GROTHENDIECK'S INEQUALITIES FOR JB*-TRIPLES: PROOF OF THE BARTON–FRIEDMAN CONJECTURE

JAN HAMHALTER, ONDŘEJ F.K. KALENDA, ANTONIO M. PERALTA, AND HERMANN PFITZNER

Abstract. We prove that, given a constant $K > 2$ and a bounded linear operator T from a JB*-triple E into a complex Hilbert space H, there exists a norm-one functional $\psi \in E^*$ satisfying

$$\|T(x)\| \leq K \,\|T\|\, \|x\|_\psi,$$

for all $x \in E$. Applying this result we show that, given $G > 8(1 + 2\sqrt{3})$ and a bounded bilinear form V on the Cartesian product of two JB*-triples E and B, there exist norm-one functionals $\varphi \in E^*$ and $\psi \in B^*$ satisfying

$$|V(x, y)| \leq G \,\|V\|\, \|x\|_\varphi \,\|y\|_\psi$$

for all $(x, y) \in E \times B$. These results prove a conjecture pursued during almost twenty years.

1. Introduction

In order to review the historical emplacement of a conjecture open for almost twenty years, we should turn back to the fifties, to a major contribution in functional analysis. Grothendieck's inequalities and Grothendieck's constants were named after A. Grothendieck, who established the first result in this direction in his celebrated "*Résumé de la théorie métrique des produits tensoriels topologiques*" (see [13]). Grothendieck's original result proves the existence of a universal constant $G > 0$ (called Grothendieck's constant), satisfying that for every couple (Ω_1, Ω_2) of compact Hausdorff spaces and every bilinear form V on $C(\Omega_1) \times C(\Omega_2)$ there exist two probability measures μ_1 and μ_2 on Ω_1 and Ω_2, respectively, such that

$$|V(f, g)| \leq G\|V\| \left(\int_{\Omega_1} |f(t)|^2 d\mu_1(t) \right)^{\frac{1}{2}} \left(\int_{\Omega_2} |g(s)|^2 d\mu_2(s) \right)^{\frac{1}{2}}$$

for all $f \in C(\Omega_1)$ and $g \in C(\Omega_2)$. In 1956, Grothendieck predicted the validity of the previous result when the space $C(\Omega)$, of all complex valued continuous functions on a compact Hausdorff space Ω, is replaced with a general C*-algebra (cf. [13, §6, Question 4]). Grothendieck's forethought was confirmed several years later. In subsequent remarkable contributions, G. Pisier [27] and U. Haagerup [14] established the so-called *non-commutative Grothendieck inequality*, which assures that for

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every bounded bilinear form V on the cartesian product of two C*-algebras A and B, there exist two states ϕ in A^* and $\psi \in B^*$ satisfying

$$|V(x,y)| \leq 4\,\|V\|\,\phi\left(\frac{xx^*+x^*x}{2}\right)^{\frac{1}{2}}\,\psi\left(\frac{yy^*+y^*y}{2}\right)^{\frac{1}{2}},$$

for all $(x,y)\in A\times B$. Briefly speaking, at the cost of replacing probability measures with states and moduli of continuous functions with absolute values of the form $|x|^2 = \frac{xx^*+x^*x}{2}$ $(x\in A)$, the Grothendieck's inequality works for bounded bilinear forms on the Cartesian product of two C*-algebras. That is, in the non-commutative setting, the pre-Hilbertian semi-norms of the form $\|x\|_\phi^2 := \phi\left(\frac{xx^*+x^*x}{2}\right)$, where ϕ runs through the set of all states on a C*-algebra A, are valid to factor all bounded bilinear forms.

There exists a class of complex Banach spaces, called JB*-triples, which are determined by the holomorphic properties of their open unit balls (see Subsection 1.1 below for details). The class of JB*-triples includes (among others) all C*-algebras, and all complex Hilbert spaces. We therefore have a strictly wider class of complex Banach spaces than that determined by all C*-algebras. The setting of JB*-triples seemed an appropriate candidate to extend the Grothendieck's inequality when in 1987 J.T. Barton and Y. Friedman explored this problem.

Although JB*-triples lack an order structure like the one appearing in the setting of C*-algebras, every JB*-triple E admits a large collection of pre-Hilbertian semi-norms which arise naturally from the geometric structure and play a similar role to those determined by the states on a C*-algebra. Barton and Friedman showed in [2] that for each norm-one functional φ in the dual, E^*, of E, and each norm-one element z in E^{**} with $\varphi(z)=1$, the mapping $x\mapsto \|x\|_\varphi = \varphi\{x,x,z\}^{\frac{1}{2}}$ defines a pre-Hilbert semi-norm on E which does not depend on the choice of the element z. Let us observe that if ϕ is a state on a C*-algebra A and 1 denotes the unit element in A^{**}, then $\phi(1)=1$ and $\|x\|_\phi^2 = \phi\{x,x,1\} = \phi\left(\frac{xx^*+x^*x}{2}\right)$ for all $x\in A$. Theorem 1.4 in [2] asserts the existence of a universal constant $K\in[2,3+2\sqrt{2}]$ satisfying the following property: for every bounded bilinear form V on the cartesian product of two JB*-triples E and F there exist norm-one functionals $\varphi\in E^*$ and $\psi\in F^*$ satisfying

$$|V(x,y)| \leq K\,\|V\|\,\|x\|_\varphi\,\|y\|_\psi, \tag{1}$$

for all $(x,y)\in E\times F$. Building upon the results published in [2], Ch.-H. Chu, B. Iochum and G. Loupias gave an alternative proof of this result in [9, Theorem 6].

Grothendieck's inequalities were revisited in the setting of real JB*-triples at the beginning of this century, and it was pointed out in [23, 25] that the proof of [2, Theorem 1.3] contains a gap affecting also the arguments and conclusions in [9]. As a consequence of these difficulties, the original statement of Grothendieck's inequality for JB*-triples in (1) can not be considered as proved, and it is nowadays known as the *Barton–Friedman conjecture*.

The main results in [23, 25, 26] show that, at the cost of replacing semi-norms of the form $\|.\|_\varphi$ and $\|.\|_\psi$ with semi-norms of the form $\|.\|_{\varphi_1,\varphi_2}$, $\|.\|_{\psi_1,\psi_2}$ for convenient norm-one functionals $\varphi_1,\varphi_2\in E^*$ and $\psi_1,\psi_2\in F^*$, the conclusion in (1) is true for $K>4(1+2\sqrt{3})$ (cf. [25, Theorem 6]). Let us remark that for $\varphi_1,\varphi_2\in E^*$ we set $\|x\|_{\varphi_1,\varphi_2}^2 := \|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2$ $(x\in E)$. This result was applied to dissipate the concerns affecting subsequent results in JB*-triple theory (for example, properties of the strong*-topology, characterization of weakly compact operators from a JB*-triple into a Banach space, etc.) whose proofs depended on the original form of Grothendieck's inequality by Barton and Friedmann. Despite these advances, the Barton–Friedman conjecture (i.e. the statement in (1)) was neither proven nor discarded.

In [24] the Barton-Friedmann conjecture was proved in some special cases – for Cartan factors and atomic JBW*-triples (i.e. ℓ_∞-sums of Cartan factors).

In 2012, G. Pisier wrote "*The problem of extending the non-commutative Grothendieck theorem from C*-algebras to JB*-triples was considered notably by Barton and Friedman around 1987, but seems to be still incomplete*" (cf. [28, Remark 8.3]). The recent monograph [7] deals with the Barton–Friedman conjecture under an equivalent reformulation in terms of the *little Grothendieck inequality* (see [7, Problem 5.10.131]). We refer to section 2 for more details on the little Grothendieck inequality. It is very well illustrated in [7, pages 337-346] how a proof to the Barton-Friedman conjecture, or equivalently, to the little Grothendieck inequality, might have important consequences and *"restore the validity"* of all subsequent works relying on the original Grothendieck inequality in (1).

In this paper we fill the gap by proving the Barton-Friedman conjecture. The main result reads as follows

Theorem 1.1. *Suppose $G > 8(1+2\sqrt{3})$. Let E and B be JB*-triples. Then for every bounded bilinear form $V : E\times B \to \mathbb{C}$ there exist norm-one functionals $\varphi \in E^*$ and $\psi \in B^*$ satisfying*

$$|V(x,y)| \leq G\ \|V\|\,\|x\|_\varphi\,\|y\|_\psi$$

for all $(x,y) \in E \times B$. □

This theorem will be proved in Theorem 6.4 below.

The paper is organized as follows. In Subsection 1.1 we provide some background on JB*-triples. Subsection 1.2 deals with a representation of JBW*-triples in the form of a suitable direct sum (see Proposition 1.3).

Section 2 is devoted to the so-called little Grothendieck inequality. We recall where the gap was and indicate the strategy of our proof.

In the three following sections we prove the little Grothendieck inequality for individual summands from Proposition 1.3 and in the last section we glue the results together and provide proofs of the main results.

Along the paper, all Banach spaces will be over the field of complex numbers, the symbols S_X and B_X will stand for the unit sphere and the closed unit ball of a Banach space X, respectively.

1.1. **Basic notions and nomenclature.** The aim of extending the celebrated Riemann mapping theorem to complex Banach spaces of arbitrary dimension led W. Kaup to classify bounded symmetric domains in arbitrary complex Banach spaces (see [21]). It was proved by L. Harris that the open unit ball of a C*-algebra is a bounded symmetric domain (cf. [17]). It should be recalled that a domain $\mathcal{D}$ in a complex Banach space is symmetric if for each a in $\mathcal{D}$ there is a biholomorphic map S_a of $\mathcal{D}$ onto itself with $S_a = S_a^{-1}$, such that a is an isolated fixed point of S_a. However, the open unit balls of all C*-algebras do not exhaust all examples, namely, infinite dimensional complex Hilbert spaces enjoy the same property, but they are never C*-algebras. The celebrated contribution due to W. Kaup shows that the biholomorphic images of the open unit balls of JB*-triples cover all possible examples of bounded symmetric domains (cf. [21] or [8, Theorem 2.5.27]).

