

# EMBEDDABILITY OF $\ell_p$ AND BASES IN LIPSCHITZ FREE $p$ -SPACES FOR $0 < p \leq 1$

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ABSTRACT. Our goal in this paper is to continue the study initiated by the authors in [3] of the geometry of the Lipschitz free  $p$ -spaces over quasimetric spaces for  $0 < p \leq 1$ , denoted  $\mathcal{F}_p(\mathcal{M})$ . Here we develop new techniques to show that, by analogy with the case  $p = 1$ , the space  $\ell_p$  embeds isomorphically in  $\mathcal{F}_p(\mathcal{M})$  for  $0 < p < 1$ . Going further we see that despite the fact that, unlike the case  $p = 1$ , this embedding need not be complemented in general, complementability of  $\ell_p$  in a Lipschitz free  $p$ -space can still be attained by imposing certain natural restrictions to  $\mathcal{M}$ . As a by-product of our discussion on bases in  $\mathcal{F}_p([0, 1])$ , we obtain the first-known examples of  $p$ -Banach spaces for  $p < 1$  that possess a basis but fail to have an unconditional basis.

## 1. INTRODUCTION

Given a pointed  $p$ -metric space  $\mathcal{M}$  ( $0 < p \leq 1$ ) it is possible to construct a unique  $p$ -Banach space  $\mathcal{F}_p(\mathcal{M})$  in such a way that  $\mathcal{M}$  embeds isometrically in  $\mathcal{F}_p(\mathcal{M})$ , and for every  $p$ -Banach space  $X$  and every Lipschitz map  $f: \mathcal{M} \rightarrow X$  that maps the base point  $0$  in  $\mathcal{M}$  to  $0 \in X$  there exists a unique linear map  $T_f: \mathcal{F}_p(\mathcal{M}) \rightarrow X$  with  $\|T_f\| = \text{Lip}(f)$ . The space  $\mathcal{F}_p(\mathcal{M})$  is known as the *Lipschitz free  $p$ -space* (or the *Arens-Eells  $p$ -space*) over  $\mathcal{M}$ . Lipschitz free  $p$ -spaces provide a canonical

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linearization process of Lipschitz maps between  $p$ -metric spaces: if we identify (through the isometric embedding  $\delta_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{F}_p(\mathcal{M})$ ) a  $p$ -metric space  $\mathcal{M}$  with a subset of  $\mathcal{F}_p(\mathcal{M})$ , then any Lipschitz map  $f$  from a  $p$ -metric space  $\mathcal{M}_1$  to a  $p$ -metric space  $\mathcal{M}_2$  which maps the base point in  $\mathcal{M}_1$  to the base point in  $\mathcal{M}_2$  extends to a continuous linear map  $L_f: \mathcal{F}_p(\mathcal{M}_1) \rightarrow \mathcal{F}_p(\mathcal{M}_2)$  and  $\|L_f\| = \text{Lip}(f)$ .

Lipschitz free  $p$ -spaces were introduced in [4], where they were used to provide for every for  $0 < p < 1$  a couple of *separable*  $p$ -Banach spaces which are Lipschitz-isomorphic without being linearly isomorphic. The study of the structure of the spaces  $\mathcal{F}_p(\mathcal{M})$ , however, has not been tackled until recently in [3], where we refer the reader for terminology and background. These spaces constitute a nice family of new  $p$ -Banach spaces which are easy to define but whose geometry seems to be difficult to understand. To carry out this task successfully one hopes to be able to count on “natural” structural results involving free  $p$ -spaces over subsets of  $\mathcal{M}$ . In [3, Section 6] we analyzed this premise and confirmed an unfortunate recurrent pattern in quasi-Banach spaces: the lack of tools can be an important stumbling block in the development of the theory. However, as we see here, not everything is lost and we still can develop specific methods that permit to shed light onto the geometry of  $\mathcal{F}_p(\mathcal{M})$ .

This paper is a continuation of the study initiated by the authors in [3]. Our aim is to delve deeper into the structure of this class of spaces and address very natural questions that arise by analogy with the case  $p = 1$ . Needless to say, the extension of such results is far from straightforward since the techniques used for metric and Banach spaces break down when the local convexity is lifted. As a consequence, our work provides a new view (and often also rather different proofs) of the structural results already known for the standard Lipschitz free spaces.

To that end, we start in § 2.1 with a method, which is an extension of known results for  $p = 1$  to the more general case of  $p \in (0, 1]$ , for constructing  $p$ -metric spaces by taking certain sums. The other two subsections of Section 2 contain results that are new even for the case when  $p = 1$ . In § 2.2 we address metric quotients. The main application here is probably that some Sobolev spaces are isometric to certain Lipschitz free spaces (see Theorem 2.11). In § 2.3 we describe the kernel of a projection induced by a Lipschitz retraction. This was already considered in [19, Proposition 1] for  $p = 1$ , but we obtain a different description using metric quotients (see Theorem 2.14).

It is known that  $\ell_1$  is isomorphic to a complemented subspace of  $\mathcal{F}(\mathcal{M})$  whenever  $\mathcal{M}$  is an infinite metric space (see [10] and [19]). This important structural property does not carry over, in general, to free

$p$ -spaces over quasimetric spaces when  $p < 1$ . Indeed,  $\ell_p$  (whose dual is  $\ell_\infty$ ) fails to be complemented in  $L_p[0, 1]$  (which is a Lipschitz free  $p$ -space by [3, Theorem 4.13]) since  $L_p[0, 1]^* = \{0\}$ . However, as we will see in Section 3, there are conditions on  $\mathcal{M}$  which ensure that  $\ell_p$  does embed complementably into  $\mathcal{F}_p(\mathcal{M})$  for every  $0 < p \leq 1$ . The most important case occurs when  $\mathcal{M}$  is not just any quasimetric space but an infinite metric space. Then  $\ell_p$  embeds complementably into  $\mathcal{F}_p(\mathcal{M})$ . We also extend the results from [19] concerning Lipschitz free  $p$ -spaces over different nets of a  $p$ -metric space (see Proposition 3.6).

The question we tackle in Section 4 is whether, by analogy with the case  $p = 1$ , we can guarantee that  $\ell_p$  embeds isomorphically in any  $\mathcal{F}_p(\mathcal{M})$  for  $0 < p < 1$ . The answer is positive, but in order to prove it we must develop a completely new set of techniques, specific of the nonlocally convex case (which, by the way, also work for  $p = 1$ ). Our results (see e.g. Theorem 4.3) provide new information even for the case of  $p = 1$ .

In the final, partially independent Section 5, we investigate Schauder bases in  $\mathcal{F}_p(\mathbb{N})$  and  $\mathcal{F}_p([0, 1])$ . One of the main results here is that  $\mathcal{F}_p([0, 1])$  has a Schauder basis. To the best of our knowledge, these provide the first examples of  $p$ -Banach spaces with a basis which do not have an unconditional basis, thus reinforcing the theoretical usefulness of Lipschitz-free  $p$ -spaces for  $p < 1$ .

Throughout this article we use standard notation in Banach space theory as can be found in [5]. We refer the reader to [18, 28] for basic facts on Lipschitz free spaces and some of their uses, and to [22] for background on quasi-Banach spaces.

## 2. BUILDING LIPSCHITZ-FREE $p$ -SPACES VIA SUMS, QUOTIENTS, AND RETRACTIONS

Let us start with an analogue to [19, Proposition 2] for  $p$ -metric spaces.

**Lemma 2.1.** *Let  $(\mathcal{M}, d, 0)$  be a pointed  $p$ -metric space,  $p \in (0, 1]$ . Suppose that  $(\mathcal{M}_\alpha)_{\alpha \in \Delta}$  is a family of subsets of  $\mathcal{M}$  such that  $0 \in \mathcal{M}_\alpha$  for every  $\alpha \in \Delta$  and  $(\mathcal{M}_\alpha \setminus \{0\})_{\alpha \in \Delta}$  is a partition  $\mathcal{M} \setminus \{0\}$ . Suppose further that there exists  $K \geq 1$  such that for all  $x \in \mathcal{M}_\alpha$  and  $y \in \mathcal{M}_\beta$  with  $\alpha, \beta \in \Delta$ ,  $\alpha \neq \beta$ , we have*

$$K^p d^p(x, y) \geq d^p(x, 0) + d^p(y, 0). \quad (2.1)$$

Then the map

$$T: \left( \bigoplus_{\alpha \in \Delta} \mathcal{F}_p(\mathcal{M}_\alpha) \right)_p \rightarrow \mathcal{F}_p(\mathcal{M}), \quad (x_\alpha)_{\alpha \in \Delta} \mapsto \sum_{\alpha \in \Delta} L_\alpha(x_\alpha),$$

where  $L_\alpha$  denotes the canonical map from  $\mathcal{F}_p(\mathcal{M}_\alpha)$  into  $\mathcal{F}_p(\mathcal{M})$ , is an onto isomorphism. Quantitatively,  $\|T\| \leq 1$  and  $\|T^{-1}\| \leq K$ .

