are Asplund Banach spaces, and $A = A_1 \times \cdots \times A_n$ where $A_i \subset X_i$ ($i = 1, \ldots, n$) are nonempty open convex sets. Let $f\colon A \to \mathbb{R}$ be a continuous partially convex (up or down) function. Then f is Fréchet differentiable on A except for a σ-lower porous Γ-null set.*

Proof. If all X_i are separable Asplund spaces (i.e., all X_i^* are separable), then X^* is separable and thus the statement of the theorem follows from Theorem 4.3 (applied with $Z := \mathbb{R}$).

Now we will prove the general case using the method of separable reduction. Our aim is to show that the set

$$N_F := \{x \in A : \ f \text{ is not Fréchet differentiable at } x\}$$

is σ-lower porous and Γ-null. Notice that [34, Theorem 2] (or [19, Corollary 3.5.5]) gives that N_F is a $G_{\delta\sigma}$ set (and hence a Souslin set) in X. Thus Theorem 2.30 implies that there exists a rich family $\mathcal{F}_1$ of closed separable subspaces of X such that for every $Y \in \mathcal{F}_1$ we have that N_F is σ-lower porous in X whenever $N_F \cap Y$ is σ-lower porous in Y.

By Lemma 2.28, the family

$$\mathcal{F}_2 := \{Y_1 \times \cdots \times Y_n : \ Y_k \text{ is a closed separable subspace of } X_k,\ 1 \le k \le n\}$$

is a rich family of closed separable subspaces in X.

By Theorem 2.29 there exists a rich family $\mathcal{F}_3$ of closed separable subspaces of X such that for every $Y \in \mathcal{F}_3$, f is Fréchet differentiable (with respect to X) at every $x \in Y \cap A$ at which its restriction to $Y \cap A$ is Fréchet differentiable (with respect to Y).

Now, by Lemma 2.27, $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ is a rich family and $\mathcal{F} \subset \mathcal{F}_1$. Consequently, by the choice of $\mathcal{F}_1$ and Theorem 2.31, it is sufficient to show that

$$N_F \cap Y \text{ is } \sigma\text{-lower porous and } \Gamma\text{-null in } Y \text{ for each } Y \in \mathcal{F}. \tag{23}$$

So consider an arbitrary $Y \in \mathcal{F}$; since $Y \in \mathcal{F}_2$, we have $Y = Y_1 \times \cdots \times Y_n$ where Y_k is a closed separable subspace of X_k, $1 \le k \le n$. If $N_F \cap Y \neq \emptyset$, set

$g := f|_{(Y\cap A)}\colon \prod_{k=1}^n (A_k \cap Y_k) \to \mathbb{R}$ and denote by M the set of points of $Y \cap A$ at which g is not Fréchet differentiable (with respect to Y). Since all Y_i are Asplund and g is continuous and partially convex (up or down), by the first part of the proof we have that M is σ-lower porous and Γ-null in Y. Since $Y \in \mathcal{F}_3$, we have $N_F \cap Y \subset M$, and thus (23) holds. □

Theorem 4.5. *Let $X = X_1 \times \cdots \times X_n$ where $X_1, \dots, X_n$ are Asplund Banach spaces, and $A = A_1 \times \cdots \times A_n$ where $A_i \subset X_i$ $(i = 1, \dots, n)$ are nonempty open convex sets. Let Z be an ordered Banach space such that Z_+ is well-based. Let $f\colon A \to Z$ be a continuous partially convex (up or down) operator. Then f is Fréchet differentiable except for a σ-lower porous Γ-null set.*

Proof. By Fact 2.6(a), Z is countably Daniell. Let $z^* \in Z_+^*$ be as in Fact 2.6(b). Since $z^* \circ f$ is clearly a continuous partially convex (up or down) real function, the assertion follows by Theorem 4.4 and Proposition 3.7. □

Theorem 4.6. *Let $X = X_1 \times \cdots \times X_n$ where $X_1, \dots, X_n$ are Asplund Banach spaces, and $A = A_1 \times \cdots \times A_n$ where $A_i \subset X_i$ $(i = 1, \dots, n)$ are nonempty open convex sets. Let Z be a countably Daniell ordered Banach space. Let $f\colon A \to Z$ be a continuous partially convex (up or down) operator and let at least one of the following two conditions is satisfied.*

(a) Z is separable.