A *JB*-triple* is a complex Banach space E equipped with a (continuous) triple product $\{.,.,.\} : E^3 \to E$, which is symmetric and bilinear in the outer variables and conjugate-linear in the middle one, and satisfies the following algebraic–analytic axioms (where given $a, b \in E$, $L(a,b)$ stands for the (linear) operator on E given by $L(a,b)(x) = \{a,b,x\}$, for all $x \in E$):

(JB^*-1) $L(x,y)L(a,b) = L(L(x,y)(a),b) - L(a,L(y,x)(b)) + L(a,b)L(x,y)$, for all $a,b,x,y \in E$; *(Jordan identity)*

(JB^*-2) The operator $L(a,a)$ is a hermitian operator with nonnegative spectrum for each $a \in E$;

(JB^*-3) $\|\{a,a,a\}\| = \|a\|^3$ for $a \in E$.

The space $B(H,K)$ of all bounded linear operators between complex Hilbert spaces H and K, which is rarely a C*-algebra, is always a JB*-triple when equipped with the triple product defined by $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$. The same triple product provides a structure of JB*-triple for every C*-algebra. Moreover, if H is a complex Hilbert space, it can be canonically identified with $B(\mathbb{C}, H)$, so the above triple triple produce induces a structure of JB*-triple on H.

Moreover, every JB*-algebra B (see, e.g. [30] or [16, Section 3.8]) is a JB*-triple under the triple product defined by $\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$ $(x,y,z \in B)$ (see [8, Lemma 3.1.6] or [6, Theorem 4.1.45]). We recall that a JB*-algebra is a complex Jordan Banach algebra A equipped with an algebra involution * satisfying the following three conditions

$$\|a \circ b\| \leq \|a\| \, \|b\|, \ \|a^*\| = \|a\|, \text{ and } \| \{a, a^*, a\} \| = \|a\|^3,$$

for all $a, b \in A$, where that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$.

A formidable result due to Kaup asserts that a linear bijection between JB*-triples is a triple isomorphism if and only if it is an isometry (cf. [21, Proposition 5.5]).

Given $a, b \in E$ the symbol $Q(a,b)$ will stand for the conjugate-linear operator given by $Q(a,b)(x) = \{a,x,b\}$. We shall write $Q(a)$ for $Q(a,a)$.

An element e in a JB*-triple E is said to be a *tripotent* if $e = \{e,e,e\}$. Every projection in a C*-algebra A is a tripotent when the latter is regarded as a JB*-triple. Actually, tripotents in A are precisely partial isometries.

For each tripotent $e \in E$, the eigenvalues of the mapping $L(e,e)$ are contained in the set $\{0, \frac{1}{2}, 1\}$. Given $i \in \{0,1,2\}$, the linear operator $P_i(e) : E \to E$ is defined by

$$P_2(e) = L(e,e)(2L(e,e) - id_E) = Q(e)^2,$$
$$P_1(e) = 4L(e,e)(id_E - L(e,e)) = 2\left(L(e,e) - Q(e)^2\right),$$
$$\text{and } P_0(e) = (id_E - L(e,e))(id_E - 2L(e,e)).$$

It is known that $P_0(e)$, $P_1(e)$ and $P_2(e)$ are contractive linear projections (see [12, Corollary 1.2]), which are called the *Peirce projections* associated with e. Furthermore, the range of $P_i(e)$ is the eigenspace, $E_i(e)$, of $L(e,e)$ corresponding to the eigenvalue $\frac{i}{2}$, and

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e)$$

is known as the *Peirce decomposition* of E relative to e (see [12], [8, Definition 1.2.37] or [6, §4.2.2] and [7, §5.7] for more details). If E is a unital C*-algebra and $e \in E$ a tripotent, then e is a partial isometry with initial projection p_i and final projection p_f. The Peirce projections are given by the following identities

$$P_2(e)(x) = p_f x p_i \,,\ P_1(e)(x) = p_f x(1-p_i)+(1-p_f)xp_i \,,\ P_0(e)(u) = (1-p_f)x(1-p_i)\,,$$

where x runs through E.

A tripotent e is called *complete* if $E_0(e) = \{0\}$. If $E = E_2(e)$, or equivalently, if $\{e,e,x\} = x$ for all $x \in E$, we say that e is *unitary*.

For each tripotent e in a JB*-triple, E, the Peirce-2 subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $a \circ_e b := \{a,e,b\}$ and involution $a^{*_e} := \{e,a,e\}$ (cf. [8, §1.2 and Remark 3.2.2]). As we noticed above, every JB*-algebra is a JB*-triple with respect to the product

$$\{a,b,c\} = (a\circ b^*)\circ c + (c\circ b^*)\circ a - (a\circ c)\circ b^*.$$

By Kaup's theorem (see [21, Proposition 5.5]) the triple product on $E_2(e)$ is uniquely determined by the expression

$$\{a,b,c\} = (a\circ_e b^{*_e})\circ_e c + (c\circ_e b^{*_e})\circ_e a - (a\circ_e c)\circ b^{*_e}, \tag{2}$$

for every $a,b,c \in E_2(e)$. Therefore, unital JB*-algebras are in one-to-one correspondence with JB*-triples admitting a unitary element.

We shall make use of the following natural partial order defined on the set of tripotents in a JB*-triple E. Two tripotents e, v in E are called *orthogonal* (denoted by $e \perp v$) if $\{e,e,v\} = 0$ ($\Leftrightarrow \{v,v,e\} = 0 \Leftrightarrow e \in E_0(v) \Leftrightarrow v \in E_0(e)$). Suppose e, u are tripotents in E, we say that $e \leq u$ if $u - e$ is a tripotent which is orthogonal to e. By [12, Corollary 1.7] $e \leq u$ if and only if any of the equivalent conditions holds:

(*a*) $P_2(e)(u) = e$;
(*b*) $\{e,u,e\} = e$;
(*c*) e is a projection (i.e. a self-adjoint idempotent) in the JB*-algebra $E_2(u)$.

A *JBW*-triple* is a JB*-triple which is also a dual Banach space. In the triple setting, JBW*-triples play the role of von Neumann algebras in the class of C*-algebras. A fundamental result in the theory of JB*-triples proves that every JBW*-triple admits a unique (isometric) predual and its product is separately weak* continuous (see [3]). JBW*-algebras, von Neumann algebras, and complex Hilbert spaces are examples of JBW*-triples for the triple products presented above (cf. [8, Example 2.5.33 and Lemma 3.1.6].

The complete tripotents of a JB*-triple E are precisely the extreme points of its closed unit ball (cf. [4, Lemma 4.1] and [22, Proposition 3.5] or [8, Theorem 3.2.3]). Therefore every JBW*-triple contains a huge set of complete tripotents.

The theory of JBW*-triples is deeply indebted with the study on the predual of JBW*-triples developed by F. Friedman and B. Russo in [12]. Among the many influencing results established in this reference, it is shown that for each non-zero functional φ in the predual, M_*, of a JBW*-triple M, there is a unique tripotent $s(\varphi) \in M$, called the *support tripotent* of φ, such that $\varphi = \varphi \circ P_2(s(\varphi))$, and $\varphi|_{M_2(s(\varphi))}$ is a faithful positive functional on the JBW*-algebra $M_2(s(\varphi))$ (cf. [12, Proposition 2], or [7, Proposition 5.10.57]). We recall that a functional φ in the dual space of a JB*-algebra B is called faithful if $\varphi(a) = 0$ for $a \geq 0$ implies $a = 0$. We know from [12, part (b) in the proof of Proposition 2] that

$$\text{if } u \text{ is a tripotent in } M \text{ with } \|\varphi\| = \varphi(u), \text{ then } u \geq e(\varphi). \tag{3}$$

It is now time to recall the definition of the pre-Hilbert semi-norm appearing in Grothendieck's inequalities, which were introduced by J.T. Barton and Y. Friedman in [2]. Suppose φ is a functional in the predual of JBW*-triple M. By [2, Proposition 1.2] the mapping $M \times M \to \mathbb{C}$, $(x, y) \mapsto \varphi\{x, y, s(\varphi)\}$ is a positive semi-definite sesquilinear form on M. In particular, the Cauchy-Schwarz inequality holds. The associated pre-Hilbert semi-norm is denoted by $\|x\|_\varphi := (\varphi\{x, x, s(\varphi)\})^{1/2}$ $(x \in M)$. It is further known that

$$\|x\|_\varphi^2 = \varphi\{x, x, s(\varphi)\} = \varphi\{x, x, z\},$$

whenever z is an element in M satisfying $\varphi(z) = \|\varphi\| = \|z\| = 1$. In particular, $\|x\|_\varphi^2 = \varphi\{x, x, u\}$ for every tripotent $u \in M$ with $u \geq s(\varphi)$. Moreover, as a consequence of the fact that $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$ for all x, y, z in a JB*-triple, we get

$$\|x\|_\varphi \leq \sqrt{\|\varphi\|}\|x\|. \tag{4}$$

1.2. **A representation of JBW*-triples.** Key tools we use to prove our main results include structure results of JBW*-triples obtained by G. Horn and E. Neher in [18, (1.7)], [19, (1.20)], and recently revisited in [15] to decompose every JBW*-triple M in a suitable way. Before formulating the variant we need give the following easy lemma on decomposing special JBW*-triples.