In order to prove Lemma 2.1 we will use the notion of *p-norming set*. We say that a subset  $Z$  of a  $p$ -Banach space  $X$  is *p-norming* with constants  $c_1$  and  $c_2$  if  $c_1^{-1}Z$  is contained in the unit closed ball  $B_X$  of  $X$  and  $c_2^{-1}B_X$  is contained in the smallest absolutely  $p$ -convex closed set containing  $Z$ , denoted by  $\text{co}_p(Z)$  (see [3, § 2.3]).

*Proof of Lemma 2.1.* Let  $\delta_\alpha$  be the canonical embedding of  $\mathcal{M}_\alpha$  into  $(\bigoplus_{\alpha \in \Delta} \mathcal{F}_p(\mathcal{M}_\alpha))_p$ . By [3, Corollary 4.11], the set

$$\bigcup_{\alpha \in \Delta} \left\{ \frac{\delta_\alpha(y) - \delta_\alpha(x)}{d(x, y)} : x, y \in \mathcal{M}_\alpha, x \neq y \right\}$$

is an isometric  $p$ -norming set for  $(\bigoplus_{\alpha \in \Delta} \mathcal{F}_p(\mathcal{M}_\alpha))_p$ . Hence, by [3, Lemma 2.7], we must prove that

$$Z := \bigcup_{\alpha \in \Delta} \left\{ \frac{\delta_{\mathcal{M}}(y) - \delta_{\mathcal{M}}(x)}{d(x, y)} : x, y \in \mathcal{M}_\alpha, x \neq y \right\}$$

is a  $p$ -norming set for  $\mathcal{F}_p(\mathcal{M})$  with constants 1 and  $K$ . To that end, invoking again [3, Corollary 4.11], it suffices to prove that

$$Z_1 := \left\{ \frac{\delta_{\mathcal{M}}(y) - \delta_{\mathcal{M}}(x)}{d(x, y)} : x \in \mathcal{M}_\alpha, y \in \mathcal{M}_\beta, \alpha \neq \beta \right\} \subseteq K \text{co}_p(Z).$$

Let  $x, y \in \mathcal{M}$  and pick  $\alpha$  and  $\beta$  such that  $x \in \mathcal{M}_\alpha$  and  $y \in \mathcal{M}_\beta$ . We have

$$\frac{\delta_{\mathcal{M}}(y) - \delta_{\mathcal{M}}(x)}{d(x, y)} = \lambda \frac{\delta_{\mathcal{M}}(y)}{d(y, 0)} + \mu \frac{\delta_{\mathcal{M}}(x)}{d(x, 0)},$$

where

$$\lambda = \frac{d(y, 0)}{d(x, y)} \text{ and } \mu = -\frac{d(x, 0)}{d(x, y)}.$$

Since  $|\lambda|^p + |\mu|^p \leq K^p$ , we are done.  $\square$

**2.1. Sums of quasimetric spaces.** We now introduce a method inspired by Lemma 2.1 for building quasimetric spaces. Let  $(M_\alpha, d_\alpha, 0)_{\alpha \in \Delta}$  be a family of pointed  $p$ -metric spaces. Consider

$$\left( \bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha \right)_p = \left\{ (x_\alpha)_{\alpha \in \Delta} \in \prod_{\alpha \in \Delta} \mathcal{M}_\alpha : (d_\alpha(x_\alpha, 0))_{\alpha \in \Delta} \in \ell_p(\Delta) \right\}.$$

Since  $(d_\alpha(x_\alpha^1, x_\alpha^2))_{\alpha \in \Delta} \in \ell_p(\Delta)$  whenever  $(x_\alpha^i)_{\alpha \in \Delta} \in (\bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha)_p$ ,  $i = 1, 2$ , we can safely define a  $p$ -distance  $d$  on  $(\bigoplus_{n=1}^\infty \mathcal{M}_n)_p$  by

$$d((x_\alpha^1)_{\alpha \in \Delta}, (x_\alpha^2)_{\alpha \in \Delta}) = \left( \sum_{\alpha \in \Delta} d_\alpha^p(x_\alpha^1, x_\alpha^2) \right)^{1/p}.$$

The base point of  $((\bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha)_p, d)$  will be the element whose entries are the base points of each  $p$ -metric space  $\mathcal{M}_\alpha$ . We consider the subset

$$\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha = \{(x_\alpha)_{\alpha \in \Delta} \in \prod_{\alpha \in \Delta} \mathcal{M}_\alpha : |\{\alpha : x_\alpha \neq 0\}| \leq 1\}.$$

of the pointed  $p$ -metric space  $((\bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha)_p, d, 0)$ . If  $\mathcal{M}_\alpha = \mathcal{M}$  for every  $\alpha \in \Delta$  we put  $(\bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha)_p = \ell_p(\mathcal{M}, \Delta)$  and  $\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha = \mathfrak{X}(\mathcal{M}, \Delta)$  (respectively,  $\ell_p(\mathcal{M})$  and  $\mathfrak{X}(\mathcal{M})$  in the case when  $\Delta = \mathbb{N}$ ). If  $\Delta$  is finite (for instance  $\Delta = \{a, b\}$ ) we write  $(\bigoplus_{\alpha \in \Delta} \mathcal{M}_\alpha)_p = \mathcal{M}_a \oplus \mathcal{M}_b$  and  $\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha = \mathcal{M}_a \mathfrak{X} \mathcal{M}_b$ .

The spaces  $\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha$  were considered in [15] in the metric space setting, although its authors preferred to use the (equivalent) supremum norm instead of the  $\ell_1$ -norm to combine the spaces. Notice that the canonical embedding  $I_\alpha$  of  $\mathcal{M}_\alpha$  into  $\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha$  is an isometry, that  $(I_\alpha(\mathcal{M}_\alpha \setminus \{0\}))_{\alpha \in \Delta}$  is a partition of  $\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha \setminus \{0\}$  and that, if  $\alpha \neq \beta$  and  $x, y \in \mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha$ ,

$$d(I_\alpha(x), I_\beta(y)) = (d^p(I_\alpha(x), 0) + d^p(I_\beta(y), 0))^{1/p}.$$

Hence, Lemma 2.1 immediately gives the following.

**Lemma 2.2.** *Let  $(\mathcal{M}_\alpha, d_\alpha, 0)_{\alpha \in \Delta}$  be a family of pointed  $p$ -metric spaces. Then*

$$\mathcal{F}_p(\mathfrak{X}_{\alpha \in \Delta} \mathcal{M}_\alpha) \simeq \left( \bigoplus_{\alpha \in \Delta} \mathcal{F}_p(\mathcal{M}_\alpha) \right)_p.$$

To be precise, the canonical map  $L_{\mathfrak{X}}$  given by

$$(x_\alpha)_{\alpha \in \Delta} \mapsto \sum_{\alpha \in \Delta} L_{I_\alpha}(x_\alpha)$$

is an isometry.

Let us next present a few applications of Lipschitz-free  $p$ -spaces over Banach spaces of continuous functions.

**Proposition 2.3.** *For every  $0 < p \leq 1$  we have  $\mathcal{F}_p(c_0) \simeq \ell_p(\mathcal{F}_p(c_0))$ .*

*Proof.* We just need to mimic the proof of [15, Proposition 4] with the aid of Lemma 2.2.  $\square$

In [15], Dutrioux and Ferenczi provided an example of two non-isomorphic Banach spaces whose corresponding Lipschitz free spaces are isomorphic. The following result is a strengthening of the main result from [15], in the sense that we make it extensive to Lipschitz free  $p$ -spaces for  $0 < p < 1$ . Like in [15], our approach uses the well-known fact from the classical linear theory that if  $K$  is an uncountable compact metric space then  $\mathcal{C}(K)$  and  $c_0$  are not Lipschitz isomorphic (see [5, Theorem 4.5.2] and [17]).

**Theorem 2.4.** *Let  $K$  be any infinite compact metric space. Then for every  $0 < p \leq 1$  we have*

$$\mathcal{F}_p(\mathcal{C}(K)) \simeq \mathcal{F}_p(c_0)$$

(with uniformly bounded Banach-Mazur distance).