(b) All X_i are separable.

Then f is Gâteaux differentiable except for a σ-lower porous Γ-null set.

Proof. (a) By Fact 2.4, Z_+^* contains a strictly positive functional z^*. Then $z^* \circ f$ is a continuous partially convex (up or down) function. So Theorem 4.4 implies that $z^* \circ f$ is Fréchet differentiable except for a σ-lower porous Γ-null set and our assertion follows by Proposition 3.7.

(b) Since the range of f is contained in a closed separable subspace of Z, the assertion follows from (a). □

5. Differentiability of saddle operators except for a small set

Saddle functions and operators have some stronger properties than general partially convex functions and operators. The aim of the present section is to show that the results of the previous section can be strengthened in this case.

Throughout the present section, X_1, X_2 are Banach spaces, $A \subset X_1$ and $B \subset X_2$ are nonempty open convex sets.

Let us start with a result which is well known in the scalar case (Rockafellar [26]) and whose vector-valued version was proved in [13]. (Recall that analogous property is false for biconvex functions: see the text before Lemma 3.5.)

Proposition 5.1 ([13, Proposition 3.1]). *Let $f\colon A\times B\to Z$ be a convex-concave operator, where Z is a normal ordered Banach space. Let $(a,b)\in A\times B$ and $(u,v)\in X_1\times X_2$ be such that both derivatives $f'_+((a,b),(u,0))$ and $f'_+((a,b),(0,v))$ exist. Then also $f'_+((a,b),(u,v))$ exists and satisfies the formula*

$$f'_+((a,b),(u,v)) = f'_+((a,b),(u,0)) + f'_+((a,b),(0,v)).$$

In particular (see Observation 2.11), *if Z is moreover countably Daniell then f admits all one-sided directional derivatives at (a,b).*

Definition 5.2. Let Z be an ordered Banach space, and $f\colon A\times B\to Z$ a continuous convex-concave operator. We define its "subdifferential" as the multivalued mapping

$$T_f\colon X_1\times X_2 \rightrightarrows \mathcal{L}(X_1,Z)\times\mathcal{L}(X_2,Z)\ (=\mathcal{L}(X_1\times X_2,Z)),$$

given by

$$T_f(x,y)=\partial\big[f(\cdot,y)\big](x)\times\partial\big[-f(x,\cdot)\big](y)\quad \text{for } (x,y)\in A\times B,$$

$T_f(x,y)=\emptyset$ otherwise. (The subdifferentials are intended as in Definition 2.13.)

Lemma 5.3. *Let T_f be as in Definition 5.2, where Z is a countably Daniell ordered Banach space.*

(a) If $Z=\mathbb{R}$ then $T_f(x,y)\neq\emptyset$ for each $(x,y)\in A\times B$.
(b) If f is Gâteaux differentiable at some $(x,y)\in A\times B$ then $T_f(x,y)$ contains exactly one element, namely the couple $(D_1f(x,y),-D_2f(x,y))$, where $D_1f(x,y)$ and $D_2f(x,y)$ denote the "partial Gâteaux derivatives" with respect to X_1 and X_2, respectively. Moreover, for $Z=\mathbb{R}$ also the vice-versa holds: if T_f is single-valued at (x,y) then f is Gâteaux differentiable at (x,y).
(c) T_f is a generalized monotone mapping.

Proof. (a) follows from Corollary 2.16, and the first part of (b) follows from Lemma 2.15.

To show the second part of (b), assume that $Z=\mathbb{R}$ and $T_f(x,y)$ is a singleton. Then, by well-known results for convex functions (see [22]), the partial functions $f(\cdot,y)$ and $f(x,\cdot)$ are Gâteaux differentiable at x and y, respectively. Theorem 3.6 now implies that f is Gâteaux differentiable at (x,y).

