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Measures of non-compactness and Sobolev-Lorentz spaces

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Abstract We show that the measure of non-compactness of the limiting embedding of Sobolev-Lorentz spaces is equal to the norm. This is a consequence of our general theorem for arbitrary Banach spaces.

Keywords Measure of non-compactness · Lorentz spaces · Sobolev spaces · Embeddings

1 Introduction

In this paper we study the measure of non-compactness of embeddings, but it can be defined for any linear mapping. This notion was firstly introduced by Gohberg, Goldenstein, and Markus (1957).

Definition 1.1 Let X and Y be Banach spaces and let T be a continuous linear mapping from X into Y. Let us denote the open unit ball in X centered at origin by B_X. We define the **measure of non-compactness** of T as

$$\beta(T) := \inf \left\{ r > 0 \colon \begin{array}{l} T(B_X) \text{ can be covered by finitely} \\ \text{many open balls with radius } r \end{array} \right\}.$$

It can be easily shown, that

$$0 \le \beta(T) \le \|T\| \tag{1}$$

and that the mapping T is compact if and only if $\beta(T) = 0$.

Measure of non-compactness can be also defined as the limit of entropy numbers e_k, where

$$e_k(T) := \inf \left\{ \varepsilon > 0 \colon \text{ there exist } c_j \in Y, \text{ such that } T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} B_Y(c_j, \varepsilon) \right\}.$$

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For the history and further properties of the entropy numbers and measure of non-compactness we refer the reader to the books (Edmunds and Evans, 1987) and (Edmunds and Triebel, 1996) and the references given there. Measure of non-compactness characterizes the geometry of the mapping and entropy numbers are closely related to eigenvalues. The study of entropy numbers was encouraged by the Carl's inequality (see (Carl and Triebel, 1980) and (Carl, 1981)), which implies that for the compact mapping there is a simple inequality between eigenvalues and entropy numbers. Some interesting results concerning embeddings and other measures of non-compactness can be found in (Lang and Musil, 2018).

Furthermore, for entropy numbers and a pair of continuous linear maps R and S it holds that $e_{k+l-1}(R\circ S) \leq e_k(R)\cdot e_l(S)$. Some interesting maps can be expressed as a composition of compact operator and embedding, hence it is useful to know more about the entropy numbers for classical embeddings. For some illustrations how can this be applied see (Edmunds and Triebel, 1996).

In particular we are concerned with the measure of non-compactness of the Sobolev embeddings. In the paper (Hencl, 2003) it was proven, that the measure of non-compactness of the classical embedding of $W_0^{k,p}(\Omega)$ into $L^{p^*}(\Omega)$ for $1 \leq p < \frac{d}{k}$ is equal to its norm, i.e.

$$\beta\left(Id\colon\ W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)\right) = \|Id\colon\ W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)\|. \tag{2}$$

In this paper we formulate and prove a general result about measure of non-compactness of embeddings of Banach spaces. It can be easily applied to simplify the proof of the case of embedding of Sobolev space into Lebesgue space. Moreover it can be applied to the case of embedding of Sobolev-Lorentz space into Lorentz space. The importance of these spaces and embeddings can be seen for example in (Stein, 1981) and (Kauhanen et al., 1999).

Theorem 1.2 (Non-compactness of embedding into Lorentz spaces) *Let $d \geq 2$, $k \in \mathbb{N}$, $k < d$, $1 \leq p < \frac{d}{k}$, denote $p^* = \frac{dp}{d-kp}$ and let $1 \leq q < \infty$. Let either $p > 1$ or $p = q = 1$. Let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. Then*

$$\beta\left(Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\right) = \|Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\|.$$

We prove this result in Section 3. In the Section 4 we show, that the measure of non-compactness can be smaller than the norm, i.e. we prove the following proposition.

Proposition 1.3 *It holds, that*

$$\beta\left(Id\colon\ W_0^{1,1}((0,1)) \to \mathcal{C}((0,1))\right) = \frac{1}{4} < \frac{1}{2} = \|Id\colon\ W_0^{1,1}((0,1)) \to \mathcal{C}((0,1))\|.$$

2 Preliminaries

2.1 Notation

We shall denote by Ω an open subset of $\mathbb{R}^d$, by $|\bullet|$ the Lebesgue measure in $\mathbb{R}^d$ and by χ_E the characteristic function of the set E.

In a Banach space X we denote the open ball with center x and radius r by $B_X(x,r)$, and the open unit ball centered at origin will be denoted by B_X, i.e. $B_X = B_X(0,1)$.