*Proof.* On the one hand  $c_0$  is complemented in  $\mathcal{C}(K)$  (see e.g. [5, Corollary 2.5.9]) so, in particular,  $c_0$  is a Lipschitz retract of  $\mathcal{C}(K)$ . On the other hand, since  $\mathcal{C}(K)$  is Lipschitz isomorphic to a subspace of  $c_0$  by [1], and  $\mathcal{C}(K)$  is an absolute Lipschitz retract (see [24, Theorem 6]) it follows that  $\mathcal{C}(K)$  is a Lipschitz retract of  $c_0$ . Then, by [3, Lemma 4.19],  $\mathcal{F}_p(c_0)$  is complemented in  $\mathcal{F}_p(\mathcal{C}(K))$  and  $\mathcal{F}_p(\mathcal{C}(K))$  is complemented in  $\mathcal{F}_p(c_0)$ . Now, taking into account Proposition 2.3, Pełczyński's decomposition method yields  $\mathcal{F}_p(\mathcal{C}(K)) \simeq \mathcal{F}_p(c_0)$ . Finally, we note that since all constants involved are independent of the compact space  $K$ , so is the isomorphism constant.  $\square$

If  $\mathcal{N}$  is a subset of a metric space  $\mathcal{M}$  and  $j$  denotes the inclusion from  $\mathcal{N}$  into  $\mathcal{M}$ , the canonical linear map  $L_j: \mathcal{F}(\mathcal{N}) \rightarrow \mathcal{F}(\mathcal{M})$  is an isometric embedding. Although this result does not carry over, in general, to Lipschitz free  $p$ -spaces for  $p < 1$  (see [3, Theorem 6.1]), it is an open question whether  $L_j: \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is always an *isomorphism* for  $p < 1$ . Lemma 2.5 sheds some light onto this matter.

**Lemma 2.5.** *Let  $0 < p \leq q \leq 1$ . We have the following dichotomy: Either there is a  $q$ -metric space  $\mathcal{M}$  and a subset  $\mathcal{N}$  of  $\mathcal{M}$  for which  $L_j: \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is not an isomorphism, or there is a universal constant  $C$  (depending only on  $p$  and  $q$ ) such that  $\|L_j^{-1}\| \leq C$  for every  $q$ -metric space  $\mathcal{M}$  and every subset  $\mathcal{N}$  of  $\mathcal{M}$ .*

*Proof.* Assume, by contradiction, that the lemma fails to be true. Then  $L_j: \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is always an isomorphism, while  $\|L_j^{-1}\|$  can be arbitrarily large. Thus there is a sequence  $(\mathcal{M}_n)_{n=1}^\infty$  of  $q$ -metric spaces and a sequence  $(\mathcal{N}_n)_{n=1}^\infty$ , where each  $\mathcal{N}_n$  is a subset of  $\mathcal{M}_n$ , such that, if  $L_n$  denotes the canonical linear map from  $\mathcal{F}_p(\mathcal{N}_n)$  into  $\mathcal{F}_p(\mathcal{M}_n)$  then  $\sup_n \|L_n^{-1}\| = \infty$ .

The  $q$ -metric space  $\mathcal{N} := \boxtimes_{n=1}^{\infty} \mathcal{N}_n$  is a subset of the  $q$ -metric space  $\mathcal{M} := \boxtimes_{n=1}^{\infty} \mathcal{M}_n$ . By assumption, the canonical map from  $\mathcal{F}_p(\mathcal{N})$  into  $\mathcal{F}_p(\mathcal{M})$  is an isomorphism. Then, by Lemma 2.2, there is an isomorphic embedding  $L_\omega$  from  $\bigoplus_{n=1}^{\infty} \mathcal{F}_p(\mathcal{N}_n)$  into  $\bigoplus_{n=1}^{\infty} \mathcal{F}_p(\mathcal{M}_n)$  given by

$$(x_n)_{n=1}^{\infty} \mapsto (L_n(x_n))_{n=1}^{\infty}.$$

Hence,  $\sup_n \|L_n^{-1}\| \leq \|L_\omega^{-1}\| < \infty$ , which is an absurdity.  $\square$

**2.2. Quotients of quasimetric spaces.** Suppose that  $(\mathcal{M}, d, 0)$  is a pointed  $p$ -metric space,  $0 < p \leq 1$ , and that  $\mathcal{N} \subset \mathcal{M}$  is a closed subset of  $\mathcal{M}$  with  $0 \in \mathcal{N}$ . Following [7] we put

$$d_{\mathcal{M}/\mathcal{N}}(x, y) := \min\{d(x, y), (d^p(x, \mathcal{N}) + d^p(y, \mathcal{N}))^{1/p}\}, \quad x, y \in \mathcal{M}.$$

It is clear that  $d_{\mathcal{M}/\mathcal{N}}(\cdot, \cdot)$  is symmetric and that  $d_{\mathcal{M}/\mathcal{N}}(x, y) = 0$  if and only if either  $x = y$  or  $x, y \in \mathcal{N}$ . Moreover, it is straightforward to check that  $d_{\mathcal{M}/\mathcal{N}}$  satisfies the  $p$ -triangle law. Hence  $((\mathcal{M} \setminus \mathcal{N}) \cup \{0\}, d_{\mathcal{M}/\mathcal{N}}, 0)$  is a pointed  $p$ -metric space, which we denote by  $\mathcal{M}/\mathcal{N}$  and that we call the *quotient of  $\mathcal{M}$  by  $\mathcal{N}$* .

Let  $Q_{\mathcal{M}/\mathcal{N}}: \mathcal{M} \rightarrow \mathcal{M}/\mathcal{N}$  be the *quotient map* given by

$$Q_{\mathcal{M}/\mathcal{N}}(x) = \begin{cases} x & \text{if } x \in \mathcal{M} \setminus \mathcal{N}, \\ 0 & \text{if } x \in \mathcal{N}. \end{cases}$$

We start our study of quotient spaces with some elementary properties.

**Proposition 2.6.** *Suppose that  $(\mathcal{M}, d, 0)$  is a pointed  $p$ -metric space,  $0 < p \leq 1$ , and that  $\mathcal{N} \subseteq \mathcal{M}$  is a closed subset of  $\mathcal{M}$  with  $0 \in \mathcal{N}$ . Given another  $p$ -metric space  $X$ , the map  $T: \text{Lip}_0(\mathcal{M}/\mathcal{N}, X) \rightarrow \text{Lip}_0(\mathcal{M}, X)$  defined by  $T(f) = f \circ Q_{\mathcal{M}/\mathcal{N}}$  is an isometric embedding with range*

$$\text{Lip}_{\mathcal{N}}(\mathcal{M}, X) := \{f \in \text{Lip}(\mathcal{M}, X) : f|_{\mathcal{N}} = 0\}.$$

Moreover, if  $X$  is  $p$ -normed,  $T$  is linear.

*Proof.* It is clear that if  $f \in \text{Lip}_0(\mathcal{M}/\mathcal{N}, X)$ ,  $f \circ Q_{\mathcal{M}/\mathcal{N}} \in \text{Lip}_{\mathcal{N}}(\mathcal{M}, X)$  and  $\text{Lip}(f \circ Q_{\mathcal{M}/\mathcal{N}}) \leq \text{Lip}(f)$ . Let  $g: \mathcal{M} \rightarrow X$  be a  $C$ -Lipschitz map with  $g(x) = 0$  for all  $x \in \mathcal{N}$ . Then, for  $x \in \mathcal{M}$ ,

$$d(g(x), 0) = \inf_{y \in \mathcal{N}} d(g(x), g(y)) \leq C \inf_{y \in \mathcal{N}} d(x, y) = Cd(x, \mathcal{N}).$$

Consequently, for all  $x, y \in \mathcal{M}$ ,

$$d(g(x), g(y)) \leq (d^p(g(x), 0) + d^p(g(y), 0))^{1/p} \leq C(d^p(x, \mathcal{N}) + d^p(y, \mathcal{N}))^{1/p}.$$

Hence  $d(g(x), g(y)) \leq Cd_{\mathcal{M}/\mathcal{N}}(x, y)$ . So, if  $f: \mathcal{M}/\mathcal{N} \rightarrow X$  is the unique map satisfying  $f \circ Q_{\mathcal{M}/\mathcal{N}} = g$ ,  $f$  is  $C$ -Lipschitz.  $\square$

Proposition 2.6 allows us to identify  $q$ -metric envelopes of quotients of  $p$ -metric spaces for  $0 < p < q \leq 1$ . We refer the reader to [3, Section 3] for a precise definition of metric envelopes, a concept that arises most naturally while investigating Lipschitz free  $p$ -spaces over quasimetric spaces.