Lemma 1.2. *Let V be a von Neumann algebra, $p \in V$ a projection and $(z_j)_{j \in J}$ an orthogonal family of projections in the center of pVp with sum equal to p. Then*

$$pV = \bigoplus_{j \in J}^{\ell_\infty} z_j V.$$

More precisely, the mapping

$$L : x \mapsto (z_j x)_{j \in J}$$

is an onto isometry witnessing the above equality.

Proof. The mapping L is clearly a one-to-one linear mapping with $\|L\| \leq 1$. Moreover, for any $a, b, c \in pV$ and $j \in J$ we have

$$\{z_j a, z_j b, z_j c\} = \frac{1}{2}(z_j a b^* z_j c + z_j c b^* z_j a) = \frac{1}{2}(z_j a b^* c + z_j c b^* a) = z_j \{a, b, c\},$$

where in the second equality we used the fact that the elements ab^* and cb^* belong to pVp and hence they commute with z_j.

It follows that L is a triple homomorphism. Since L is injective, it is an isometry by [8, Theorem 3.4.1].

Finally, it is clear that the range contains all elements with only finitely many nonzero coordinates. Since L is weak*-to-weak* continuous, it follows that L is onto. □

The promised representation result follows. For definitions and basic results on types of projections in von Neumann algebras we refer to [29, Chapter V].

Proposition 1.3. *Let M be any JBW*-triple. Then M is (isometrically) JB*triple isomorphic to a JBW*-triple of the form*

$$\left(\bigoplus_{k\in\Lambda}^{\ell_\infty} L^\infty(\mu_k, C_k)\right) \oplus^{\ell_\infty} N \oplus^{\ell_\infty} p_1V \oplus^{\ell_\infty} p_2V \oplus^{\ell_\infty} p_3V,$$

where

- $(\mu_k)_{k\in\Lambda}$ *is a (possibly empty) family of probability measures;*
- *Each* C_k *is a finite dimensional JB*-triple (actually a finite dimensional Cartan factor) for any* $k \in \Lambda$*;*
- N *is a JBW*-algebra;*
- V *is a von Neumann algebra,* $p_1, p_2, p_3 \in V$ *are projections such that* p_1 *is properly infinite,* p_2Vp_2 *is a von Neumann algebra of type* II_1 *and* p_3Vp_3 *is a finite von Neumann algebra of type* I*.*

Proof. By [15, Proposition 9.1] M is (isometrically) JB*triple isomorphic to a JBW*-triple of the form

$$\left(\bigoplus_{k\in\Lambda}^{\ell_\infty} L^\infty(\mu_k, C_k)\right) \oplus^{\ell_\infty} N \oplus^{\ell_\infty} pV,$$

where $(\mu_k)_{k\in\Lambda}$, $(C_k)_{k\in\Lambda}$ and N have the properties given in the statement and, moreover, V is a von Neumann algebra and $p \in V$ is a projection.

It remains to refine this decomposition a bit. The summand pV can be decomposed as a direct sum of two summands of the form p_1V and $p_2'V$, where p_1 is a properly infinite projection and p_2' is a finite projection (cf. [20, Proposition 6.3.7] or [15, Theorem 10.1]).

Further, by [29, Theorem V.1.19] there are orthogononal central projections z_1, z_2 in $p_2'Vp_2'$ with $z_1 + z_2 = p$ such that $z_1p_2'Vp_2'$ is of type I and $z_2p_2'Vp_2'$ of type II_1. To complete the proof set $p_2 = z_2p_2'$, $p_3 = z_1p_2'$ and use Lemma 1.2. □

2. Little Grothendieck inequality

The difficulties around Barton-Friedman conjecture are essentially due to a gap in the proof of the so-called *little Grothendieck inequality* stated in [2, Theorem 1.3]. As pointed out in [25] only the following statement was actually proved.

Lemma 2.1. ([25, Lemma 3], [2, Theorem 1.3]) *Let M be a complex JBW*-triple, H a complex Hilbert space, and let $T : M \to H$ be a norm-attaining weak*-continuous linear operator. Then there exists a norm-one normal functional $\varphi \in M_*$ satisfying*

$$\|T(x)\| \le \sqrt{2}\ \|T\|\ \|x\|_\varphi,$$

for all $x \in M$. □

In [25] it was observed that the assumption of norm-attaining, tacitly used in [2], need not to be satisfied. Via approximating operators by norm-attaining ones the following perturbed version of [2, Theorem 1.3] was proved.

Theorem 2.2. [25, Theorem 3] *Let $K > \sqrt{2}$ and $\varepsilon > 0$. Then, for every JBW*-triple M, every complex Hilbert space H, and every weak*-continuous linear operator $T : M \to H$, there exist norm-one functionals $\varphi_1, \varphi_2 \in M_*$ such that the* *inequality*

$$\|T(x)\| \leq K\,\|T\|\,\sqrt{\|x\|_{\varphi_1}^2 + \varepsilon\,\|x\|_{\varphi_2}^2}$$

holds for all $x \in M$. □

This version is enough for many structure results on JBW*-triples, but the question whether the perturbation is necessary, remained to be challenging. We can get rid of the perturbation if we assume that the JBW*-triple M contains a unitary element, or equivalently, when M is a (unital) JBW*-algebra, as witnessed by the following theorem.

Theorem 2.3. [25, Theorem 4] *Let $K > 2$ and let M be a JBW*-triple admitting a unitary element u. Then for every complex Hilbert space and every weak*-continuous linear operator $T : M \to H$ there exists a norm-one functional $\varphi \in M_*$ such that $s(\varphi) \leq u$ and*

$$\|T(x)\| \leq K\,\|T\|\,\|x\|_\varphi,$$

for all $x \in M$. □

We are going to extend this theorem to general JBW*-triples by analyzing behaviour of the seminorms $\|\cdot\|_{\varphi_1,\varphi_2}$ for a pair of normal functionals which do not have necessarily norm one. More specifically, we are going to prove the following theorem.

Theorem 2.4. *Let M be a JBW*-triple. Then given any two functionals φ_1, φ_2 in M_*, there exists a norm-one functional $\psi \in M_*$ such that*

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2}\cdot\sqrt{\|\varphi_1\| + \|\varphi_2\|}\cdot\|x\|_\psi\,,$$

for all $x \in M$. Furthermore, given $K > 2$, for every complex Hilbert space H, and every weak-to-weak continuous linear operator $T : M \to H$, there exists a norm-one functional $\psi \in M_*$ satisfying*

$$\|T(x)\| \leq K\,\|T\|\,\|x\|_\psi$$

for all $x \in M$.

This theorem will be proved in Theorem 6.1 below.

Observe that, once we establish the first estimate in this theorem, the second part follows easily from Theorem 2.2 (note that $\sqrt{\|x\|_{\varphi_1}^2 + \varepsilon\,\|x\|_{\varphi_2}^2} = \|x\|_{\varphi_1,\varepsilon\varphi_2}$).

The first estimate will be proved using the representation from Proposition 1.3. We will prove it for individual summands and then we will glue the results together using the following proposition which is a finer version of [24, Theorem 2.12].

Proposition 2.5. *Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a family of JBW*-triples for which there exists a positive constant G satisfying that for every $\alpha \in \Lambda$, and every couple of normal functionals $\varphi_{1,\alpha}, \varphi_{2,\alpha} \in (M_\alpha)_*$ there exists a norm-one functional $\varphi_\alpha \in (M_\alpha)_*$ satisfying*

$$\|x\|_{\varphi_{1,\alpha},\varphi_{2,\alpha}} \leq G\sqrt{\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\|}\,\|x\|_{\varphi_\alpha},$$

for all $x \in M_\alpha$. Let $M = \bigoplus_{\alpha\in\Lambda}^{\ell_\infty} M_\alpha$. Then for every couple of normal functionals $\varphi_1, \varphi_2 \in M_*$ there exists a norm-one functional $\varphi \in M_*$ satisfying

$$\|x\|_{\varphi_1,\varphi_2} \leq G\sqrt{\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\|}\,\|x\|_\varphi,$$

for all $x \in M$.

Proof. Let $\varphi_1, \varphi_2 \in M_*$ be given. For $\alpha \in \Lambda$ and $j = 1, 2$ denote by $\varphi_{j,\alpha}$ the restriction of φ_j to M_α (or, more precisely, the composition of φ_j with the canonical embedding of M_α into M). By the assumption there is a norm-one functional $\varphi_\alpha \in (M_\alpha)_*$ with

$$\|x\|_{\varphi_{1,\alpha},\varphi_{2,\alpha}} \leq G\sqrt{\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\|}\, \|x\|_{\varphi_\alpha}, \text{ for } x \in M_\alpha.$$

Further, set

$$c_\alpha = \frac{\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\|}{\|\varphi_1\| + \|\varphi_2\|}, \quad \alpha \in \Lambda,$$

and observe that $\sum_{\alpha\in\Lambda} c_\alpha = 1$. Thus the functional $\varphi \in M_*$ defined by

$$\varphi((x_\alpha)_{\alpha\in\Lambda}) = \sum_{\alpha\in\Lambda} c_\alpha \varphi_\alpha(x_\alpha) \text{ for } x = (x_\alpha)_{\alpha\in\Lambda} \in M,$$

has norm one. Moreover, for each $x \in M$ we have

$$\begin{aligned} \|x\|^2_{\varphi_1,\varphi_2} &= \sum_{\alpha\in\Lambda} \|x_\alpha\|^2_{\varphi_{1,\alpha},\varphi_{2,\alpha}} \leq \sum_{\alpha\in\Lambda} G^2(\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\|)\, \|x_\alpha\|^2_{\varphi_\alpha} \\ &= G^2(\|\varphi_1\| + \|\varphi_2\|) \sum_{\alpha\in\Lambda} c_\alpha \|x_\alpha\|^2_{\varphi_\alpha} = G^2(\|\varphi_1\| + \|\varphi_2\|)\, \|x\|^2_\varphi . \end{aligned}$$

□

The individual summands will be addressed in the three following sections, in the last section we glue the results together and show that a solution to the Barton–Friedman conjecture follows.