Let γ be a multi-index, i.e. a finite sequence of non-negative integers. If we denote $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, then we define the norm of γ by

$$|\gamma| := \gamma_1 + \gamma_2 + \cdots + \gamma_d.$$

For suitable $f\colon\ \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ we denote the weak (distributional) derivative by

$$D^\gamma f(x) := \frac{\partial^{|\gamma|} f}{\partial^{\gamma_1} x_1 \partial^{\gamma_2} x_2 \cdots \partial^{\gamma_d} x_d}(x).$$

By $W_0^{k,p}(\Omega)$ we denote the set of functions form the Sobolev space $W^{k,p}(\Omega)$ with zero traces. If not stated otherwise we use the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\gamma| \le k} \int_\Omega |D^\gamma g(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Let f be a measurable function from Ω to $\mathbb{R}$. By f_* we denote the distribution function, that is

$$f_*(s) := |\, \{x \in \Omega\colon\ |f(x)| > s\} \,|,$$

where $|\bullet|$ denotes the Lebesgue measure in $\mathbb{R}^d$. By f^* we denote the non-increasing rearrangement, that is

$$f^*(t) := \inf \{s > 0\colon\ f_*(s) \le t\}.$$

For further properties of non-increasing rearrangement see for example (Stein and Weiss, 1971). By f^{**} we denote the double star operator defined as

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s), \quad t \in (0,\infty).$$

Furthermore we denote $\{f > s\} := \{x \in \Omega\colon\ f(x) > s\}$. We denote it analogously for other types of (in)equalities $(<, \ge, \le, =)$.

Let $m, q \in \mathbb{R}$. If $1 \le q \le m < \infty$ then we use in the Lorentz space $L^{m,q}(\Omega)$ the norm

$$\|f\|_{L^{m,q}(\Omega)} := \left(\int_0^\infty \left(t^{\frac{1}{m}} \cdot f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}$$

and if $1 \le m < q < \infty$ then we use in the Lorentz space $L^{m,q}(\Omega)$ the norm

$$\|f\|_{L^{(m,q)}(\Omega)} := \left(\int_0^\infty \left(t^{\frac{1}{m}} \cdot f^{**}(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

For introduction to Lorentz spaces see e.q. (Stein and Weiss, 1971).

In the Sobolev-Lorentz space $W^k L^{m,q}(\mathbb{R}^d)$ we use the norm

$$\|f\|_{W^k L^{m,q}(\Omega)} := \left(\sum_{|\gamma| \le k} \|D^\gamma f\|^q_{L^{m,q}(\Omega)} \right)^{\frac{1}{q}}.$$

If $1 \le m < q < \infty$ we use in this definition $\|\bullet\|_{L^{(m,q)}(\Omega)}$ instead of $\|\bullet\|_{L^{m,q}(\Omega)}$. For $1 \le m, q < \infty$ we denote by $W_0^k L^{m,q}(\Omega)$ the set

$$W_0^k L^{m,q}(\Omega) := \left\{ f\colon\ \Omega \to \mathbb{R}\colon\ \widetilde{f} \in W^k L^{m,q}(\mathbb{R}^d) \right\},$$

where

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega \text{ and} \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

2.2 Sobolev and Lorentz spaces

We recall this well known fact about the Sobolev spaces.

Proposition 2.1 *Let $f\colon\ \mathbb{R}^d \to \mathbb{R}$ have weak derivatives up to the order k and let $|\gamma| \le k$. Let us denote $f_K(x) := f(Kx)$ for $K \in (0, \infty)$. Then*

$$D^\gamma f_K(x) = K^{|\gamma|}(D^\gamma f)(Kx) = K^{|\gamma|}(D^\gamma f)_K(x).$$

We need the following easy observation about the non-increasing rearrangement and double star operator.

Proposition 2.2 *Let $g\colon\ \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Let $K \in (0, \infty)$ and let us denote $g_K(x) := g(Kx)$. Then*

$$\begin{aligned} (g_K)^*(t) &= g^*(K^d t) \quad \text{and} \\ (g_K)^{**}(t) &= g^{**}(K^d t). \end{aligned}$$

Proof From the definition of the distribution function and change of variables $y = Kx$ it follows, that

$$\begin{aligned} (g_K)_*(s) &= |\{|g_K| > s\}| \\ &= \int_{\{|g_K(x)|>s\}} 1 \,\mathrm{d}x = \int_{\{|g(Kx)|>s\}} 1 \,\mathrm{d}x \\ &= \int_{\{|g(y)|>s\}} \frac{1}{K^d} \,\mathrm{d}y \\ &= \frac{g_*(s)}{K^d}. \end{aligned}$$

Therefore

$$\begin{aligned} (g_K)^*(t) &= \inf\{s > 0\colon\ (g_K)_*(s) \le t\} \\ &= \inf\left\{ s > 0\colon\ g_*(s) \le K^d t \right\} = g^*(K^d t). \end{aligned}$$