**Corollary 2.7.** *Let  $0 < p \leq q \leq 1$ . Suppose  $(\mathcal{M}, d, 0)$  is a pointed  $p$ -metric space and that  $\mathcal{N} \subseteq \mathcal{M}$  is a closed subset of  $\mathcal{M}$  with  $0 \in \mathcal{N}$ . Let  $\mathcal{N}^q$  be the closure of  $J_q(\mathcal{N})$  in the  $q$ -metric envelope  $(\widetilde{\mathcal{M}}^q, J_q)$  of  $\mathcal{M}$ . Then the  $q$ -metric envelope of  $\mathcal{M}/\mathcal{N}$  is  $(\widetilde{\mathcal{M}}^q/\mathcal{N}^q, J_{q,\mathcal{N}})$ , where  $J_{q,\mathcal{N}}$  is the unique map for which the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{J_q} & \widetilde{\mathcal{M}}^q \\ Q_{\mathcal{M}/\mathcal{N}} \downarrow & & \downarrow Q_{\widetilde{\mathcal{M}}^q/\mathcal{N}^q} \\ \mathcal{M}/\mathcal{N} & \xrightarrow{J_{q,\mathcal{N}}} & \widetilde{\mathcal{M}}^q/\mathcal{N}^q \end{array} \quad (2.2)$$

*commutes.*

*Proof.* Proposition 2.6 applied to  $Q_{\widetilde{\mathcal{M}}^q/\mathcal{N}^q} \circ J_q \in \text{Lip}_{\mathcal{N}}(\mathcal{M}, \widetilde{\mathcal{M}}^q/\mathcal{N}^q)$  gives the existence and uniqueness of the map  $J_{q,\mathcal{N}}$  with  $\text{Lip}(J_{q,\mathcal{N}}) = 1$  that makes the diagram (2.2) commutative. Let  $f: \mathcal{M}/\mathcal{N} \rightarrow X$  be a  $C$ -Lipschitz map into a  $q$ -metric space. If we set  $f(0)$  as the base point of  $X$ , applying first the universal property of  $q$ -metric envelopes and then that of quotients, we obtain unique  $C$ -Lipschitz maps  $g$  and  $h$  such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{J_q} & \widetilde{\mathcal{M}}^q \\ Q_{\mathcal{M}/\mathcal{N}} \downarrow & & \downarrow Q_{\widetilde{\mathcal{M}}^q/\mathcal{N}^q} \\ \mathcal{M}/\mathcal{N} & & \widetilde{\mathcal{M}}^q/\mathcal{N}^q \\ f \downarrow & \swarrow g & \searrow h \\ X & & \end{array} \quad (2.3)$$



commutes. Using that  $Q_{\mathcal{M}/\mathcal{N}}$  is onto we deduce that (2.2) and (2.3) merge in the commutative diagram

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{J_q} & \widetilde{\mathcal{M}}^q \\
 Q_{\mathcal{M}/\mathcal{N}} \downarrow & & \downarrow Q_{\widetilde{\mathcal{M}}^q/\mathcal{N}^q} \\
 \mathcal{M}/\mathcal{N} & \xrightarrow{J_{q,\mathcal{N}}} & \widetilde{\mathcal{M}}^q/\mathcal{N}^q \\
 f \downarrow & \swarrow h & \\
 X & & 
 \end{array} \tag{2.4}$$

The uniqueness of  $g$  and  $h$  in (2.3) gives that  $h$  is unique in (2.4).  $\square$

**Corollary 2.8.** *Let  $0 < p \leq q \leq 1$ . Suppose that  $(\mathcal{M}, d, 0)$  is a pointed  $p$ -metric space and that  $\mathcal{N} \subseteq \mathcal{M}$  is a closed subset of  $\mathcal{M}$  with  $0 \in \mathcal{N}$ . Then the  $q$ -Banach envelope of  $\mathcal{F}_p(\mathcal{M}/\mathcal{N})$  is  $\mathcal{F}_q(\widetilde{\mathcal{M}}^q/\mathcal{N}^q)$ .*

*Proof.* Just combine Corollary 2.7 with [3, Proposition 4.20].  $\square$

**Corollary 2.9.** *Let  $0 < p \leq 1$ . Suppose that  $(\mathcal{M}, d, 0)$  is a pointed  $p$ -metric space and that  $\mathcal{N} \subseteq \mathcal{M}$  is a closed subset of  $\mathcal{M}$  with  $0 \in \mathcal{N}$ . Then the dual space of  $\overline{\mathcal{F}_p}(\mathcal{M}/\mathcal{N})$  is  $\text{Lip}_{\mathcal{N}}(\mathcal{M}, \mathbb{R})$  under the dual pairing given by  $\langle f, \delta_{\mathcal{M}/\mathcal{N}}(x) \rangle = f(x)$  for all  $f \in \text{Lip}_{\mathcal{N}}(\mathcal{M}, \mathbb{R})$  and all  $x \in \mathcal{M} \setminus \mathcal{N}$ .*

*Proof.* Just combine Proposition 2.6 with [3, Proposition 4.23].  $\square$

Lipschitz free spaces over quotients yield an alternative description of some Sobolev spaces. Prior to state and prove this result, we write down an elementary functional lemma that we will need. We omit the proof.

**Lemma 2.10.** *Let  $X$  and  $Y$  be Banach spaces,  $X_0 \subseteq X$  be a dense subspace and  $T: X_0 \rightarrow Y$  be a linear map. Assume that there is an isomorphism  $S: Y^* \rightarrow X^*$  such that  $S(y^*)(x) = y^*(T(x))$  for every  $y^* \in Y^*$  and  $x \in X_0$ . Then  $T$  extends to an isomorphism from  $X$  onto  $Y$ . Moreover, if  $S$  is an isometry, so is  $T$ .*

For the definition of the Sobolev space  $W^{-1,1}(U)$  for open bounded subset of  $\mathbb{R}^d$  we refer to [26, Definitions 2.2 and 2.3].

**Theorem 2.11.** *Let  $d \in \mathbb{N}$  and  $U \subseteq \mathbb{R}^d$  be an open bounded set. Then  $\mathcal{F}(\mathbb{R}^d/U^c) \simeq W^{-1,1}(U)$  isometrically.*

*Proof.* Let

$$\langle \cdot, \cdot \rangle_0: \text{Lip}_{UC}(\mathbb{R}^d) \times \mathcal{F}(\mathbb{R}^d/U^c) \rightarrow \mathbb{R}$$

be the dual pairing between  $\text{Lip}_{U^c}(\mathbb{R}^d)$  and  $\mathcal{F}(\mathbb{R}^d/U^c)$  provided by Corollary 2.9, and let  $\delta(x)$  denote the Dirac measure on  $U$  at the point  $x \in U$ . It is known (see [26, Proposition 8.7]) that the dual of the Sobolev space  $W^{-1,1}(U)$  is isometric to  $\text{Lip}_{U^c}(\mathbb{R}^d)$  and that, if  $\langle \cdot, \cdot \rangle_1$  denotes the associated dual pairing, we have  $\langle f, \delta(x) \rangle_1 = f(x)$  for every  $f \in \text{Lip}_{U^c}(\mathbb{R}^d)$  and every  $x \in U$ . Therefore, if we define the linear map

$$T: \mathcal{P}(\mathbb{R}^d/U^c) \rightarrow W^{-1,1}(U), \quad \delta_{\mathbb{R}^d/U^c}(x) \mapsto \delta(x),$$

we have  $\langle f, \mu \rangle_0 = \langle f, T(\mu) \rangle_1$ . By Lemma 2.10,  $T$  extends to an isometry from  $\mathcal{F}(\mathbb{R}^d/U^c)$  onto  $W^{-1,1}(U)$ .  $\square$

*Remark 2.12.* After consulting with experts on the field, we found out that the literature is not unified concerning the right definition of the Sobolev space  $W^{-1,1}$ . We used the definition from [26], because in this case  $W^{-1,1}$  is a canonical predual of  $\text{Lip}_U(\mathbb{R}^d, \mathbb{R})$ . In general, the proof above shows that any “natural predual” of the space  $\text{Lip}_U(\mathbb{R}^d, \mathbb{R})$  is isometric to  $\mathcal{F}(\mathbb{R}^d/U^c)$ .

Since  $\mathcal{F}(\mathcal{M}) \simeq \mathcal{F}(\mathbb{R}^n)$  whenever  $\mathcal{M} \subset \mathbb{R}^d$  has nonempty interior (see [23, Corollary 3.5]) the following question, which could be of interest for experts working with Sobolev spaces, arises.

*Question 2.13.* Let  $n \in \mathbb{N}$ ,  $B$  be an open unit ball in  $\mathbb{R}^n$ , and  $U \subseteq \mathbb{R}^n$  be a bounded open set. Are the Lipschitz free spaces over metric quotients  $\mathbb{R}^d/U^c$  and  $\mathbb{R}^d/B^c$  isomorphic?

**2.3. Lipschitz retractions.** It is known that if  $\mathcal{N}$  is a Lipschitz retract of  $\mathcal{M}$  then the space  $\mathcal{F}_p(\mathcal{N})$  is a complemented subspace of  $\mathcal{F}_p(\mathcal{M})$  via the canonical linear map from  $\mathcal{F}_p(\mathcal{N})$  into  $\mathcal{F}_p(\mathcal{M})$  (see [3, Lemma 4.19]). The following result identifies a complement of  $\mathcal{F}_p(\mathcal{N})$  in  $\mathcal{F}_p(\mathcal{M})$ .