The proof for the summands N and $p_1 V$ is given in Corollary 3.4 and it is done by a refinement of the proof of Theorem 2.3 using some ideas from [15]. The proof for the remaining cases is done by showing that in these cases any seminorm of the form $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on B_M and then applying Lemma 2.1. The last step of this approach is explained in the following lemma.

Lemma 2.6. *Let $\varphi_1, \varphi_2 \in M_*$ be two normal functionals such that the seminorm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on B_M. Then there is a norm-one functional $\psi \in M_*$ such that*

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2}\sqrt{\|\varphi_1\| + \|\varphi_2\|}\|x\|_\varphi$$

for all $x \in M$.

Proof. Set

$$N_{\varphi_1,\varphi_2} = \{x \in M : \|x\|_{\varphi_1,\varphi_2} = 0\}.$$

On the quotient space $M/N_{\varphi_1,\varphi_2}$, the semi-norm $\|\cdot\|_{\varphi_1,\varphi_2}$ becomes a pre-Hilbert norm. Let H_{φ_1,φ_2} be the completion of the so-defined pre-Hilbert space and let $\pi_{\varphi_1,\varphi_2}$ be the natural quotient map viewed as a map from M into H_{φ_1,φ_2}. The separate weak*-continuity of the triple product and (4) ensure that $\pi_{\varphi_1,\varphi_2}$ is a weak*-continuous linear operator with norm at most $\sqrt{\|\varphi_1\| + \|\varphi_2\|}$. Finally, we may apply Lemma 2.1 to the operator $T = \pi_{\varphi_1,\varphi_2}$. □

3. JBW*-triples in which Peirce-2 subspaces of tripotents are upward directed

In this section we particularize our study to JBW*-triples satisfying that Peirce-2 subspaces of tripotents are upward directed by inclusion. The idea stems from [15] where such JBW*-triples were considered in order to have a mild substitute for the lack of an order, see e.g. [15, Proposition 6.5]. Let us begin with a series of technical lemmata.

Lemma 3.1. *Let φ_1, φ_2 be two functionals in the predual of a JBW*-triple M. Suppose there exists a tripotent p in M such that $s(\varphi_1) \le p$ and $s(\varphi_2) \le p$. Then the functional $\psi = \frac{\varphi_1+\varphi_2}{\|\varphi_1\|+\|\varphi_2\|}$ satisfies $\|\psi\| = 1$, $s(\psi) \le p$, and*

$$\|x\|_{\varphi_1,\varphi_2} = \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_{\psi}, \quad x \in M.$$

Proof. Set $e = s(\varphi_2)$ and $u = s(\varphi_1)$. By the assumption we have $u \le p$ and $e \le p$. Furhter, $\varphi_2(p) = \varphi_2(e) = \|\varphi_2\|$ and $\varphi_1(p) = \varphi_1(u) = \|\varphi_1\|$, so $\psi(p) = 1$. Since clearly $\|\psi\| \le 1$, we deduce that $\|\psi\| = \psi(p) = 1$ and hence $s(\psi) \le p$ (cf. (3)).

Finally, for $x \in M$ we have

$$\begin{aligned}\|x\|_{\psi}^2 = \psi(\{x,x,s(\psi)\}) = \psi(\{x,x,p\}) &= \frac{\varphi_1(\{x,x,p\}) + \varphi_2(\{x,x,p\})}{\|\varphi_1\| + \|\varphi_2\|} \\ = \frac{\varphi_1(\{x,x,u\}) + \varphi_2(\{x,x,e\})}{\|\varphi_1\| + \|\varphi_2\|} &= \frac{\|x\|_{\varphi_1,\varphi_2}^2}{\|\varphi_1\| + \|\varphi_2\|}.\end{aligned}$$

□

In our next proposition we show that the semi-norm given by a normal functional whose support tripotent is contained in the Peirce-2 subspace of another tripotent p in a JBW*-triple M can be bounded by the semi-norm given by a positive functional in the predual of the JBW*-algebra $M_2(p)$.

Proposition 3.2. *Let M be a JBW*-triple and let $\varphi \in M_*$. Assume that $p \in M$ is a tripotent such that $s(\varphi) \in M_2(p)$. Then there exists a functional $\tilde{\varphi} \in M_*$ such that $\|\tilde{\varphi}\| = \|\varphi\|$, $s(\tilde{\varphi}) \le p$ and $\|x\|_{\varphi} \le \sqrt{2}\,\|x\|_{\tilde{\varphi}}$ for all $x \in M$.*

Proof. We mimic the approach in the proof of [15, Lemma 7.7]. By the arguments in the first paragraph in the proof of [5, Proposition 2.4] (see also [10, Lemma 3.9]) we can find a unital JB*-algebra B and an isometric triple embedding π of M into B such that $\pi(p)$ is a projection in B. We can therefore assume that M is a JB*-subtriple of B and p is a projection in B. The triple product in B (and in M) is uniquely determined by the expressions $\{a,b,c\} = (a\circ b^*)\circ c + (c\circ b^*)\circ a - (a\circ c)\circ b^*$ $(a,b,c \in B)$.

Set $u = s(\varphi)$. Define $G : B \to B$ by $G(x) = P_2(u)(x \circ u)$. Having in mind that $1 - p \perp u$, and hence for each $x \in M$, we have $P_2(u)(x \circ u) = P_2(u)\{x,1,u\} = P_2(u)\{x,p,u\} \in M$, we deduce that G maps M into $M_2(u)$ and its restriction to M is weak*-to-weak* continuous.

Set $\tilde{\varphi} = \varphi \circ G|_M$. Then $\tilde{\varphi} \in M_*$ and $\|\tilde{\varphi}\| \le \|\varphi\|$ (as Peirce projections are contractive and hence clearly $\|G\| \le 1$).

Moreover,

$$\tilde{\varphi}(p) = \varphi(P_2(u)(p \circ u)) = \varphi P_2(u)\{p,p,u\} = \varphi(u) = \|\varphi\|,$$

hence $\|\tilde{\varphi}\| = \|\varphi\|$ and $s(\tilde{\varphi}) \le p$ (see (3)).

Finally, it is explicitly shown in the proof of [15, Lemma 7.7] that for each $x \in B$ we have $P_2(u)(\{x,x,u\} + \{x^*,x^*,u\}) = 2G(P_2(p)\{x,x,p\})$, and hence

$$\begin{aligned}\|x\|_{\tilde{\varphi}}^2 &= \tilde{\varphi}(\{x,x,s(\tilde{\varphi})\} = \tilde{\varphi}(\{x,x,p\}) = \tilde{\varphi}(P_2(p)\{x,x,p\}) = \varphi(G(P_2(p)\{x,x,p\})) \\ &= \frac{1}{2}\varphi(P_2(u)(\{x,x,u\} + \{x^*,x^*,u\})) \ge \frac{1}{2}\|x\|_{\varphi}^2.\end{aligned}$$

This completes the argument. □

We can next combine Lemma 3.1 and Proposition 3.2 to obtain a strengthened conclusion.

Proposition 3.3. *Let φ_1, φ_2 be two functionals in the predual of a JBW*-triple M. Assume there exists a tripotent $p \in M$ such that $\{s(\varphi_1), s(\varphi_2)\} \subseteq M_2(p)$. Then there is a norm-one functional $\psi \in M_*$ such that $s(\psi) \le p$ and*

$$\|x\|_{\varphi_1,\varphi_2} \le \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_{\psi},$$

for all $x \in M$.

Proof. Find, via Proposition 3.2, two functionals $\tilde{\varphi_1}$ and $\tilde{\varphi_2}$ in M_* such that $\|\tilde{\varphi_j}\| = \|\varphi_j\|$, $s(\tilde{\varphi_j}) \le p$ and $\|x\|_{\varphi_j} \le \sqrt{2}\|x\|_{\tilde{\varphi}_j}$ for all $x \in M$ and $j \in \{1,2\}$. Take $\psi = \frac{\tilde{\varphi_1}+\tilde{\varphi_2}}{\|\varphi_1\|+\|\varphi_2\|} \in M_*$. Lemma 3.1 implies that $s(\psi) \le p$ and

$$\|x\|_{\varphi_1,\varphi_2}^2 = \|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2 \le 2\left(\|x\|_{\tilde{\varphi}_1}^2 + \|x\|_{\tilde{\varphi}_2}^2\right) = 2(\|\varphi_1\| + \|\varphi_2\|)\|x\|_{\psi}^2,$$

for all $x \in M$. □

Corollary 3.4. *Let M be a JBW*-triple in which Peirce-2 subspaces of tripotents are upward directed by inclusion. Then given any φ_1, φ_2 in M_*, there exists a norm-one functional $\psi \in M_*$ such that*

$$\|x\|_{\varphi_1,\varphi_2} \le \sqrt{2}\sqrt{\|\varphi_1\| + \|\varphi_2\|}\,\|x\|_{\psi},$$

for all $x \in M$. This holds, in particular, when M is either a JBW-algebra or a JBW*-triple of the form pV, where V a von Neumann algebra and $p \in V$ a properly infinite projection.*

Proof. The first statement in a straight consequence of the previous Proposition 3.3. The second statement follows from [15, Remark 9.13]. □

4. Finite dimensional Cartan factors

In this section we shall deal with JBW*-triples of the form $L^\infty(\mu, C)$, where μ is a probability measure and C is a finite dimensional Cartan factor. In fact, the results work in a slightly more general setting – if C is a finite-dimensional JB*-triple. Henceforth, let C be such a JB*-triple. Since C is finite dimensional, every bounded linear operator from C into a Hilbert space attains its norm. In particular, any seminorm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the unit ball B_C. We will show that this property can be carried over to the space $L^\infty(\mu, C)$. This goal will be obtained after a series of lemmata.