The second statement follows, as

$$\begin{aligned} (g_K)^{**}(t) &:= \frac{1}{t}\int_0^t (g_K)^*(s) \,\mathrm{d}s = \frac{1}{t}\int_0^t g^*(K^d s) \,\mathrm{d}s \\ &= \frac{1}{K^d}\frac{1}{t}\int_0^{K^d t} g^*(s) \,\mathrm{d}s =: g^{**}(K^d t). \end{aligned}$$

$\square$

Corollary 2.3 *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ have weak derivatives up to the order k, $|\gamma| \leq k$ and let $K > 0$. Let us denote $g_K(x) := g(K \cdot x)$. Then by Proposition 2.1 we have*

$$(D^\gamma(g_K))^*(t) = \left(K^{|\gamma|} \cdot (D^\gamma g)_K\right)^*(t) = K^{|\gamma|} \cdot (D^\gamma g)^*(K^d t) \quad \textit{and}$$
$$(D^\gamma(g_K))^{**}(t) = \left(K^{|\gamma|} \cdot (D^\gamma g)_K\right)^{**}(t) = K^{|\gamma|} \cdot (D^\gamma g)^{**}(K^d t).$$

Lemma 2.4 *Let f and g be two measurable functions from $\Omega \subseteq \mathbb{R}^d$ to $\mathbb{R}$ with disjoint supports and let $s > 0$. Then*

$$(f+g)_*(s) = f_*(s) + g_*(s).$$

Proof Clearly

$$\begin{aligned}(f+g)_*(s) &= |\,\{|f+g| > s\}\,| \\ &= |\,\{|f| > s\} \cup \{|g| > s\}\,| \\ &= |\,\{|f| > s\}\,| + |\,\{|g| > s\}\,| \\ &= f_*(s) + g_*(s).\end{aligned}$$

□

The key element in the proof of the main Theorem 1.2 is the following proposition. The proof can be found in (Malý, 2003, Lemma 3.10). For the convenience of the reader we include it. It is well-known that for $m, q \in [1, \infty)$ it holds that

$$\|f\|^q_{L^{m,q}(\Omega)} = m \int_0^\infty s^{q-1} [f_*(s)]^{\frac{q}{m}} \,\mathrm{d}s. \tag{3}$$

Proposition 2.5 *Let $\Omega \subseteq \mathbb{R}^d$, $1 \leq q \leq m$ and let f_1 and f_2 be two functions from $L^{m,q}(\Omega)$ with disjoint support. Then*

$$\|f_1\|^m_{L^{m,q}(\Omega)} + \|f_2\|^m_{L^{m,q}(\Omega)} \leq \|f_1 + f_2\|^m_{L^{m,q}(\Omega)}.$$

Proof If $q = m$, then $\|\bullet\|_{L^{m,q}(\Omega)} = \|\bullet\|_{L^m(\Omega)}$ and the inequality holds because for f_1 and f_2 with disjoint supports we have

$$\|f_1 + f_2\|^m_{L^m(\Omega)} = \int_\Omega |f_1 + f_2|^m = \int_\Omega |f_1|^m + \int_\Omega |f_2|^m = \|f_1\|^m_{L^m(\Omega)} + \|f_2\|^m_{L^m(\Omega)}.$$

So we may assume that $q < m$. From Lemma 2.4 we know that

$$(f_1)_* + (f_2)_* = (f_1 + f_2)_*.$$

Hölder's inequality for measure $s^{q-1}\,\mathrm{d}s$ yields

$$\begin{aligned}&\left(\int_0^\infty s^{q-1} (f_j)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}} \\ &\quad= \left(\int_0^\infty s^{q-1} \left((f_j)_*^{\frac{q}{m}}(s)(f_1+f_2)_*^{\frac{q(q-m)}{m^2}}(s)\right)\left((f_1+f_2)_*^{\frac{q(m-q)}{m^2}}\right) \mathrm{d}s\right)^{\frac{m}{q}} \\ &\quad\leq \left(\int_0^\infty s^{q-1} (f_j)_*(s)(f_1+f_2)_*^{\frac{q}{m}-1}(s)\,\mathrm{d}s\right)\left(\int_0^\infty s^{q-1}(f_1+f_2)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}-1}\end{aligned}$$

for $j = 1, 2$. We apply (3) and sum over j to get with the help of $q < m$ that

$$\begin{aligned}
m^{-\frac{m}{q}} \sum_{j=1}^{2} \|f_j\|^m_{L^{m,q}(\Omega)} &= \sum_{j=1}^{2} \left(\int_0^\infty s^{q-1} (f_j)_*^{\frac{q}{m}}(s) \,\mathrm{d}s \right)^{\frac{m}{q}} \\
&\leq \left(\int_0^\infty s^{q-1} (f_1 + f_2)_*^{\frac{q}{m}}(s) \,\mathrm{d}s \right)^{\frac{m}{q}-1} \sum_{j=1}^{2} \left(\int_0^\infty s^{q-1} (f_j)_*(s) (f_1 + f_2)_*^{\frac{q}{m}-1}(s) \,\mathrm{d}s \right) \\
&= \left(\int_0^\infty s^{q-1} (f_1 + f_2)_*^{\frac{q}{m}}(s) \,\mathrm{d}s \right)^{\frac{m}{q}} \\
&= m^{-\frac{m}{q}} \|f_1 + f_2\|^m_{L^{m,q}(\Omega)}.
\end{aligned}$$

□

For the case $q > m$ analogous statement holds as well, but with different power. For that we need the elementary inequality

$$(a + b)^p \geq a^p + b^p, \tag{4}$$

which holds for $a, b \geq 0$ and $1 \leq p < \infty$.