(c) is well known for $Z=\mathbb{R}$ (see e.g. [22, Proposition 3.29]); the proof of the general case is done in the same way, as follows. Given $(x_1,y_1),(x_2,y_2)\in A\times B$

and $(L_i, M_i) \in T_f(x_i, y_i)$ $(i = 1, 2)$, we have:

$$\begin{aligned} f(x_2, y_2) - f(x_2, y_1) &\geq M_2(y_1 - y_2), \\ f(x_2, y_1) - f(x_1, y_1) &\geq L_1(x_2 - x_1), \\ f(x_1, y_1) - f(x_1, y_2) &\geq M_1(y_2 - y_1), \\ f(x_1, y_2) - f(x_2, y_2) &\geq L_2(x_1 - x_2). \end{aligned}$$

Summing up these inequalities, one easily obtains that

$$(L_1 - L_2)(x_1 - x_2) + (M_1 - M_2)(y_1 - y_2) \geq 0,$$

which is what was needed. □

Theorem 5.4 (scalar case)**.** *Let $f\colon A \times B \to \mathbb{R}$ be a continuous convex-concave function. Then:*

(a) if the spaces X_1, X_2 are separable, then f is Gâteaux differentiable except for a sparse (and hence Γ-null) set;
(b) if $X_1 = X_2 = \mathbb{R}$, then f is Gâteaux differentiable except for a DC sparse set;
(c) if X_1, X_2 either both are Gâteaux smooth or both are subspaces of Asplund generated spaces, then f is Gâteaux differentiable except for a σ-cone supported set;
(d) if X_1, X_2 are Asplund spaces, then f is Fréchet differentiable except for a set which is angle-small and Γ-null.

Proof. Let $T_f\colon X_1 \times X_2 \rightrightarrows X_1^* \times X_2^* = (X_1 \times X_2)^*$ be the corresponding "subdifferential" mapping from Definition 5.2. By Lemma 5.3, T_f is a monotone operator with domain $D(T_f) = A \times B$. Let N_G (resp. N_F) be the set of points of $A \times B$ at which f is not Gâteaux (resp. Fréchet) differentiable. By Lemma 5.3, N_G coincides with the set of points of $A \times B$ at which T_f is not single-valued. By Theorem 2.22, the set N_G is sparse in the case (a), and even DC sparse in the case (b). By (5), N_G is Γ-null as well.

If X_1, X_2 are Gâteaux smooth then $X_1 \times X_2$, when equipped with the ℓ_2 product norm, is Gâteaux smooth as well (this is standard). If X_1, X_2 are subspaces of Asplund generated spaces then $X_1 \times X_2$ is isomorphic to a subspace of an Asplund generated space (see [8, Theorem 1.3.6(iii)]). Since T_f is locally bounded at the points of the open set $A \times B$ (see e.g. [22, Theorem 2.28]), we can apply Theorem 2.23 to obtain (c).

To prove (d), first assume that X_1, X_2 are separable Asplund spaces (i.e., X_1^*, X_2^* are separable). By Theorem 4.3, N_F is Γ-null. We claim that N_F is contained in the set of points of $A \times B$ at which T_f is not both single-valued and u.s.c. To show this, let $(x, y) \in N_F$. By Theorem 3.6, f cannot be "partially Fréchet differentiable" at (x, y). Assume that, for example, $f(\cdot, y)$ is not Fréchet differentiable at x. It is well known (see e.g. [22]) that then the subdifferential $\partial[f(\cdot,y)]$ is not both single-valued and u.s.c. at x, and hence neither T_f is both single-valued and u.s.c. at (x,y). Thus our claim holds. By Theorem 2.25, such points form an angle-small set.

To show (d) for general Asplund spaces, we proceed by the method of separable reduction by repeating the second part of the proof of Theorem 4.4 with $n = 2$ and with "angle-small" instead of "σ-lower porous". □

Theorem 5.5 (Gâteaux differentiability)**.** *Let $f\colon A\times B \to Z$ be a continuous convex-concave operator, where Z is a countably Daniell ordered Banach space. Let N_G be the set of points in $A\times B$ at which f is not Gâteaux differentiable. Then:*

(a) if the spaces X_1, X_2 are separable, then N_G is sparse (and hence Γ-null);
(b) if $X_1 = X_2 = \mathbb{R}$, then N_G is DC sparse;
(c) if the duals X_1^*, X_2^* are separable, then N_G is angle-small.