Lemma 4.1. *The mapping $C \times C^* \to [0,\infty)$, $(x,\varphi) \mapsto \|x\|_{\varphi}$ is continuous.*

Proof. The set

$$A := \{(\varphi, e) \in C^* \times S_C : \varphi(e) = \|\varphi\|, \{e, e, e\} = e\}$$

is clearly closed. Moreover, the mapping $\Phi : C \times A \to [0, \infty)$ given by

$$\Phi(x, \varphi, e) = \varphi\{x, x, e\}$$

is continuous and $\Phi(x, \varphi, e) = \|x\|_\varphi^2$ for $x \in C$ and $(\varphi, e) \in A$.

Assume now that (x_n, φ_n) is a sequence in $C \times C^*$ converging to an element (x, φ). We will show that $\|x_n\|_{\varphi_n} \to \|x\|_\varphi$. Otherwise, up to passing to a subsequence, we may assume that $\|x_n\|_{\varphi_n} \to c \neq \|x\|_\varphi$ (note that the sequence $(\|x_n\|_{\varphi_n})_n$ is bounded). Let $e_n = s(\varphi_n)$ for $n \in \mathbb{N}$. We may assume, without loss of generality, that the sequence (e_n) converges to some $e \in C$. Since $(\varphi_n, e_n) \in A$ for each n, necessarily $(\varphi, e) \in A$ as well. Thus

$$\|x_n\|_{\varphi_n}^2 = \Phi(x_n, \varphi_n, e_n) \to \Phi(x, \varphi, e) = \|x\|_\varphi^2,$$

a contradiction which completes the proof. □

Lemma 4.2. *The set valued mapping $\Psi : (C^*)^2 \to 2^{B_C}$ defined by*

$$\Psi(\varphi_1, \varphi_2) = \{x \in B_C;\ \|x\|_{\varphi_1,\varphi_2} = \max_{y \in B_C} \|y\|_{\varphi_1,\varphi_2}\}$$

is upper semi-continuous and compact-valued. Consequently, there is a Borel-measurable selection H from Ψ.

Proof. Taking into account that S_{C^*} is compact, by [11, Lemma 3.1.1] it is enough to show that the set

$$\{(\varphi_1, \varphi_2, x) \in (C^*)^2 \times B_C;\ \|x\|_{\varphi_1,\varphi_2} = \max_{y \in B_C} \|y\|_{\varphi_1,\varphi_2}\}$$

is closed. But this easily follows from Lemma 4.1 as this set equals

$$\bigcap_{y \in B_C} \{(\varphi_1, \varphi_2, x) \in (C^*)^2 \times B_C;\ \|x\|_{\varphi_1,\varphi_2} \geq \|y\|_{\varphi_1,\varphi_2}\}.$$

Since Ψ has clearly nonempty values, the final statement follows, for example, from the Kuratowski-Ryll-Nardzewski theorem (see [1, Theorem 18.13]). □

Let (Ω, Σ, μ) be a probability space, and let $M = L^\infty(\mu, C)$. Then M is a JBW*-triple (with the triple product defined pointwise) and $M_* = L^1(\mu, C^*)$.

We need a more concrete description of the elements in M_*. Assume $g \in M_* = L^1(\mu, C^*)$. Let $u = s(g)$. Then u is a tripotent in M, hence $u(\omega)$ is a tripotent in C for almost all $\omega \in \Omega$. Under these circumstances we have

$$\begin{aligned}\|g\| = g(u) = \operatorname{Re} g(u) = \operatorname{Re} \int \langle g(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) = \int \operatorname{Re} \langle g(\omega), u(\omega)\rangle \,\mathrm{d}\mu(\omega) \\ \leq \int |\langle g(\omega), u(\omega)\rangle| \,\mathrm{d}\mu(\omega) \leq \int \|g(\omega)\| \cdot \|u(\omega)\| \,\mathrm{d}\mu(\omega) \leq \int \|g(\omega)\| \,\mathrm{d}\mu(\omega) = \|g\|\end{aligned}$$

So, we have everywhere equalities, hence $\langle g(\omega), u(\omega)\rangle = \|g(\omega)\|$ almost everywhere, and thus $u(\omega) \geq s(g(\omega))$ almost everywhere (cf. (3)).

It follows that for almost all ω we have

$$\|x\|_{g(\omega)}^2 = \langle g(\omega), \{x, x, u(\omega)\}\rangle, \text{ for all } x \in C.$$

Therefore, given $f \in M$ we have

$$\|f\|_g^2 = \langle g, \{f, f, u\}\rangle = \int \langle g(\omega), \{f(\omega), f(\omega), u(\omega)\}\rangle \, \mathrm{d}\mu(\omega) = \int \|f(\omega)\|_{g(\omega)}^2 \, \mathrm{d}\mu(\omega).$$

Let $g_1, g_2 \in L^1(\mu, C^*)$. Let H be the Borel-measurable selection from Ψ given by 4.2. We set $f(\omega) = H(g_1(\omega), g_2(\omega))$. Then $\|f\|_\infty \le 1$. Let $h \in L^\infty(\mu, C)$ be any element of the unit ball. Then

$$\|h\|_{g_1,g_2}^2 = \int \|h(\omega)\|_{g_1(\omega),g_2(\omega)}^2 \, \mathrm{d}\mu(\omega) \le \int \|f(\omega)\|_{g_1(\omega),g_2(\omega)}^2 \, \mathrm{d}\mu(\omega) = \|f\|_{g_1,g_2}^2 .$$

Therefore the pre-Hilbert semi-norm $\|\cdot\|_{g_1,g_2}$ attains its maximum on the closed unit ball of $L^\infty(\mu, C)$ (at f).

The previous arguments combined with Lemma 2.6 provide the following solution to the little Grothendieck problem for JBW*-triples of the form $L^\infty(\mu, C)$.

Proposition 4.3. *Let (Ω, Σ, μ) be a probability space, and let $M = L^\infty(\mu, C)$, where C is a finite dimensional JB*-triple. Then for every couple of normal functionals $g_1, g_2 \in M_*$ the pre-Hilbert semi-norm $\|\cdot\|_{g_1,g_2}$ attains its maximum on the closed unit ball of $L^\infty(\mu, C)$, and thus there exists a norm-one functional $h \in M_*$ satisfying*

$$\|f\|_{g_1,g_2} \le \sqrt{2}\,\sqrt{\|g_1\| + \|g_2\|}\,\|f\|_h,$$

for all $f \in M$.

5. Right ideals associated with finite projections in a von Neumann algebra

The aim of this section is to solve the little Grothendieck problem for the summands p_2V and p_3V from Proposition 1.3. They require different methods, but some tools are common for both cases. The first lemma shows how to express the hilbertian semi-norms using polar decomposition of the functional.

Lemma 5.1. *Let V be a von Neumann algebra, $p \in V$ a finite projection and $\varphi \in (pV)_*$. Then there is a positive functional ψ on pVp and a unitary element $u \in V$ such that $\|\psi\| = \|\varphi\|$, $\varphi(x) = \psi(xup)$ for $x \in pV$, $s(\psi)u^* = s(\varphi)$, and*

$$\|x\|_\varphi^2 = \frac{1}{2}\left(\psi(xx^*) + \psi(pu^*x^*xup)\right) \text{ for all } x \in pV.$$

Proof. Let $v = s(\varphi)$. Then v, being a tripotent in pV, is a partial isometry in V with final projection $q \le p$. Denote by r the initial projection. Further, since p is finite, q is finite as well, hence v can be extended to a unitary operator $\tilde{v} \in V$ (cf. [29, Proposition V.1.38]).

Set $\psi(x) := \varphi(x\tilde{v})$ for $x \in pV$. Since $x \mapsto x\tilde{v}$ is an isometry of pV onto pV, we deduce that $\|\psi\| = \|\varphi\|$. Further, since

$$\psi(q) = \varphi(q\tilde{v}) = \varphi(v) = \|\varphi\| = \|\psi\|,$$

we deduce that $s(\psi) \le q$ (cf. (3)), hence $\psi|_{pVp}$ is a positive functional on pVp. It remains to observe that one can take $u = \tilde{v}^*$. Indeed, for any $x \in pV$ we have

$$\psi(xup) = \psi(x\tilde{v}^*p) = \psi(qx\tilde{v}^*pq) = \psi(qx\tilde{v}^*q) = \psi(qxr\tilde{v}^*) = \varphi(qxr) = \varphi(x).$$

In particular,

$$\|\varphi\| = \|\psi\| = \psi(s(\psi)) = \psi(s(\psi)p) = \psi(s(\psi)u^*up) = \varphi(s(\psi)u^*),$$

which shows that $s(\psi)u^* \geq s(\varphi)$ (cf. (3)). But $s(\psi) \leq q$ implies that $s(\psi)u^* = s(\varphi)$.

Finally, for any $x \in pV$ we have

$$\|x\|_\varphi^2 = \varphi(\{x,x,v\}) = \frac{1}{2}\varphi(xx^*v + vx^*x) = \frac{1}{2}\psi(xx^*vup + vx^*xup)$$
$$= \frac{1}{2}(\psi(xx^*) + \psi(pu^*x^*xup)),$$

where in the last equality we used that $vup = v\tilde{v}^*p = qp = q$ and $s(\psi) \leq q$ to obtain the first term and

$$\psi(vx^*xup) = \psi(qu^*x^*xup) = \psi(qpu^*x^*xup) = \psi(pu^*x^*xup),$$

to obtain the second term. □

The key result for algebras of type II_1 is established in the next lemma.