Proposition 2.6 *Let $\Omega \subseteq \mathbb{R}^d$, $1 \leq m < q < \infty$ and let f and g be two functions from $L^{m,q}(\Omega)$ with disjoint supports. Then*

$$\|f\|^q_{L^{m,q}(\Omega)} + \|g\|^q_{L^{m,q}(\Omega)} \leq \|f + g\|^q_{L^{m,q}(\Omega)}.$$

Proof Thanks to Lemma 2.4, (3) and (4) for $p := \frac{q}{m} > 1$ we have

$$\begin{aligned}
\|f + g\|^q_{L^{m,q}(\Omega)} &= m \int_0^\infty s^{q-1} (f + g)_*^{\frac{q}{m}}(s) \,\mathrm{d}s \\
&= m \int_0^\infty s^{q-1} (f_* + g_*)^{\frac{q}{m}}(s) \,\mathrm{d}s \\
&\geq m \int_0^\infty s^{q-1} (f_*)^{\frac{q}{m}}(s) \,\mathrm{d}s + m \int_0^\infty s^{q-1} (g_*)^{\frac{q}{m}}(s) \,\mathrm{d}s \\
&= \|f\|^q_{L^{m,q}(\Omega)} + \|g\|^q_{L^{m,q}(\Omega)}.
\end{aligned}$$

□

The embeddings between Sobolev-Lorentz spaces and Lorentz spaces we study in the next sections are ensured by (Peetre, 1966) or (Malý and Pick, 2002).

Theorem 2.7 (Sobolev-Lorentz embedding) *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with Lipschitz boundary, $d \geq 2$, $k \in \mathbb{N}$, $k < d$. Let $1 < p < \frac{d}{k}$ and $1 \leq q \leq \infty$. Denote $p^* := \frac{dp}{d-kp}$. Then*

$$W^k_0 L^{p,q}(\Omega) \hookrightarrow L^{p^*,q}(\Omega).$$

Remark 2.8
– The embedding holds even for $p = q = 1$.
– Let $q > p > 1$. The functionals $\|\bullet\|_{L^{p,q}(\Omega)}$ and $\|\bullet\|_{L^{(p,q)}(\Omega)}$ are equivalent and thus we may consider either of them in the definition of Lorentz or Sobolev-Lorentz space in the embedding.

3 Embeddings into Lorentz spaces

3.1 General theorem

To prove the main Theorem 1.2 we need a geometric assumption on the target Lorentz space.

Definition 3.1 Let X be a Banach space of functions from $\Omega \subseteq \mathbb{R}^d$ to $\mathbb{R}$ (set of functions which form a Banach space) and let $1 \le m < \infty$. We say that X is **disjointedly m-superadditive**, if there is a constant $M > 0$ such that for any finite sequence of functions $\{f_i\}_{i=1}^k \subseteq X$ with disjoint supports it holds, that

$$\sum_{i=1}^{k} \|f_i\|_X^m \le M \left\| \sum_{i=1}^{k} f_i \right\|_X^m .$$

Furthermore we say that X is **monotone** if restricting decreases norm, that is if $E \subseteq \Omega$ and $f \in X$, then

$$\|f \cdot \chi_E\|_X \le \|f\|_X .$$

Remark 3.2 The Lebesgue spaces L^m and Lorentz spaces $L^{m,q}$ are for $1 \le m < \infty$ clearly monotone.

If $q \le m$, then Proposition 2.5 implies that $L^{m,q}$ (and therefore L^m) is disjointedly m-superadditive with $M = 1$.

For $q > m$ we must be a bit more careful, because the functional $\|\bullet\|_{L^{m,q}(\Omega)}$ is not a norm. But for $m > 1$ it is equivalent to the norm $\|\bullet\|_{L^{(m,q)}(\Omega)}$, and thanks to Proposition 2.6 we know that for $q < \infty$

$$\sum_{i=1}^{k} \|f_i\|_{L^{(m,q)}}^q \le \sum_{i=1}^{k} v^q \|f_i\|_{L^{m,q}}^q \le v^q \left\| \sum_{i=1}^{k} f_i \right\|_{L^{m,q}}^q \le v^q V \left\| \sum_{i=1}^{k} f_i \right\|_{L^{m,q}}^q ,$$

where v and V are the constants from the equivalence of the functionals. Therefore for $\infty > q > m > 1$ the space $L^{m,q}$ equipped with the norm $\|\bullet\|_{L^{(m,q)}}$ is disjointedly q-superadditive.