Proof. Let X_1, X_2 be separable. Since $Z_1 := \overline{\operatorname{span}}\, f(A\times B)$, considered in the ordering inherited from Z, is a separable ordered Banach space which is clearly countably Daniell, we can (and do) assume that Z is separable. Then, by Fact 2.4, Z_+^* contains a strictly positive functional z^*. Then $z^*\circ f$ is clearly a continuous convex-concave function and, by Proposition 3.7, N_G coincides with the set of points at which $z^*\circ f$ is not Gâteaux differentiable. The rest now follows from Theorem 5.4(a,b,d). □

Remark 5.6. In the same way as in the proof of Theorem 5.5, we can obtain the following vector-valued versions of the "nonseparable" Theorem 5.4(c).

Let X_1, X_2 either both be Gâteaux smooth or both be subspaces of Asplund generated spaces, and let $f\colon A\times B\to Z$ be a continuous convex-concave operator, where Z is a countably Daniell ordered Banach space. If there exists a strictly positive functional $z^*\in Z_+^*$, then f is Gâteaux differentiable except for a σ-cone supported set.

Notice that existence of z^* is assured in any of the following cases: (a) if Z is separable (see Fact 2.4); (b) if Z_+ is well-based (see Fact 2.6(b)); (c) if Z is an order continuous Banach lattice with a weak (order) unit (see [20, Proposition 1.b.15], for the definition of a weak unit see [20, p. 9]).

Theorem 5.7 (Fréchet differentiability)**.** *Let $f\colon A\times B\to Z$ be a continuous convex-concave operator, where Z is a countably Daniell ordered Banach space. Let at least one of the following two conditions be satisfied.*

(a) $\mathcal{L}(X_1, Z)$ and $\mathcal{L}(X_2, Z)$ are separable.
(b) X_1, X_2 are Asplund spaces and Z_+ is well-based.

Then f is Fréchet differentiable on $A\times B$ except for a set which is angle-small and Γ-null.

Proof. (a) Since Z is always assumed to be nontrivial, the duals X_1^*, X_2^* are separable. By Theorem 5.5(c), f is Gâteaux differentiable except for an angle-small set N_G. Let $T_f\colon X_1 \times X_2 \rightrightarrows \mathcal{L}(X_1 \times X_2, Z)$ be the generalized monotone mapping from Lemma 5.3. Consider the geralized monotone mapping $T(x,y) := T_f(x,y)$ if $(x,y) \in (A \times B) \setminus N_G$, $T(x,y) := \emptyset$ otherwise. By Lemma 5.3(b), the domain of T is $D(T) = (A \times B) \setminus N_G$ and T is single-valued at each point of $D(T)$. Moreover, $D(T)$ is clearly dense in $A \times B$ since N_G is a first category set. By Theorem 2.25, there exists an angle-small set $M \subset D(T)$ such that T is upper semicontinuous at each point of $D(T) \setminus M$. This implies, via Lemma 5.3, that the Gâteaux derivative of f, defined on the dense set $D(T)$, is continuous at each point of $D(T) \setminus M$. By Corollary 3.9, f is Fréchet-differentiable at each such point. It follows that f is Fréchet differentiable at each point that does not belong to the angle-small set $N_G \cup M$. Moreover, f is Fréchet differentiable except for a Γ-null set by Theorem 4.3.

(b) Let $z^* \in Z_+^*$ be as in Fact 2.6(b). Since $z^* \circ f$ is clearly a convex-concave real function, this case immediately follows by Theorem 5.4(d) and Proposition 3.7. $\square$

6. Examples and applications

In this last section, we present some applications of our general results. In particular, we present (following [29, Section 5]) *several consequences of our "supergeneric" results (and "Γ-null" results) which are formulated in standard terms and do not follow from generic results* – Examples 6.2 and 6.3 are of this type. Other applications are *"joint differentiability results"* in Example 6.1.