**Theorem 2.14.** *Let  $(\mathcal{M}, d, 0)$  be a pointed  $p$ -metric space,  $0 < p \leq 1$ , and  $\mathcal{N} \subseteq \mathcal{M}$  be a Lipschitz retract. Then*

$$\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N}) \oplus \mathcal{F}_p(\mathcal{M}/\mathcal{N}).$$

*Proof.* Let  $r: \mathcal{M} \rightarrow \mathcal{M}$  be a Lipschitz retraction with  $r(\mathcal{M}) = \mathcal{N}$  and set  $L = \text{Lip}(r)$ . Let  $f: \mathcal{M} \rightarrow \mathcal{F}_p(\mathcal{N} \boxtimes \mathcal{M}/\mathcal{N})$  be defined as

$$f(x) = \begin{cases} \delta_{\boxtimes}(r(x), 0) + \delta_{\boxtimes}(0, x) & \text{if } x \in \mathcal{M} \setminus \mathcal{N}, \\ \delta_{\boxtimes}(x, 0) & \text{if } x \in \mathcal{N}, \end{cases}$$

where  $\delta_{\mathfrak{X}}$  denotes the natural isometric embedding of  $\mathcal{N}\bowtie\mathcal{M}/\mathcal{N}$  into  $\mathcal{F}_p(\mathcal{N}\bowtie\mathcal{M}/\mathcal{N})$ . Since  $r(x) = x$  for  $x \in \mathcal{N}$  we obtain

$$\|f(x) - f(y)\|^p \leq \begin{cases} A^p(x, y) + B^p(x, y) & \text{if } x, y \in \mathcal{M} \setminus \mathcal{N}, \\ A^p(x, y) & \text{if } x, y \in \mathcal{N}, \\ A^p(x, y) + B^p(x, 0) & \text{if } x \in \mathcal{M} \setminus \mathcal{N} \text{ and } y \in \mathcal{N}, \end{cases}$$

where  $A(x, y) = d(r(x), r(y))$  and  $B(x, y) = d_{\mathcal{M}/\mathcal{N}}(x, y)$ . Since, for all  $x, y \in \mathcal{M}$ ,

$$A(x, y) \leq L d(x, y), \quad B(x, y) \leq d(x, y),$$

and, if  $y \in \mathcal{N}$ ,

$$B(x, 0) \leq d(x, \mathcal{N}) \leq d(x, y),$$

the function  $f$  is  $(L^p + 1)^{1/p}$ -Lipschitz. So, by [3, Theorem 4.5], there is a linear operator  $T: \mathcal{F}_p(\mathcal{M}) \rightarrow \mathcal{F}_p(\mathcal{N}\bowtie\mathcal{M}/\mathcal{N})$  such that  $\|T\| \leq (L^p + 1)^{1/p}$  and  $T \circ \delta_{\mathcal{M}} = f$ .

Conversely, we define a map  $g: \mathcal{N}\bowtie\mathcal{M}/\mathcal{N} \rightarrow \mathcal{F}_p(\mathcal{M})$  as

$$g(x_1, x_2) = \begin{cases} \delta_{\mathcal{M}}(x_1) & \text{if } x_2 = 0, \\ \delta_{\mathcal{M}}(x_2) - \delta_{\mathcal{M}}(r(x_2)) & \text{if } x_2 \neq 0. \end{cases}$$

Let  $(x_1, x_2), (y_1, y_2) \in \mathcal{N}\bowtie\mathcal{M}/\mathcal{N}$ . Since  $y_1 = 0$  if  $y_2 \neq 0$ , we deduce that  $\|g(x_1, x_2) - g(y_1, y_2)\|^p$  is bounded above by

$$\begin{cases} d^p(x_1, y_1) & \text{if } x_2 = y_2 = 0, \\ d^p(x_2, y_2) + A^p(x_2, y_2) & \text{if } x_2, y_2 \neq 0, \\ d^p(x_1, y_1) + d^p(x_2, y_2) + A^p(x_2, y_2) & \text{if } x_2 = 0, y_2 \neq 0. \end{cases}$$

Hence for all  $(x_1, y_1)$  and  $(x_2, y_2) \in \mathcal{N}\bowtie\mathcal{M}/\mathcal{N}$  we have

$$\begin{aligned} \|g(x_1, x_2) - g(y_1, y_2)\|^p &\leq (1 + L^p)(d^p(x_1, y_1) + d^p(x_2, y_2)) \\ &\leq (1 + L^p)(d^p(x_1, y_1) + d_{\mathcal{M}/\mathcal{N}}^p(x_2, y_2)). \end{aligned}$$

So, there is a linear map  $S: \mathcal{F}_p(\mathcal{N}\bowtie\mathcal{M}/\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  such that  $S \circ \delta_{\mathfrak{X}} = g$  and  $\|S\| \leq (1 + L^p)^{1/p}$ .

Finally, it is easy to verify that  $S \circ T \circ \delta_{\mathcal{M}} = \delta_{\mathcal{M}}$  and  $S \circ T \circ \delta_{\mathcal{N}\bowtie\mathcal{M}/\mathcal{N}} = \delta_{\mathcal{N}\bowtie\mathcal{M}/\mathcal{N}}$ , and so  $S \circ T = \text{Id}_{\mathcal{F}_p(\mathcal{M})}$  and  $S \circ T = \text{Id}_{\mathcal{F}_p(\mathcal{N}\bowtie\mathcal{M}/\mathcal{N})}$ .

Summarizing and appealing to Lemma 2.2 we obtain

$$\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N}\bowtie\mathcal{M}/\mathcal{N}) \simeq \mathcal{F}_p(\mathcal{N}) \oplus \mathcal{F}_p(\mathcal{M}/\mathcal{N}). \quad \square.$$

Let us mention an application which we will use later. Recall that a subset  $\mathcal{N}$  of quasimetric space  $(\mathcal{M}, d)$  is said to be a *net* if

$$\inf\{d(x, y) : x, y \in \mathcal{N}, x \neq y\} > 0 \quad \text{and} \quad \sup_{x \in \mathcal{M}} d(x, \mathcal{N}) < \infty.$$

**Corollary 2.15.** *Suppose  $\mathcal{M}$  is a uniformly separated  $p$ -metric space,  $0 < p \leq 1$ , and that  $\mathcal{N} \subseteq \mathcal{M}$  is a net. Then*

$$\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N}) \oplus \ell_p(|\mathcal{M} \setminus \mathcal{N}|).$$

*Proof.* Given  $x \in \mathcal{M} \setminus \mathcal{N}$  let  $z(x) \in \mathcal{N}$  such that  $d(x, z(x)) \leq 2d(x, \mathcal{N})$ . The map

$$x \mapsto \begin{cases} x & \text{if } x \in \mathcal{N} \\ z(x), & \text{if } x \notin \mathcal{N}. \end{cases}$$

is a Lipschitz retract of  $\mathcal{M}$  onto  $\mathcal{N}$ . Indeed, if  $x$  and  $y$  are different points in  $\mathcal{M} \setminus \mathcal{N}$ ,

$$d^p(z(x), z(y)) \leq d^p(x, y) + 2^p(d^p(x, \mathcal{N}) + d^p(y, \mathcal{N})) \leq \left(1 + 2^p \frac{R^p}{\varepsilon^p}\right) d^p(x, y),$$

where  $R = \sup_{x \in \mathcal{M}} d(x, \mathcal{N})$  and  $\varepsilon = \inf\{d(x, y) : x, y \in \mathcal{M}, x \neq y\} > 0$ . By Theorem 2.14,

$$\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N}) \oplus \mathcal{F}_p(\mathcal{M}/\mathcal{N}).$$

Since  $\mathcal{M}/\mathcal{N}$  is uniformly separated and bounded, [3, Theorem 4.14] yields  $\mathcal{F}_p(\mathcal{M}/\mathcal{N}) \simeq \ell_p(|\mathcal{M} \setminus \mathcal{N}|)$ .  $\square$

### 3. COMPLEMENTABILITY OF $\ell_p$ IN $\mathcal{F}_p(\mathcal{M})$ FOR $0 < p \leq 1$ AND LIPSCHITZ FREE $p$ -SPACES OVER NETS

Our main result on complementability of  $\ell_p$  in Lipschitz-free  $p$ -spaces is the following generalization of [10, Theorem 1.1] and [19, Proposition 3]. We will obtain Theorem 3.1 from Theorems 3.8 and 3.9 below. Given a topological space  $\mathcal{M}$ ,  $\text{dens } \mathcal{M}$  will denote the density character of  $\mathcal{M}$ , i.e., the minimal cardinality of a dense subset of  $\mathcal{M}$ .