Lemma 3.3 *Let $1 \le m < \infty$, $\alpha > 0$. Let X and Y be Banach spaces and let Y be disjointedly m-superadditive and monotone function space. Let $T\colon\ X \to Y$ be a continuous linear map. Assume that there exists a sequence of points $\{x_i\}_{i=1}^\infty \subseteq X$, such that the supports of $T(x_i)$ are pairwise disjoint and that*

$$\begin{aligned} \|x_i\|_X < 1 \quad &\textit{and} \\ \|T(x_i)\|_Y \ge \alpha. & \end{aligned} \tag{5}$$

Then $\beta(T) \ge \alpha$.

Proof Denote $f_i := T(x_i)$. From the continuity we know that

$$\|f_i\|_Y = \|T(x_i)\|_Y \le \|T\| \cdot \|x_i\|_X \le \|T\|. \tag{6}$$

Suppose (for contradiction) that $\beta(T) < \alpha$. We can clearly find $\varepsilon > 0$ such that $\beta(T) < \alpha - \varepsilon$. Let us fix $n \in \mathbb{N}$ big enough, such that $\big(\|T\| + \alpha\big)^m < \frac{n}{M} \cdot \varepsilon^m$, where M is the constant from the disjoint m-superadditivity.

From the definition of measure of non-compactness we know, that $T(B_X)$ is covered by finitely many balls. Therefore for some functions $\{c_j\}_{j=1}^k \subseteq Y$ we have

$$\{f_i\}_{i=1}^{\infty} \subseteq T(B_X) \subseteq \bigcup_{j=1}^{k} B_Y(c_j, \alpha - \varepsilon). \tag{7}$$

We claim that for every $j \in \{1, \dots, k\}$ there are at most $n-1$ functions f_i, such that $f_i \in B_Y(c_j, \alpha - \varepsilon)$.

Indeed, suppose for contradiction that there are n distinct numbers $i_1, \dots, i_n$ and in fact any ball with center C and radius $(\alpha - \varepsilon)$ such that

$$f_{i_1}, \dots, f_{i_n} \in B_Y(C, \alpha - \varepsilon). \tag{8}$$

Let S_r denote the support of f_{i_r}, $S := \bigcup_{1 \le r \le n} S_r$. Put $\widetilde{C} = C \cdot \chi_S$ and note that clearly $\|f_i - \widetilde{C}\|_Y \le \|f_i - C\|_Y$ because of the monotonicity of Y. Therefore without loss of generality we may assume, that C is supported in S.

We observe that S_r are disjoint and therefore we can write C as sum of functions $C_r := C \cdot \chi_{S_r}$ which have disjoint supports, i.e. $C = \sum_{1 \le r \le n} C_r$.

The monotonicity of Y and (8) give us

$$\|f_{i_r} - C_r\|_Y \le \|f_{i_r} - C\|_Y \le (\alpha - \varepsilon).$$

Using this and (5) we estimate for each $1 \le r \le n$

$$\|C_r\|_Y \ge \|f_{i_r}\|_Y - \|f_{i_r} - C_r\|_Y \ge \alpha - (\alpha - \varepsilon) = \varepsilon.$$

Thanks to the disjoint m-superadditivity of Y we obtain the estimate

$$\|C\|_Y^m = \left\| \sum_{r=1}^{n} C_r \right\|_Y^m \ge \frac{1}{M} \sum_{r=1}^{n} \|C_r\|_Y^m \ge \frac{1}{M} n\varepsilon^m > \big(\|T\| + \alpha\big)^m.$$

Using this, (8) and (6) we get

$$\alpha - \varepsilon \ge \|C - f_{i_1}\|_Y \ge \|C\|_Y - \|f_{i_1}\|_Y \ge \big(\|T\| + \alpha\big) - \|T\| = \alpha,$$

which is a contradiction.

We proved that inside every ball in (7) there are at most $n-1$ functions f_i. But that contradicts the fact that there are infinitely many functions f_i and finitely many balls. $\square$

Remark 3.4 Let $X(\mathbb{R}^d)$ be a space of functions from $\mathbb{R}^d$ to $\mathbb{R}$ and let Ω be an open subset of $\mathbb{R}^d$. We denote

$$X_0(\Omega) := \left\{ f \in X(\mathbb{R}^d) \colon \ f(x) = 0 \text{ for } x \in \mathbb{R}^d \setminus \Omega \right\}.$$

We furthermore denote $\|f\|_{X_0(\Omega)} := \|f\|_{X_0(\mathbb{R}^d)}$.