Lemma 5.2. *Let V be a von Neumann algebra of type II_1 and let $p \in V$ be a projection. Then for each couple of functionals $\varphi_1, \varphi_2 \in (pV)_*$ the pre-Hilbert semi-norm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the closed unit ball of pV.*

Proof. For $j = 1, 2$ let ψ_j be a positive functional in $(pVp)_*$ and $u_j \in V$ a unitary element provided by Lemma 5.1 for φ_j. Then

$$\|x\|_{\varphi_1,\varphi_2}^2 = \frac{1}{2}(\psi_1(xx^*) + \psi_1(pu_1^*x^*xu_1p) + \psi_2(xx^*) + \psi_2(pu_2^*x^*xu_2p))$$

for any $x \in pV$.

By the Krein-Milman theorem and the weak*-compactness of the the closed unit ball of pV, the supremum of this semi-norm on the closed unit ball of pV is attained if and only if it is attained at an extreme point of this closed unit ball. Note that a tripotent in pV is a partial isometry in V with final projection below p, the tripotent is complete (i.e. it is an extreme point of the closed unit ball) if and only if its final projection equals p. Therefore the supremum of the semi-norm over the unit ball equals $\sqrt{C}$, where

$$C = \sup\left\{\frac{1}{2}(\psi_1(xx^*) + \psi_1(pu_1^*x^*xu_1p) + \psi_2(xx^*) + \psi_2(pu_2^*x^*xu_2p));\ xx^* = p\right\}$$
$$= \sup\left\{\frac{1}{2}(\psi_1(p) + \psi_2(p)) + \frac{1}{2}(\psi_1(pu_1^*x^*xu_1p) + \psi_2(pu_2^*x^*xu_2p));\ xx^* = p\right\}.$$

Let T be the center-valued trace on V (cf. [29, Theorem V.2.6]). If $x \in V$ is such that $xx^* = p$, then $0 \leq x^*x \leq 1$ and $T(x^*x) = T(p)$. Hence

$$C \leq \sup\left\{\frac{1}{2}(\psi_1(p) + \psi_2(p)) + \frac{1}{2}(\psi_1(pu_1^*yu_1p) + \psi_2(pu_2^*yu_2p));\ \begin{array}{c} 0 \leq y \leq 1, \\ T(y) = T(p) \end{array}\right\}.$$

The supremum on the right-hand side is attained, as it is a supremum of an affine weak*-continuous functional over the convex weak*-compact set

$$K = \{y \in V;\, 0 \leq y \leq 1, T(y) = T(p)\}.$$

So, the supremum is attained at an extreme point of K. Now, we claim that every extreme point of K is a projection. Indeed, assume that, say, $y \in K$ is not a projection. Since $0 \leq y \leq 1$, we may consider the spectral measure E of y. Since y is not a projection, there is some $\delta \in (0, \frac{1}{2})$ such that $q = E([\delta, 1-\delta]) \neq 0$. Since V is of type II_1, there is a projection $r \leq q$ with $r \sim q - r$. Set

$$v = y + \delta(2r - q), \quad w = y - \delta(2r - q).$$

Then $y = \frac{1}{2}(v+w)$, $T(v) = T(w) = T(y) = T(p)$ (as $T(r) = \frac{1}{2}T(q)$ by [29, Corollary V.2.8]). Moreover

$$v \geq y - \delta q \geq 0, \text{ and } v \leq y + \delta q \leq 1,$$

and similarly for w. It follows that $v, w \in K$, so y is not an extreme point of K. This finishes the proof of the claim.

Fix $y \in \operatorname{ext} K$ where the supremum is attained. Then y is a projection satisfying $T(y) = T(p)$, so $y \sim p$ by [29, Corollary V.2.8]. Therefore there is $x \in V$ with $xx^* = p$ and $x^*x = y$. We finally observe that the supremum C is attained at this element x. □

The following technical lemma enables us, roughly speaking, to reduce the case pV for a finite projection p to the case pV where the whole V is finite.

Lemma 5.3. *Let V be a von Neumann algebra and $p \leq t$ two projections in V such that p is finite. Consider the JBW*-triple $M = pV$ and its subtriple $N = pVt$. Let $\varphi_1, \varphi_2 \in M_*$ be two functionals such that $s(\varphi_j) \in N$ for $j = 1, 2$. Then*

$$\sup\{\|x\|_{\varphi_1,\varphi_2} ; x \in B_M\} = \sup\{\|x\|_{\varphi_1,\varphi_2} ; x \in B_N\}.$$

Proof. We use some ideas from the proof of [24, Proposition 2.8]. Let $W = tVt$. Then W is a von Neumann algebra, a C*-subalgebra of V and t is its unit. Set

$$u_j' = s(\varphi_j) \text{ for } j = 1, 2.$$

Both these tripotents are partial isometries in W with final projection below p. Since p is finite, by [29, Proposition V.1.38] these partial isometries can be extended to unitary elements $u_1'', u_2'' \in W$. Set

$$u_j = pu_j'' \text{ for } j = 1, 2.$$

Then u_1, u_2 are partial isometries in W with final projection equal to p. In particular, they are complete tripotents in N and also in M.

Moreover,

$$u_j' \leq u_j \text{ for } j = 1, 2,$$

where we use the standard order on tripotents. Indeed, it is enough to observe that

$$\{u_j', u_j, u_j'\} = u_j' u_j^* u_j' = u_j'(u_j'')^* p u_j' = u_j'(u_j'')^* p_f(u_j') u_j' = u_j'(u_j')^* u_j' = u_j'.$$

Further, define functionals $\zeta_j \in W_*$ by $\zeta_j(x) = \varphi_j(u_j x)$ for $x \in W$. Clearly $\|\zeta_j\| \leq \|\varphi_j\|$ and, moreover,

$$\zeta_j(t) = \varphi_j(u_j t) = \varphi_j(u_j) = \|\varphi_j\|,$$

hence ζ_j is positive (and $s(\zeta_j) \leq t$).

Given $x \in M$, set $x_1 = xt$ and $x_2 = x(1-t)$. Note that

$$\{x, x, u_j\} = \frac{1}{2}(xx^* u_j + u_j x^* x) = \frac{1}{2}(x_1 x_1^* u_j + x_2 x_2^* u_j + u_j x_1^* x_1 + u_j x_1^* x_2)$$

where we used that $x_1 x_2^* = x_2 x_1^* = 0$ and $u_j x_2^* = 0$ (the initial and the final projections of u_j both are below t). Since $\frac{1}{2} u_j x_1^* x_2 \in pV(1-t) \subset M_1(u_j)$, we deduce

$$P_2(u_j)\{x, x, u_j\} = \frac{1}{2} P_2(u_j)(x_1 x_1^* u_j + x_2 x_2^* u_j + u_j x_1^* x_1)$$

Using the fact that $s(\varphi_j) = u_j' \leq u_j$ we infer that

$$\|x\|^2_{\varphi_1,\varphi_2} = \frac{1}{2}\varphi_1(x_1x_1^*u_1 + x_2x_2^*u_1 + u_1x_1^*x_1) + \frac{1}{2}\varphi_2(x_1x_1^*u_2 + x_2x_2^*u_2 + u_2x_1^*x_1)$$
$$= \frac{1}{2}(\varphi_1(x_1x_1^*u_1 + x_2x_2^*u_1) + \zeta_1(x_1^*x_1) + \varphi_2(x_1x_1^*u_2 + x_2x_2^*u_2) + \zeta_2(x_1^*x_1)).$$

By the Krein-Milman theorem and the weak*-compactness of B_M (and B_N), the supremum of this semi-norm over any of these balls equals the supremum over its extreme points, i.e., over completes tripotents. Further note that a complete tripotent in M (in N) is a partial isometry in V (in W) with final projection equal to p, i.e, an element $x \in M$ ($x \in N$) satisfying $xx^* = p$. Since for $x \in M$ we have $xx^* = x_1x_1^* + x_2x_2^*$, we have

$$\begin{aligned}
&\sup\{\|x\|^2_{\varphi_1,\varphi_2}\,;\, x \in B_N\} \leq \sup\{\|x\|^2_{\varphi_1,\varphi_2}\,;\, x \in B_M\}\\
&\quad= \sup\{\|x\|^2_{\varphi_1,\varphi_2}\,;\, x \in M, xx^* = p\}\\
&\quad= \frac{1}{2}\sup\{\varphi_1(pu_1) + \varphi_2(pu_2) + \zeta_1(x_1^*x_1) + \zeta_2(x_1^*x_1);\, x \in M, x_1x_1^* + x_2x_2^* = p\}\\
&\quad\leq \frac{1}{2}\sup\{\varphi_1(pu_1) + \varphi_2(pu_2) + \zeta_1(y^*y) + \zeta_2(y^*y);\, y \in N, yy^* \leq p\}\\
&\quad\leq \frac{1}{2}\sup\{\varphi_1(pu_1) + \varphi_2(pu_2) + \zeta_1(y^*y) + \zeta_2(y^*y);\, y \in B_N\}\\
&\quad= \frac{1}{2}\sup\{\varphi_1(pu_1) + \varphi_2(pu_2) + \zeta_1(y^*y) + \zeta_2(y^*y);\, y \in N, yy^* = p\}\\
&\quad\leq \sup\{\|x\|^2_{\varphi_1,\varphi_2}\,;\, x \in B_N\},
\end{aligned}$$

where we used that $y \mapsto (\zeta_1(y^*y)+\zeta_2(y^*y))^{1/2}$ is a weak*-continuous pre-hilbertian semi-norm, hence the supremum can be computed over extreme points. □

We are now in a position to present a solution to the little Grothendieck problem for the summand p_2V from Proposition 1.3.