**Theorem 3.1.** *Let  $p \in (0, 1]$ . Suppose that  $(\mathcal{M}, d)$  is either*

- (a) *a metric space, or*
- (b) *a  $p$ -metric space containing  $\text{dens } \mathcal{M}$ -many isolated points.*

*Then for every  $C > 2^{1/p}$ ,  $\ell_p(\text{dens } \mathcal{M})$  is  $C$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ .*

We do not attempt to achieve an optimal quantitative estimate here, but what we consider interesting is that it does not depend on the space  $\mathcal{M}$ . Let us note that this seems to be a new result even for  $p = 1$  and nonseparable metric spaces. For separable metric spaces and  $p = 1$ , quantitative estimates other than  $C = 1$  are not important because by [14], whenever a Banach space  $X$  has a complemented subspace isomorphic to  $\ell_1$ , then for each  $\varepsilon > 0$  it has a  $(1 + \varepsilon)$ -complemented subspace  $(1 + \varepsilon)$ -isomorphic to  $\ell_1$ .

Sometimes it is even possible to have a more precise information about the complemented copy of  $\ell_p$  in the space  $\mathcal{F}_p(\mathcal{M})$  at the cost of losing the quantitative estimate. For instance, if  $\mathcal{M}$  is a uniformly separated infinite  $p$ -metric space containing a net  $\mathcal{N}$  with  $|\mathcal{M} \setminus \mathcal{N}| = |\mathcal{M}|$ , the proof of Corollary 2.15 allows us to identify the complemented copy of  $\ell_p(\mathcal{M})$  inside  $\mathcal{F}_p(\mathcal{M})$  as a Lipschitz free  $p$ -space on a quotient space. In the following result, which seems to be new even for the case of  $p = 1$ , we identify the aforementioned copy of  $\ell_p$  inside  $\mathcal{F}_p(\mathcal{M})$  as a Lipschitz free  $p$ -space on a subset of  $\mathcal{M}$ .

**Theorem 3.2.** *Let  $p \in (0, 1]$ . Suppose that  $(\mathcal{M}, d)$  is either*

- (a) *a metric space, or*
- (b) *a uniformly separated uncountable  $p$ -metric space.*

*Then there exists  $\mathcal{N} \subset \mathcal{M}$  such that*

- (i)  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p$ ,
- (ii)  $L_j: \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  *is an isomorphic embedding and*
- (iii)  $L_j(\mathcal{F}_p(\mathcal{N}))$  *is complemented in  $\mathcal{F}_p(\mathcal{M})$ .*

Before tackling the proof of Theorems 3.1 and 3.2, let us highlight some interesting applications.

**Corollary 3.3.** *Let  $0 < p \leq 1$ . There is a constant  $C$  such that for every  $n \in \mathbb{N}$  and every infinite metric space  $\mathcal{M}$ ,*

$$\mathcal{F}_p(\mathcal{M}) \simeq_C \mathcal{F}_p(\mathcal{M}) \oplus \ell_p^n \simeq_C \mathcal{F}_p(\mathcal{M}) \oplus \ell_p.$$

**Proposition 3.4.** *Let  $(\mathcal{M}, d)$  be an infinite metric space and  $0 < p \leq 1$ . Then  $\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{M} \setminus \{x_0\})$  for every  $x_0 \in \mathcal{M}$ .*

*Proof.* If  $r = \inf_{x \in \mathcal{M} \setminus \{x_0\}} d(x_0, x) = 0$  the result follows from [3, Proposition 4.17]. Assume that  $r > 0$  and pick an arbitrary point  $0 \in \mathcal{M} \setminus \{x_0\}$ . If  $x \in \mathcal{M} \setminus \{x_0\}$  we have

$$d^p(0, x_0) + d^p(0, x) \leq 2d^p(0, x_0) + d^p(x_0, x) \leq \left(1 + 2\frac{d^p(0, x_0)}{r^p}\right) d^p(x_0, x).$$

Then, by Lemma 2.1 and Corollary 3.3,

$$\begin{aligned} \mathcal{F}_p(\mathcal{M}) &\simeq \mathcal{F}_p(\mathcal{M} \setminus \{x_0\}) \oplus \mathcal{F}_p(\{0, x_0\}) \\ &\simeq \mathcal{F}_p(\mathcal{M} \setminus \{x_0\}) \oplus \mathbb{R} \\ &\simeq \mathcal{F}_p(\mathcal{M} \setminus \{x_0\}). \end{aligned} \quad \square$$

Further, we obtain the following extensions of [19, Theorem 4 and Proposition 5] to the whole range of values of  $p \in (0, 1]$ . Recall that there are examples of non-Lipschitz isomorphic equivalent nets in  $\mathbb{R}^2$  (see e.g. [8], [25] or [6, p. 242]).

**Theorem 3.5.** *Let  $0 < p \leq 1$ . Suppose that  $\mathcal{M}$  is a uniformly separated  $p$ -metric space and that  $\mathcal{N} \subseteq \mathcal{M}$  is a net such that  $|\mathcal{N}| = |\mathcal{M}|$ . Then  $\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N})$ .*

*Proof.* By Theorem 3.1, we have  $\mathcal{F}_p(\mathcal{N}) \simeq X \oplus \ell_p(|\mathcal{N}|)$  for some  $p$ -Banach space  $X$ . Then, appealing to Theorem 2.15,

$$\begin{aligned} \mathcal{F}_p(\mathcal{M}) &\simeq X \oplus \ell_p(|\mathcal{N}|) \oplus \ell_p(|\mathcal{M} \setminus \mathcal{N}|) \\ &\simeq X \oplus \ell_p(|\mathcal{M}|) \\ &= X \oplus \ell_p(|\mathcal{N}|) \\ &\simeq \mathcal{F}_p(\mathcal{N}). \end{aligned} \quad \square$$

**Proposition 3.6.** *Let  $\mathcal{M}$  be a  $p$ -metric space,  $0 < p \leq 1$ . Suppose  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are nets in  $\mathcal{M}$  of the same cardinality  $\text{dens } \mathcal{M}$ . Then  $\mathcal{F}_p(\mathcal{N}_1) \simeq \mathcal{F}_p(\mathcal{N}_2)$ .*

*Proof.* The proof follows from Theorem 3.5 exactly in the same way as [19, Proposition 5] follows from [19, Theorem 4].  $\square$

Before presenting the results on which Theorems 3.1 and 3.2 are based, let us note that for  $p = 1$  and for separable spaces the result has a quantitative improvement in [11], where it is proved that  $\ell_1$  is isometric to a 1-complemented subspace of  $\mathcal{F}(\mathcal{M})$  whenever  $\mathcal{M}$  has an accumulation point or contains an infinite ultrametric space. However, by the recent result [27], there is a metric space  $\mathcal{M}$  such that  $\ell_1$  does not isometrically embed into  $\mathcal{F}(\mathcal{M})$ . These advances suggest several natural areas for further research. Let us mention two sample questions. For which nonseparable metric spaces  $\mathcal{M}$  is  $\ell_1(\text{dens } \mathcal{M})$  isometric to a 1-complemented subspace of  $\mathcal{F}(\mathcal{M})$ ? Is  $\ell_p$  isometric to a subspace of  $\mathcal{F}_p(\mathcal{M})$  whenever the metric space  $\mathcal{M}$  has an accumulation point?

Finally, in what remains of this section we provide results which imply Theorem 3.1 and Theorem 3.2. The arguments in our proofs are inspired by [10], [11] and [19].

Given a map  $f: X \rightarrow Y$ , where  $X$  is a set and  $Y$  is a vector space, we shall denote the set  $f^{-1}(Y \setminus \{0\})$  by  $\text{supp}_0(f)$ . What follows is an application of the following general result.

**Lemma 3.7.** *Let  $(\mathcal{M}, d)$  be a  $p$ -metric space,  $C > 0$ ,  $(x_\gamma)_{\gamma \in \Gamma}$  and  $(y_\gamma)_{\gamma \in \Gamma}$  be sequences in  $\mathcal{M}$ , and  $(f_\gamma)_{\gamma \in \Gamma}$  be a sequence of  $C$ -Lipschitz maps from  $(\mathcal{M}, d)$  into  $\mathbb{R}$  such that  $(\text{supp}_0(f_\gamma))_{\gamma \in \Gamma}$  is a pairwise disjoint sequence,  $f_\gamma(x_\gamma) \neq 0$  for every  $\gamma \in \Gamma$ ,  $f_{\gamma_1}(x_{\gamma_2}) = 0$  for every  $\gamma_1, \gamma_2 \in \Gamma$  with  $\gamma_1 \neq \gamma_2$ , and  $f_{\gamma_1}(y_{\gamma_2}) = 0$  for every  $\gamma_1, \gamma_2 \in \Gamma$ . Finally, suppose*

there exists  $t > 0$  with

$$\frac{f_\gamma(x_\gamma)}{d(x_\gamma, y_\gamma)} \geq \frac{1}{t}, \quad \gamma \in \Gamma.$$

Then  $\ell_p(\Gamma)$  is  $2^{1/p}Ct$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ .

Moreover, if we have  $y_\gamma = 0$  for every  $\gamma \in \Gamma$ , then for the set  $\mathcal{N} = \{x_\gamma : \gamma \in \Gamma\} \cup \{0\}$  we have:

- (i)  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p(\Gamma)$ ;
- (ii)  $L_j : \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is an isomorphic embedding; and
- (iii)  $L_j(\mathcal{F}_p(\mathcal{N}))$  is complemented in  $\mathcal{F}_p(\mathcal{M})$ .

*Proof.* For each  $\gamma \in \Gamma$  put

$$b_\gamma = \frac{\delta_{\mathcal{M}}(x_\gamma) - \delta_{\mathcal{M}}(y_\gamma)}{d(x_\gamma, y_\gamma)} \in \mathcal{F}_p(\mathcal{M}).$$

Since  $\|b_\gamma\|_{\mathcal{F}_p(\mathcal{M})} = 1$ , there is a norm-one linear operator  $S : \ell_p(\Gamma) \rightarrow \mathcal{F}_p(\mathcal{M})$  such that  $S(\mathbf{e}_\gamma) = b_\gamma$  for all  $\gamma \in \Gamma$ .

Define

$$f : \mathcal{M} \rightarrow \ell_p(\Gamma), \quad x \mapsto \sum_{\gamma \in \Gamma} \frac{d(x_\gamma, y_\gamma)}{f_\gamma(x_\gamma)} f_\gamma(x) \mathbf{e}_\gamma.$$

The map  $f$  is  $2^{1/p}Ct$ -Lipschitz. Indeed for every  $x, y \in \mathcal{M}$  there are  $\gamma_1, \gamma_2 \in \Gamma$  such that  $f_\gamma(x) = f_\gamma(y) = 0$  if  $\gamma \notin \{\gamma_1, \gamma_2\}$ . So,

$$\|f(x) - f(y)\| \leq t \left( \sum_{j=1}^2 |f_{\gamma_j}(x) - f_{\gamma_j}(y)|^p \right)^{1/p} \leq 2^{1/p}Ctd(x, y).$$

Hence, there is a linear bounded map  $P : \mathcal{F}_p(\mathcal{M}) \rightarrow \ell_p(\Gamma)$  such that  $P \circ \delta_{\mathcal{M}} = f$  and  $\|P\| \leq 2^{1/p}Ct$ . We have

$$P(b_\gamma) = \frac{P(\delta_{\mathcal{M}}(x_\gamma)) - P(\delta_{\mathcal{M}}(y_\gamma))}{d(x_\gamma, y_\gamma)} = \frac{f(x_\gamma) - f(y_\gamma)}{d(x_\gamma, y_\gamma)} = \mathbf{e}_\gamma,$$

and so  $P \circ S = \text{Id}_{\ell_p(\Gamma)}$ .

For the ‘‘moreover’’ part, we consider the norm-one linear operator  $S' : \ell_p(\Gamma) \rightarrow \mathcal{F}_p(\mathcal{N})$  such that

$$S'(\mathbf{e}_\gamma) = \frac{\delta_{\mathcal{N}}(x_{\gamma,1})}{d(x_{\gamma,1}, 0)}, \quad \gamma \in \Gamma.$$

Then  $S = L_j \circ S'$  and so by the above we have  $P \circ L_j \circ S' = \text{Id}_{\ell_p(\Gamma)}$ . In particular,  $L_j \circ S'$  is an isomorphism between  $\ell_p(\Gamma)$  and  $\mathcal{F}_p(\mathcal{N})$ , so we have that  $L_j$  is an isomorphism as well.  $\square$

In general, there is no tool for building nontrivial Lipschitz maps from a quasimetric space into the real line. In fact, there are quasimetric spaces  $\mathcal{M}$ , such as  $(\mathbb{R}, |\cdot|^{1/p})$ , for which  $\text{Lip}_0(\mathcal{M}, \mathbb{R}) = \{0\}$  (see e.g. [2, Lemma 2.7]). The first case when the situation is quite different is if the quasimetric space contains isolated points. Indeed, if  $\ell_p$  embeds complementably in  $\mathcal{F}_p(\mathcal{M})$  then  $\ell_\infty$  must embed complementably in  $\text{Lip}_0(\mathcal{M}, \mathbb{R}) = \mathcal{F}_p(\mathcal{M})^*$ !

**Theorem 3.8.** *Suppose  $(\mathcal{M}, d)$  is a  $p$ -metric space and let  $\kappa$  be the cardinality of the set  $\{x: d(x, \mathcal{M} \setminus \{x\}) > 0\}$  of isolated points of  $\mathcal{M}$ . Then for every  $C > 2^{1/p}$ , the space  $\ell_p(\kappa)$  is  $C$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ .*

*Proof.* Let us enumerate the set  $\{x: d(x, \mathcal{M} \setminus \{x\}) > 0\}$  as  $\{x_i: i < \kappa\}$ . Now consider the map  $f_i := d(x_i, \mathcal{M} \setminus \{x_i\}) \cdot \chi_{\{x_i\}}$  and, for  $t > 1$  fixed, we let  $y_i \in \mathcal{M}$  be an arbitrary point with  $d(x_i, y_i) \leq td(x_i, \mathcal{M} \setminus \{x_i\})$ . Each  $f_i$  is 1-Lipschitz and

$$\frac{f_i(x_i)}{d(x_i, y_i)} \geq \frac{1}{t}.$$

Hence, by Lemma 3.7,  $\ell_p(\kappa)$  is  $2^{1/p}t$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ . Since  $t > 1$  was arbitrary, this finishes the proof.  $\square$

Another case when Lipschitz functions are available are metric spaces. Indeed, if  $(\mathcal{M}, d)$  is a metric space then the map  $x \mapsto d(x, y)$  is 1-Lipschitz for every  $y \in \mathcal{M}$ . This fact is one of the foundations of the following result.

**Theorem 3.9.** *Let  $(\mathcal{M}, d)$  be an infinite metric space and  $p \in (0, 1]$ . Then for every  $C > 2^{1/p}$ , the space  $\ell_p(\text{dens } \mathcal{M})$  is  $C$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ .*

*Proof.* It is well-known and not very difficult to prove (for a reference see e.g. [21, Theorem 8.1]) that in any metric space  $\mathcal{M}$  we may find  $(\text{dens } \mathcal{M})$ -many disjoint balls. By Theorem 3.8, we may assume that there do not exist  $(\text{dens } \mathcal{M})$ -many isolated points in  $\mathcal{M}$ . So we may pick non-isolated points  $(x_i)_{i < \text{dens } \mathcal{M}}$  in  $\mathcal{M}$  and positive numbers  $(r_i)_{i < \text{dens } \mathcal{M}}$  such that the balls  $B(x_i, r_i)$ ,  $i < \text{dens } \mathcal{M}$  are pairwise disjoint. For each  $i < \kappa$  we pick  $y_i \in B(x_i, r_i) \setminus \{x_i\}$  and define the 1-Lipschitz function  $f_i: \mathcal{M} \rightarrow \mathbb{R}$  by

$$f_i(x) = \max\{d(x_i, y_i) - d(x, x_i), 0\}, \quad x \in \mathcal{M}.$$

Then  $\text{supp}_0 f_i = B(x_i, d(x_i, y_i)) \subset B(x_i, r_i)$ . Since these sets are pairwise disjoint we obviously have  $f_i(y_j) = 0$  for  $i, j < \text{dens } \mathcal{M}$ . Finally, we have  $f_i(x_i) = d(x_i, y_i)$  and so the assumptions of Lemma 3.7 are



satisfied with  $t = 1$ . Thus we obtain that the space  $\ell_p(\text{dens } \mathcal{M})$  is even  $2^{1/p}$ -complemented in  $\mathcal{F}_p(\mathcal{M})$ .  $\square$

*Proof of Theorem 3.1.* Just combine Theorems 3.8 and 3.9.  $\square$

Uniformly separated  $p$ -metric spaces also have non-trivial Lipschitz functions. Indeed, if  $(\mathcal{M}, d)$  is  $r$ -separated, then the map  $x \mapsto d^p(x, y)$  is Lipschitz, with constant  $r^{p-1}$ , for every  $y \in \mathcal{M}$ .