Theorem 3.5 (Non-compactness of embedding) *Let Ω be an open subset of $\mathbb{R}^d$ and let $X_0(\Omega)$ and $Y(\Omega)$ be two Banach spaces of functions from Ω to $\mathbb{R}$. Let $a \in (0, \infty)$ and assume the following conditions:*

(i) The space $X_0(\Omega)$ is continuously embedded into $Y(\Omega)$ and

$$\|Id\colon\ X_0(\Omega) \to Y(\Omega)\| = a. \tag{9}$$

(ii) The space $X_0(B)$ is continuously embedded into $Y(B)$ for any open ball $B \subseteq \Omega$ and

$$\|Id\colon\ X_0(B) \to Y(B)\| = a. \tag{10}$$

(iii) The space $Y(\Omega)$ is monotone and disjointedly m-superadditive.

Denote by I the embedding of $X_0(\Omega)$ into $Y(\Omega)$. (The condition (i) states that it is continuous and $\|I\| = a$.) Then

$$\beta(I) = \|I\|.$$

Proof We claim that $\beta(I) \geq a$. To prove that we find sequence of pairwise disjoint balls $B_i(x_i, r_i) \subseteq \Omega$. Fix $\delta > 0$. For every $i \in \mathbb{N}$ there is a function $g_i \in X(B_i)$, such that

$$\|g_i\|_{X_0} < 1 \quad \text{and}$$
$$\|g_i\|_Y > a - \delta.$$

The space $Y(\Omega)$ is monotone and disjointedly m-superadditive, so we can harness Lemma 3.3 applied to $T = I$, $x_i = g_i$ and $\alpha = a - \delta$ to get $\beta(I) \geq a - \delta$. We conclude by sending δ to 0.

Thanks to inequalities (1), i.e. $\beta(I) \leq \|I\|$, we can conclude with

$$a \leq \beta(I) \leq \|I\| = a.$$

□

3.2 Embedding of Sobolev-Lorentz spaces

The measure of non-compactness depends on the norm. If $q \leq m$ we consider the norm $\|\bullet\|_{L^{m,q}}$ and if $q > m$ we use $\|\bullet\|_{L^{(m,q)}}$.

Proof (of Theorem 1.2) Let us denote the embedding of $W_0^k L^{p,q}(\Omega)$ into $L^{p^*,q}(\Omega)$ by I. Let us denote

$$a_r := \|Id\colon\ W_0^k L^{p,q}\left(B(c, r)\right) \to L^{p^*,q}(B(c, r))\|$$

for $B(c, r) \subseteq \mathbb{R}^d$. Clearly a_r does not depend on $c \in \mathbb{R}^d$. Clearly we have $a_r \geq a_s$ for $r > s > 0$. We claim that $a_r = a_s$ and we denote this value by a (that is for example $a := a_1$).

Assume that we already know that $a_r = a$ for every $r > 0$. The claims (i) and (ii) from Theorem 3.5 follow immediately from Theorem 2.7 and the assumption $a_r = a$.

The condition (iii) follows from the fact that $L^{p^*,q}(\Omega)$ is monotone and disjointedly m-superadditive thanks to Remark 3.2, where $m = p^*$ for $p^* \geq q$ and $m = q$ for $p^* < q$.

It remains to prove, that for $r > s > 0$ we have $a_r \leq a_s$, that is

$$\begin{aligned} &\|Id\colon W_0^k L^{p,q}(B(0,r)) \to L^{p^*,q}(B(0,r))\| \\ &\qquad \leq \|Id\colon W_0^k L^{p,q}(B(0,s)) \to L^{p^*,q}(B(0,s))\|. \end{aligned} \tag{11}$$

Because of different norms in Lorentz spaces we need to split the proof into three parts depending on the value of q with respect to p and p^*, where $p < p^*$.

Part 1: $\mathbf{q \leq p < p^*}$

In this case on $W_0^k L^{p,q}$ resp. $L^{p^*,q}$ we have the norm $\|\bullet\|_{W^k L^{p,q}}$ resp. $\|\bullet\|_{L^{p^*,q}}$. Let $r > s > 0$ and fix $\varepsilon > 0$. Then we find $g \in W_0^k L^{p,q}(B(0,r))$ such that

$$\begin{aligned} \|g\|_{W^k L^{p,q}(B(0,r))} &= 1 \quad \text{and} \\ \|g\|_{L^{p^*,q}(B(0,r))} &> a_r - \varepsilon \end{aligned}$$

and let us denote

$$h\colon\ B(0,s) \to \mathbb{R}, \quad h(x) = cg\left(\frac{r}{s}x\right),$$

where c is a positive constant such that $\|h\|_{W^k L^{p,q}(B(0,s))} = 1$. From Corollary 2.3 it follows, that

$$(D^\gamma h)^*(t) = c \cdot \left(\frac{r}{s}\right)^{|\gamma|} \cdot (D^\gamma g)^* \left(\left(\frac{r}{s}\right)^d t\right).$$