Example 6.1. Let $p, q \in (1, \infty)$ and assume that either

$$Z = \ell_q \text{ and } \mathcal{X} = \{\ell_p : \; q < p < \infty\} \cup \{c_0\} \cup \{L_p[0,1] : \; q < \min(p,2)\}$$

or

$$Z = L_q[0,1] \text{ and } \; \mathcal{X} = \{\ell_p : \; \max(q,2) < p < \infty\} \cup \{c_0\}.$$

Then the following assertions follow from Theorem 4.3 and Theorem 5.7.

(a) *If $f\colon X = X_1 \times \cdots \times X_n \to Z$ is a continuous partially convex (up or down) operator and $X_i \in \mathcal{X}$, $i = 1, \ldots, n$, then f is Fréchet differentiable except for a σ-lower porous, Γ-null set.*

(b) *If $f\colon X = X_1 \times X_2 \to Z$ is a continuous convex-concave operator and $X_1 \in \mathcal{X}$, $X_2 \in \mathcal{X}$, then f is Fréchet differentiable except for an angle small, Γ-null set.*

Indeed, in all above cases, $\mathcal{L}(X, Z)$ is separable (see [29, Example 5.5]).

Moreover, from (a), we obtain two "joint differentiability results".

Namely, under the assumptions of (a), suppose that $g\colon X \to T$, where T is a Banach space with the Radon-Nikodým property, is a Lipschitz mapping, then *at some points of X (in fact, at all points of X except for a Γ-null set), both f is Fréchet differentiable and g is Gâteaux differentiable.* Indeed, by [18, Theorem 2.5] g is Gâteaux differentiable except for a Γ-null set.

Furthermore, if h is a Lipschitz real function on X, then *at some points of X both f and h are Fréchet differentiable.* This clearly follows from [19, Theorem 12.1.1] which implies that h is Fréchet differentiable at all points of a set which is not σ-upper porous, and from the fact that each σ-lower porous set is σ-upper porous.

The following examples are easy extensions of Examples 5.6 and 5.7 of [29], and so we present the proof for the first one only.

Example 6.2. Let Γ_1, Γ_2 be infinite (countable or uncountable) sets, $p, q \in (1, \infty)$, Z a separable ordered Banach space which is countably Daniell, and $f\colon \ell_p(\Gamma_1) \times \ell_q(\Gamma_2) \to Z$ a continuous convex-concave operator.

Then the set of all $(x, y) \in \ell_1(\Gamma_1) \times \ell_1(\Gamma_2) \subset \ell_p(\Gamma_1) \times \ell_q(\Gamma_2)$ at which f is Gâteaux differentiable is uncountable and dense in $\ell_p(\Gamma_1) \times \ell_q(\Gamma_2)$.

Proof. Since $\ell_p(\Gamma_1) \times \ell_q(\Gamma_2)$ is an Asplund space, Remark 5.6 implies that the set $N_G(f)$ of all $(x, y) \in \ell_p(\Gamma_1) \times \ell_q(\Gamma_2)$ at which f is not Gâteaux differentiable is σ-cone supported. Let $\emptyset \neq H \subset \ell_p(\Gamma_1) \times \ell_q(\Gamma_2)$ be an open set. Then [35, Lemma 4] (used with $Y = \ell_1(\Gamma_1) \times \ell_1(\Gamma_2)$, $D = G = H \cap (\ell_1(\Gamma_1) \times \ell_1(\Gamma_2))$, and $g = id\colon \ell_1(\Gamma_1) \times \ell_1(\Gamma_2) \to X := \ell_p(\Gamma_1) \times \ell_q(\Gamma_2))$ implies that $H \cap (\ell_1(\Gamma_1) \times \ell_1(\Gamma_2))$ is not σ-cone supported (in X). So the set $\big[(\ell_1(\Gamma_1) \times \ell_1(\Gamma_2)) \setminus N_G(f)\big] \cap H$ is not σ-cone supported, and so it is uncountable. $\square$

Example 6.3. Let Z be an ordered Banach space which is countably Daniell, $f\colon X = C[0, 1] \times C[0, 1] \to Z$ a continuous convex-concave operator. Then there exist increasing real analytic functions $x \in C[0, 1]$, $y \in C[0, 1]$ such that f is Gâteaux differentiable at (x, y).

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