Proposition 5.4. *Let V be a von Neumann algebra and $p \in V$ a projection such that pVp is of type II_1. Then for any $\varphi_1, \varphi_2 \in (pV)_*$ the semi-norm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the unit ball of pV and therefore there exists a norm-one functional $\psi \in (pV)_*$ satisfying*

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_\psi\,, \text{ for all } x \in pV.$$

Proof. For $j = 1, 2$ let ψ_j be a positive functional on pVp and $u_j \in V$ a unitary element provided by Lemma 5.1 for φ_j. Set

$$t = p \vee u_1pu_1^* \vee u_2pu_2^*$$

and $W = tVt$. Then t, being the supremum of three projections equivalent to p, is a finite projection (cf. [29, Theorem V.1.37]). Moreover, the central carrier (also called the central support) of p in W equals $t = 1_W$ (just observe that if z is a central projection in W with $zp = 0$, then $zu_jpu_j^* = zu_jpu_j^*z = 0$ for all $j = 1, 2$, and hence $z = 0$).

We claim that W is of type II_1. Indeed, assume that $r \in W$ is a nonzero abelian projection. Since the central carrier of p equals 1_W [29, Lemma V.1.25] yields a nonzero projections $r_1 \leq r$ such that $r_1 \sim p$. Since r_1 is abelian, p is abelian, too, which contradicts the assumption that pVp is of type II_1.

Moreover, for $j = 1,2$ we have $s(\varphi_j) = s(\psi_j)u_j^*$, so the inital projection is $u_j s(\psi_j)u_j^* \leq u_j p u_j^* \leq t$, hence $s(\varphi_j) \in pVt = pW$. By Lemma 5.2 the pre-Hilbert semi-norm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the closed unit ball of pVt. We deduce from Lemma 5.3 that $\|\cdot\|_{\varphi_1,\varphi_2}$ actually attains its maximum on the closed unit ball of pV. Thus, by Lemma 2.6, there is a norm-one functional $\psi \in (pV)_*$ such that

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_{\psi}, \qquad x \in pV.$$

□

So, we have solved the case of the summand p_2V from Proposition 1.3 and we turn our attention to the remaining summand p_3V.

Henceforth, for each natural n, the symbol M_n will stand for the C*-algebra of all $n \times n$-matrices with complex entries. Given $1 \leq k \leq n$, we shall denote by $\mathcal{U}(M_n)$ the set of all unitary matrices in M_n, and by $P_k(M_n)$ the set of all projections of rank k.

Lemma 5.5. *The following assertions hold:*

(*a*) *Any two projections $q_1, q_2 \in P_k(M_n)$ are unitarily equivalent;*
(*b*) *$P_k(M_n)$ is a compact set;*
(*c*) *given $r \in \mathcal{P}_k(M_n)$ there is a Borel measurable function $\upsilon : P_k(M_n) \to \mathcal{U}(M_n)$ such that*

$$r = \upsilon(q)^* q \upsilon(q) \text{ for all } q \in P_k(M_n).$$

Proof. (*a*) This is well known and easy to see.

(*b*) It is clear that $\mathcal{U}(M_n)$ is a compact set and that the mapping

$$u \mapsto uru^*, \quad u \in \mathcal{U}(M_n),$$

where $r \in P_k(M_n)$ is fixed, is a continuous map of $\mathcal{U}(M_n)$ onto $P_k(M_n)$. Thus, $P_k(M_n)$ is compact.

(*c*) Fix $r \in P_k(M_n)$ and consider the continuous mapping used in (*b*). The inverse of this mapping admits a Borel measurable selection by the Kuratowski-Ryll-Nardzewski theorem (cf. [1, Theorem 18.13]). Denote the selection by υ. Then

$$\upsilon(q) r \upsilon(q)^* = q \text{ for all } q \in P_k(M_n),$$

hence the assertion follows. □

Lemma 5.6. *Let $W = L^\infty(\mu, M_n)$ for a probability measure μ and $n \in \mathbb{N}$.*

(*a*) *An element $f \in W$ is a projection if and only if $f(\omega)$ is a projection in M_n for μ-almost all ω;*
(*b*) *Any projection $f \in W$ is unitarily equivalent to a projection $g \in W$ such that $g(\omega) \in \{0, r_1, \ldots, r_{n-1}, I\}$ for μ-almost all ω, where $r_j \in M_n$ is a fixed projection of rank j for $1 \leq j < n$.*

Proof. (*a*) This assertion follows immediately from definitions.

(*b*) Let $f \in W$ be a projection. For $k \in \{0, \ldots, n\}$ let

$$A_k = \{\omega; \dim \operatorname{ran} f(\omega) = k\}.$$

By Lemma 5.5(b) each A_k is μ-measurable, being a preimage of a compact set. Further, for each $k \in \{1, \dots, n\}$ let $\upsilon_k : P_k(M_n) \to \mathcal{U}(M_n)$ be the mapping provided by Lemma 5.5(c) for the projection r_k. Set

$$u(\omega) = \begin{cases} I & \omega \in A_0 \cup A_n, \\ \upsilon_k(\omega) & \omega \in A_k, 0 < k < n. \end{cases}$$

Then u is a unitary element of W and $g = u^* f u$ is a projection satisfying the required properties. □

Lemma 5.7. *Let $W = L^\infty(\mu, M_n)$ for a probability measure μ and $n \in \mathbb{N}$. Let $p \in W$ be a projection. Then the JB*-triple pW is JB*-triple isomorphic to*

$$\bigoplus_{1 \le k \le n}^{\ell_\infty} L^\infty(\mu_k, r_k M_n),$$

where μ_k is a finite non-negative measure and $r_k \in M_n$ is a projection of rank k for each $k \in \{1, \dots, n\}$.

Proof. For each $k \in \{0, \dots, n\}$ let $r_k \in M_n$ be a projection of rank k (note that $r_0 = 0$ and $r_n = I$). By Lemma 5.6 p is unitarily equivalent to a projection g such that $g(\omega) \in \{r_0, \dots, r_n\}$ μ-almost everywhere. Then pW is triple-isomorphic to gW. Further, for $k = 0, \dots, n$ set

$$A_k = \{\omega;\, g(\omega) = r_k\}.$$

Then

$$gW = \bigoplus_{1 \le k \le n}^{\ell_\infty} L^\infty(\mu|_{A_k}, r_k M_n),$$

which completes the proof. □

Lemma 5.8. *Let V be a finite von Neumann algebra of type I and let $p \in V$ be a projection. Then the JB*-triple pV is JB*-triple isomorphic to*

$$\bigoplus_{j \in J}^{\ell_\infty} L^\infty(\mu_j, p_j M_{n_j}),$$

where μ_j is a probability measure, $n_j \in \mathbb{N}$ and $p_j \in M_{n_j}$ is a projection for $j \in J$.

Proof. By combining [29, Theorem V.1.27] and [29, Corollary V.2.9] we get an orthogonal family $(z_\alpha)_{\alpha \in \Lambda}$ of central projections in V with sum equal to 1 such that $z_\alpha V$ is isomorphic to $A_\alpha \overline{\otimes} M_{n_\alpha}$, where A_α is a σ-finite abelian von Neumann algebra and $n_\alpha \in \mathbb{N}$ for $\alpha \in \Lambda$. Each A_α, being σ-finite, is isomorphic to $L^\infty(\mu_\alpha)$ for some probability measure μ_α. Thus $pV = \bigoplus_{\alpha \in \Lambda} p z_\alpha V$ is isomorphic to

$$\bigoplus_{\alpha \in \Lambda}^{\ell_\infty} z_\alpha p L^\infty(\mu_\alpha, M_{n_\alpha}).$$

We conclude by applying Lemma 5.7 to each summand. □

The following proposition solves the case of the summand $p_3 V$ from Proposition 1.3.

Proposition 5.9. *Let V be a von Neumann algebra and $p \in V$ a finite projection such that pVp is of type I. Then for any normal functionals $\varphi_1, \varphi_2 \in (pV)_*$ the semi-norm $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the unit ball of pV and therefore there exists a norm-one functional $\psi \in (pV)_*$ satisfying*

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_\psi \quad \textit{for } x \in pV.$$

Proof. For $j = 1, 2$ let ψ_j be a positive functional on pVp and $u_j \in V$ a unitary element provided by Lemma 5.1 for φ_j. Set

$$t = p \vee u_1 p u_1^* \vee u_2 p u_2^*$$

and $W = tVt$. Then t, being the supremum of three projections equivalent to p, is a finite projection. Moreover, the central carrier of p in W equals $t = 1_W$.

We claim that W is of type I. Indeed, assume that $r \in W$ is a nonzero projection. Since the central carrier of p equals 1_W, [29, Lemma V.1.7] yields that there are two nonzero projections $r_1 \leq r$ and $p_1 \leq p$ such that $r_1 \sim p_1$. Since pVp is of type I, there is a nonzero abelian projection $p_2 \leq p_1$. Then there is a projection $r_2 \leq r_1$ equivalent to p_2. Therefore r_2 is abelian and $r_2 \leq r_1 \leq r$, which completes the proof of the claim.