This and the change of variables $T = \left(\frac{r}{s}\right)^d t$ give us

$$\begin{aligned} 1 &= \|h\|^q_{W^k L^{p,q}(B(0,s))} \\ &= \sum_{|\gamma| \leq k} \int_0^\infty \left(t^{\frac{1}{p}} \left(D^\gamma h\right)^*(t)\right)^q \frac{\mathrm{d}t}{t} \\ &= c^q \sum_{|\gamma| \leq k} \int_0^\infty t^{\frac{q}{p}-1} \left[\left(\frac{r}{s}\right)^{|\gamma|} \cdot (D^\gamma g)^* \left(\left(\frac{r}{s}\right)^d t\right)\right]^q \mathrm{d}t \\ &= c^q \sum_{|\gamma| \leq k} \int_0^\infty \left(\left(\frac{s}{r}\right)^d T\right)^{\frac{q}{p}-1} \left(\frac{r}{s}\right)^{|\gamma| q} \left[\left(D^\gamma g\right)^*(T)\right]^q \left(\frac{s}{r}\right)^d \mathrm{d}t \\ &= \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}} \sum_{|\gamma| \leq k} \int_0^\infty \left(\frac{r}{s}\right)^{|\gamma| q - kq} \left(T^{\frac{1}{p}} (D^\gamma g)^*(T)\right)^q \frac{\mathrm{d}T}{T}. \end{aligned}$$

Because $|\gamma| q - kq \leq 0$, $\frac{r}{s} > 1$ and $\|g\|^q_{W_0^k L^{p,q}(B(0,r))} = 1$ we can continue with

$$1 \leq \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}} \sum_{|\gamma| \leq k} \int_0^\infty \left(T^{\frac{1}{p}} (D^\gamma g)^*(T)\right)^q \frac{\mathrm{d}T}{T} = \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}}. \tag{12}$$

From Proposition 2.2 it follows, that

$$h^*(t) = c \cdot g^* \left(\left(\frac{r}{s}\right)^d t\right).$$

This combined with inequality (12) and change of variables $T = \left(\frac{r}{s}\right)^d t$ give us

$$\begin{aligned}
\|h\|^q_{L^{p^*,q}} &= \int_0^\infty \left(t^{\frac{1}{p^*}} \cdot h^*(t)\right)^q \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty t^{\frac{q}{p^*}-1} \cdot \left(c \cdot g^*\left(\left(\frac{r}{s}\right)^d t\right)\right)^q \mathrm{d}t \\
&= \int_0^\infty c^q \left(\left(\frac{s}{r}\right)^d T\right)^{\frac{q}{p^*}-1} \left(g^*(T)\right)^q \left(\frac{s}{r}\right)^d \mathrm{d}t \\
&= \frac{c^q}{\left(\frac{r}{s}\right)^{d\left(\frac{q}{p^*}-1\right)+d}} \|g\|^q_{L^{p^*,q}} \\
&\geq \left(\frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}}\right) (a_r - \varepsilon)^q \geq (a_r - \varepsilon)^q.
\end{aligned} \tag{13}$$

Therefore the function h proves that $a_s \geq a_r - \varepsilon$. Sending $\varepsilon \to 0$ gives us (11).

Part 2: $\mathbf{p < q \leq p^*}$

In this case we have the same norm on $L^{p^*,q}$, but on $W_0^k L^{p,q}$ we have the norm $\|\bullet\|_{W^k L^{(p,q)}}$. The proof is the same as in the first case, just everywhere we wrote $\|\bullet\|_{W^k L^{p,q}}$ we now write $\|\bullet\|_{W^k L^{(p,q)}}$ and up to the equation (12) we use the double-star operator ** instead of the rearrangement *. Note that by Corollary 2.3 the double star operator ** scales in the same way as the rearrangement *.

Part 3: $\mathbf{p < p^* < q}$

In this case on $W_0^k L^{p,q}$ resp. $L^{p^*,q}$ we have the norm $\|\bullet\|_{W^k L^{(p,q)}}$ resp. $\|\bullet\|_{L^{(p^*,q)}}$. The proof is again the same as in the second case, now we replace $\|\bullet\|_{L^{p^*,q}}$ with $\|\bullet\|_{L^{(p^*,q)}}$ and we use ** instead of * everywhere.

Remark 3.6 Let $1 \leq q \leq Q < \infty$. Theorem 1.2 holds even for the embedding of $W_0^k L^{p,q}(\Omega)$ into $L^{p^*,Q}(\Omega)$, i.e. it's measure of non-compactness is equal to it's norm.