Moreover, for $j = 1, 2$ we have $s(\varphi_j) = s(\psi_j) u_j^*$, so the inital projection is $u_j s(\psi_j) u_j^* \leq u_j p u_j^* \leq t$, hence $s(\varphi_j) \in pVt = pW$. By Lemma 5.8 $pW = pWt$ is JB*-triple isomorphic to $\bigoplus_{j\in J}^{\ell_\infty} L^\infty(\mu_j, p_j M_{n_j})$, where μ_j is a probability measure, $n_j \in \mathbb{N}$ and $p_j \in M_{n_j}$ is a projection for $j \in J$. For each $j \in J$, let $\varphi_{1,j} = \varphi_1|_{L^\infty(\mu_j, p_j M_{n_j})}$ and $\varphi_{2,j} = \varphi_2|_{L^\infty(\mu_j, p_j M_{n_j})}$. Proposition 4.3 assures that the pre-Hilbert semi-norm $\|.\|_{\varphi_1,\varphi_2}|_{L^\infty(\mu_j, p_j M_{n_j})} = \|.\|_{\varphi_{1,j},\varphi_{2,j}}$ attains its maximum on the closed unit ball of $L^\infty(\mu_j, p_j M_{n_j})$ at some point x_j. It follows that the semi-norm $\|.\|_{\varphi_1,\varphi_2}$ attains its maximum on the closed unit ball of $pW = pVt$ at the point $(x_j)_{j\in J}$. We can therefore apply Lemma 5.3 to deduce that $\|.\|_{\varphi_1,\varphi_2}$ attains its maximum on the closed unit ball of pV. Finally, Lemma 2.6 yields a norm-one functional $\psi \in (pV)_*$ such that

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2}\sqrt{\|\varphi_1\| + \|\varphi_2\|}\ \|x\|_\psi\,, \text{ for all } x \in pV.$$

□

6. Proof of Grothendieck's inequalities for JB*-triples

Now we are ready to prove the Barton-Friedmann conjecture. We start by restating and proving the little Grothendieck inequality given in Theorem 2.4.

Theorem 6.1. *Let M be a JBW*-triple. Then given any two functionals φ_1, φ_2 in M_*, there exists a norm-one functional $\psi \in M_*$ such that*

$$\|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_\psi\,,$$

for all $x \in M$. Furthermore, given $K > 2$, for every complex Hilbert space H, and every weak*-to-weak continuous linear operator $T : M \to H$, there exists a norm-one functional $\psi \in M_*$ satisfying

$$\|T(x)\| \leq K\,\|T\|\,\|x\|_\psi$$

for all $x \in M$.

Proof. The first statement follows from the results of the previous section. Indeed, consider the decomposition of M from Proposition 1.3. The statement for individual summands follows from Proposition 4.3, Corollary 3.4, Proposition 5.4, and Proposition 5.9, respectively. Finally, Proposition 2.5 completes the argument.

Let us prove the second statement. Fix $K > 2$. Let $\varepsilon > 0$ be such that $K > 2(1+\varepsilon)$. By Theorem 2.2 there are norm-one functionals $\varphi_1, \varphi_2 \in M_*$ such that for any $x \in M$ we have

$$\|T(x)\| \leq \sqrt{K}\,\|T\|\,\sqrt{\|x_1\|_{\varphi_1}^2 + \varepsilon\,\|x\|_{\varphi_2}^2} = \sqrt{K}\,\|T\|\,\|x\|_{\varphi_1,\varepsilon\varphi_2}.$$

By the first part of the theorem we get a norm-one functional $\psi \in M_*$ such that for $x \in M$ we have

$$\|x\|_{\varphi_1,\varepsilon\varphi_2} \leq \sqrt{2}\sqrt{\|\varphi_1\| + \|\varepsilon\varphi_2\|}\,\|x\|_\psi = \sqrt{2(1+\varepsilon)}\,\|x\|_\psi.$$

By combining the two inequalities we get

$$\|T(x)\| \leq \sqrt{2(1+\varepsilon)K}\,\|x\|_\psi \leq K\,\|x\|_\psi$$

for $x \in M$. This completes the proof. □

Given a bounded linear operator T from a JB*-triple E into a complex Hilbert space H we can always consider its bitranspose $T^{**}: E^{**} \to H$, which is a weak*-continuous linear operator from a JBW*-triple into a complex Hilbert space. We therefore arrive, via Theorem 6.1, to a proof of the little Grothendieck inequality with one control functional.

Theorem 6.2. *Let E be a JB*-triple, H a complex Hilbert space, and $K > 2$. Then for every bounded linear operator $T : E \to H$, there exists a norm-one functionals $\psi \in E^*$ satisfying*

$$\|T(x)\| \leq K\,\|T\|\,\|x\|_\psi,$$

for all $x \in E$. □

The previous Theorems 6.1 and 6.2 restore the equilibrium and the validity of original statements concerning the little Grothendieck inequality in the case of JB*-triples in [2, 9]. It also provides a complete solution to [7, Problem 5.10.131], [25, Remark 3], and [28, Remark 8.3]. We shall next trace back the original sources to see how our results can be also employed to provide a complete proof to the Barton–Friedmann conjecture concerning Grothendieck's inequality for bilinear forms on JB*-triples.

Theorem 6.3. *Suppose $G > 8(1+2\sqrt{3})$. Let M and N be JBW*-triples. Then for every separately weak*-continuous bilinear form $V : M \times N \to \mathbb{C}$ there exist norm-one functionals $\varphi \in M_*$ and $\psi \in N_*$ satisfying*

$$|V(x,y)| \leq G\,\|V\|\,\|x\|_\varphi\,\|y\|_\psi$$

for all $(x,y) \in M \times N$.

Proof. Thanks to our previous Theorem 6.1 we can recover a trick from [9, Theorem 6] and [25, Remark 3]. A brief argument is included here for completeness reasons. Let us find a weak*-to-weak continuous linear operator $R : M \to N_*$ defined by $V(a,b) = \langle R(a), b\rangle$ $((a,b) \in M \times N)$. Clearly $\|R\| \leq \|V\|$. By [9, Lemma 5] R factors through a complex Hilbert space, more precisely, there exists a complex Hilbert space H and bounded linear operators $T : M \to H$, $S : H \to N_*$ satisfying

$R = S \circ T$ and $\|T\|\ \|S\| \le 2(1+2\sqrt{3})\ \|R\|$. It is further shown in the proof of [25, Theorem 6] that we can choose H in such a way that S is injective and T is weak*-to-weak continuous.

Let $\tilde{G} = \left(\dfrac{G}{2(1+2\sqrt{3})}\right)^{\frac{1}{2}} > 2$. By applying Theorem 6.1 to the weak*-continuous linear operators $T : M \to H$ and $S^* : N \to H$ we find two norm-one functionals $\varphi \in M_*$ and $\psi \in N_*$ satisfying

$$\|T(x)\| \le \tilde{G}\,\|T\|\,\|x\|_\varphi, \text{ and } \|S^*(y)\| \le \tilde{G}\,\|S^*\|\,\|y\|_\psi$$

for all $(x, y) \in M \times N$. We therefore have

$$\begin{aligned}|V(x,y)| = |\langle R(x), y\rangle| = |\langle T(x), S^*(y)\rangle| &\le \tilde{G}^2\,\|T\|\,\|S\|\,\|x\|_\varphi\,\|y\|_\psi \\ &\le G\,\|V\|\,\|x\|_\varphi\,\|y\|_\psi\end{aligned}$$

for all $(x, y) \in M \times N$. □

Since every bounded bilinear form on the cartesian product of two JB*-triples admits a norm-preserving separately weak*-continuous extension to the cartesian product of the corresponding bidual spaces (cf. [25, Lemma 1]), Theorem 6.3 implies the following statement (restating of Theorem 1.1 from Introduction).

Theorem 6.4. *Suppose $G > 8(1+2\sqrt{3})$. Let E and B be JB*-triples. Then for every bounded bilinear form $V : E \times B \to \mathbb{C}$ there exist norm-one functionals $\varphi \in E^*$ and $\psi \in B^*$ satisfying*

$$|V(x,y)| \le G\ \|V\|\,\|x\|_\varphi\,\|y\|_\psi$$

for all $(x, y) \in E \times B$. □

Remark 6.5. The optimal values of the constants in question remain to be unknown. However, it seems that our method cannot give a better constant in Theorem 6.1. One factor $\sqrt{2}$ appears due to the use of Lemma 2.1 and a second factor $\sqrt{2}$ appears due to estimates of semi-norms $\|\cdot\|_{\varphi_1,\varphi_2}$ by a semi-norm generated by one functional. Let us consider a JBW*-triple represented as in Proposition 1.3. The individual summands have different behaviour.

(*i*) The JBW*-algebra N is covered by the already known Theorem 2.3.
(*ii*) The summand $p_1 V$ is covered by Corollary 3.4. This approach can be applied to N as well (note that Corollary 3.4 can be viewed as a generalization of Theorem 2.3).
(*iii*) The remaining summand, i.e.,

$$\left(\bigoplus_{k\in\Lambda}^{\ell_\infty} L^\infty(\mu_k, C_k)\right) \oplus^{\ell_\infty} p_2 V \oplus^{\ell_\infty} p_3 V,$$

has a special property. It follows from our arguments that in this case $\|\cdot\|_{\varphi_1,\varphi_2}$ attains its maximum on the unit ball for any two normal functionals φ_1, φ_2.

This analysis confirms that there are two basic tools – attaining the norm and some kind of order on tripotents.

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Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Technicka 2, 166 27, Prague 6, Czech Republic
E-mail address: hamhalte@math.feld.cvut.cz

Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 86, 186 75 Praha 8, Czech Republic
E-mail address: kalenda@karlin.mff.cuni.cz

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.
E-mail address: aperalta@ugr.es

Université d'Orléans, BP 6759, F-45067 Orléans Cedex 2, France
E-mail address: pfitzner@univ-orleans.fr