Proof The validity of the embedding follows from Theorem 2.7 and embeddings between Lorentz spaces, because

$$W_0^k L^{p,q}(\Omega) \hookrightarrow L^{p^*,q}(\Omega) \hookrightarrow L^{p^*,Q}(\Omega).$$

The rest of the proof is analogous to the proof of Theorem 1.2, we only need to raise the inequality (12) to the power of $\frac{Q}{q}$ and to replace q by Q in inequalities (13).

4 Embedding into the space of continuous functions

In this section we show that the measure of non-compactness of an embedding can be smaller than its norm. For that we consider the Sobolev space $W_0^{1,1}((0,1))$ equipped with the norm $\|u\|_{1,1} := \int_0^1 |u'(x)| \,\mathrm{d}x$ (where u' is the weak derivative), and the space of continuous functions $\mathcal{C}((0,1))$ equipped with the supremum norm $\|u\|_\infty = \sup_{x\in(0,1)} |f(x)|$.

Proof (of Proposition 1.3) To show that the norm of the embedding is less or equal $\frac{1}{2}$ we observe that the norm in the Sobolev space is essentially total variation and that the functions in $W_0^{1,1}((0,1))$ have zero at the boundary, therefore their supremum norm is at most $\frac{1}{2}$. To show that the norm is at least $\frac{1}{2}$ it is enough to consider the function $\frac{1}{2} - \left|x - \frac{1}{2}\right| \in \overline{B}_{W_0^{1,1}((0,1))}$.

To show that the measure of non-compactness is smaller or equal than $\frac{1}{4}$ it is enough to show that for every $\varepsilon > 0$ there are finitely many open balls in $\mathcal{C}((0,1))$ with radius $\frac{1}{4} + \varepsilon$ that cover $B_{W_0^{1,1}((0,1))}$.

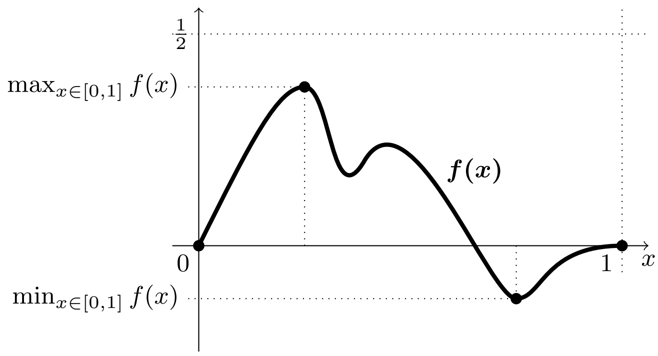
We can choose finitely many points c_i in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, such that any interval of length $\frac{1}{2}$ inside $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is contained in one of the balls (intervals)

$$B_{\mathbb{R}}\left(c_i, \frac{1}{4} + \varepsilon\right) = \left(c_i - \frac{1}{4} - \varepsilon, c_i + \frac{1}{4} + \varepsilon\right).$$

We claim that the balls

$$B_{\mathcal{C}((0,1))}\left(c_i, \frac{1}{4} + \varepsilon\right)$$

(where c_i is a constant function) cover $B_{W_0^{1,1}((0,1))}$.



Consider arbitrary function $f \in B_{W_0^{1,1}((0,1))}$. Thanks to (Brezis, 2013, Theorem 8.2 and Theorem 8.12) we can without loss of generality assume that f is (absolutely) continuous and $f(0) = f(1) = 0$. Because the total variation of function f is at most 1 we can easily deduce that $\max_{x\in[0,1]} f(x) - \min_{x\in[0,1]} f(x) \le \frac{1}{2}$. Therefore the range of f is an interval of length at most $\frac{1}{2}$ inside $[-\frac{1}{2}, \frac{1}{2}]$, so thanks to the choice of the points c_i there is i such that $f \in B_{\mathcal{C}((0,1))}(c_i, \frac{1}{4} + \varepsilon)$.

It remains to prove that the measure of non-compactness is bigger or equal to $\frac{1}{4}$. To show that we need to prove that for every ε the set $B_{W_0^{1,1}((0,1))}$ cannot be covered by finitely many balls in $\mathcal{C}((0,1))$ with radius $\frac{1}{4} - \varepsilon$.

Consider sequence of functions f_n, $n \in \mathbb{N}$, such that

$$f_n\left(\frac{1}{k}\right) = \begin{cases} 0 & k \in \mathbb{N},\ n \neq k \text{ and} \\ \frac{1}{2} & k \in \mathbb{N},\ n = k, \end{cases}$$

and each f_n is linear between points $\frac{1}{k}$, $k \in \mathbb{N}$. It can be easily verified that the norm of these functions in $W_0^{1,1}((0,1))$ is one. And in every ball in $\mathcal{C}((0,1))$ with radius $\frac{1}{4} - \varepsilon$ is at most 1 of these functions, because for $n \neq k$ it holds that $\|f_n - f_k\|_\infty = \frac{1}{2}$. $\square$

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