**Lemma 3.10.** *Let  $p \in (0, 1]$  and  $(\mathcal{M}, d)$  be a pointed  $p$ -metric space. Assume there are sequences  $(x_\gamma)_{\gamma \in \Gamma}$  in  $\mathcal{M}$  and  $(r_\gamma)_{\gamma \in \Gamma}$  in  $(0, +\infty)$  such that there exists  $t > 0$  with*

$$\frac{r_\gamma}{d(x_\gamma, 0)} \geq \frac{1}{t}, \quad \gamma \in \Gamma. \quad (3.5)$$

Further, assume that, with the convention  $r_0 = 0$  and  $x_0 = 0$ ,

(a) either  $d$  is a metric on  $\mathcal{M}$  and

$$d(x_{\gamma_1}, x_{\gamma_2}) \geq r_{\gamma_1} + r_{\gamma_2}, \quad \gamma_1, \gamma_2 \in \Gamma \cup \{0\}, \gamma_1 \neq \gamma_2. \quad (3.6)$$

(b) or  $(\mathcal{M}, d)$  is uniformly separated and

$$d^p(x_{\gamma_1}, x_{\gamma_2}) \geq r_{\gamma_1} + r_{\gamma_2}, \quad \gamma_1, \gamma_2 \in \Gamma \cup \{0\}, \gamma_1 \neq \gamma_2. \quad (3.7)$$

Then for the set  $\mathcal{N} = \{x_\gamma : \gamma \in \Gamma\} \cup \{0\}$  we have:

- (i)  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p(\Gamma)$ ;
- (ii)  $L_j : \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is an isomorphic embedding; and
- (iii)  $L_j(\mathcal{F}_p(\mathcal{N}))$  is complemented in  $\mathcal{F}_p(\mathcal{M})$ .

*Proof.* Consider functions  $f_\gamma : \mathcal{M} \rightarrow \mathbb{R}$  given by

$$f_\gamma(x) = \max\{r_\gamma - d(x, x_\gamma), 0\}, \quad x \in \mathcal{M}$$

if we are in the case (a), or by

$$f_\gamma(x) = \max\{r_\gamma - d^p(x, x_\gamma), 0\}, \quad x \in \mathcal{M}$$

if we are in the case (b). Those functions are 1-Lipschitz if  $\mathcal{M}$  is metric and  $r^{p-1}$ -Lipschitz if  $\mathcal{M}$  is  $r$ -separated, respectively. It follows from either the condition (3.6) or the condition (3.7) that  $\text{supp}_0 f_\gamma$  are pairwise disjoint sets and that  $f_\gamma(0) = 0$  for every  $\gamma \in \Gamma$ . Finally, we have  $f_\gamma(x_\gamma) = r_\gamma \geq t^{-1}d(x_\gamma, 0)$ . Thus, an application of the ‘‘Moreover’’ part of Lemma 3.7 finishes the proof.  $\square$

If  $p < 1$  and conditions (3.5) and (3.7) hold, then  $d(0, x_\gamma) \leq t^{1/(1-p)}$  for every  $\gamma \in \Gamma$ . So, if  $\mathcal{M}$  is not a metric space, Lemma 3.10 only applies when  $\mathcal{N}$  is bounded and  $\mathcal{M}$  is uniformly separated. Conversely,

if  $\mathcal{N}$  is uniformly separated and bounded and  $\mathcal{M}$  is metric (respectively uniformly separated) then, if 0 is an arbitrary point of  $\mathcal{N}$  and we set

$$s = \inf\{d(x, y) : x, y \in \mathcal{N}, x \neq y\}, \quad R = \sup\{d(0, x) : x \in \mathcal{N}\},$$

the conditions of Lemma 3.10 hold with  $\Gamma = \mathcal{N} \setminus \{0\}$ ,  $r_x = s/2$  (resp.  $r_x = s^p/2$ ) for every  $x \in \mathcal{N} \setminus \{0\}$  and  $t = 2Rs^{-1}$  (resp.  $t = 2Rs^{-p}$ ). This observation immediately yields the following result.

**Lemma 3.11.** *Let  $p \in (0, 1]$  and  $(\mathcal{M}, d)$  be a metric space or a uniformly separated  $p$ -metric space. If  $\mathcal{N} \subset \mathcal{M}$  is uniformly separated and bounded then:*

- (i)  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p(|\mathcal{N}| - 1)$ ;
- (ii)  $L_j : \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is an isomorphic embedding; and
- (iii)  $L_j(\mathcal{F}_p(\mathcal{N}))$  is complemented in  $\mathcal{F}_p(\mathcal{M})$ .

We are almost ready to prove the second main result of this section. Prior to do it, we write down an easy lemma which will be used several times throughout the paper.

**Lemma 3.12.** *Assume that a  $p$ -metric space  $(\mathcal{M}, d)$  either is unbounded or its completion has a limit point. Then for every  $t > 1$  there are  $(x_n)_{n=1}^\infty$  in  $\mathcal{M}$ ,  $x_0$  in the completion of  $\mathcal{M}$  and a monotone sequence  $(r_n)_{n=1}^\infty$  in  $(0, \infty)$  such that, with the convention  $r_0 = 0$ ,*

$$d^p(x_n, x_m) \geq r_n + r_m, \quad \frac{|r_n - r_m|}{d^p(x_n, x_m)} \geq \frac{1}{t}$$

for every  $m, n \in \mathbb{N} \cup \{0\}$  with  $m \neq n$ .

*Proof.* Choose  $0 < s < 1$  such that  $\sqrt{t} = (1+s)/(1-s)$ . In the case when  $\mathcal{M}$  is unbounded, if  $x_0$  is an arbitrary point of  $\mathcal{M}$  there is  $(x_n)_{n=1}^\infty$  such that

$$\sup_{n \in \mathbb{N}} \frac{d(x_n, x_0)}{d(x_{n+1}, x_0)} < s^{1/p}.$$

In the case when there is a limit point  $x_0$  in the completion of  $\mathcal{M}$ , we pick a sequence  $(x_n)_{n=1}^\infty$  in  $\mathcal{M}$  such that

$$\sup_{n \in \mathbb{N}} \frac{d(x_{n+1}, x_0)}{d(x_n, x_0)} < s^{1/p}.$$

In both cases, if we set  $r_n = t^{-1/2}d^p(x_n, x_0)$ , we have

$$r_n + r_m \leq \sqrt{t}|r_n - r_m|, \quad m, n \in \mathbb{N} \cup \{0\}, m \neq n.$$

Therefore, if  $m, n \in \mathbb{N} \cup \{0\}$  are such that  $n \neq m$ ,

$$d^p(x_n, x_m) \geq |d^p(x_n, x_0) - d^p(x_m, x_0)| = \sqrt{t}|r_n - r_m| \geq r_n + r_m$$

and

$$d^p(x_n, x_m) \leq d^p(x_n, x_0) + d^p(x_m, x_0) = \sqrt{t}(r_n + r_m) \leq t|r_n - r_m|. \quad \square$$

*Proof of Theorem 3.2.* Since the case when  $\mathcal{M}$  is finite is trivial, we assume that  $\mathcal{M}$  is infinite. Assume that  $\text{dens } \mathcal{M}$  is uncountable. For each  $n \in \mathbb{N}$ , let  $\mathcal{M}_n$  be a maximal  $(1/n)$ -separated set in  $B(0, n)$ . Then  $|\mathcal{M}_n| \leq \text{dens } \mathcal{M}$  for every  $n \in \mathbb{N}$ , and  $\mathcal{N} := \cup_{n=1}^{\infty} \mathcal{M}_n$  is dense in  $\mathcal{M}$ . Therefore, there exists  $n \in \mathbb{N}$  with  $\mathcal{M}_n$  infinite. An application of Lemma 3.11 yields (b). Now, in order to prove (a), it suffices to consider the case when  $\mathcal{M}$  is separable. By Lemma 3.11, we may assume that  $\mathcal{M}$  does not contain an infinite uniformly separated bounded subset. We infer that either  $\mathcal{M}$  is unbounded or the completion of  $\mathcal{M}$  has a limit point. Then, given  $t > 1$ , we use Lemma 3.12 to pick  $(x_n)_{n=1}^{\infty}$  in  $\mathcal{M}$ ,  $(r_n)_{n=1}^{\infty}$  in  $(0, \infty)$  and  $x_0$  in the completion of  $\mathcal{M}$  such that, if we choose  $x_0$  as the base point of  $\mathcal{M} \cup \{x_0\}$ , (3.5) and (3.6) hold for  $\Gamma = \mathbb{N}$ . This way, if  $x_0 \in \mathcal{M}$  the result follows from Lemma 3.10. If  $x_0 \notin \mathcal{M}$  the result follows by Lemma 3.10 in combination with [3, Proposition 4.17].  $\square$

Let us note that we do not know whether Theorem 4.3 holds for separable uniformly separated  $p$ -metric spaces.

*Question 3.13.* Let  $p \in (0, 1)$  and  $(\mathcal{M}, d)$  be an infinite countable uniformly separated  $p$ -metric space. Does there exist  $\mathcal{N} \subset \mathcal{M}$  such that  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p$  and  $\mathcal{F}_p(\mathcal{N})$  is complemented in  $\mathcal{F}_p(\mathcal{M})$ ?

Notice that in the case when  $\text{dens } \mathcal{M}$  has uncountable cofinality, the proof of Theorem 4.3 yields a subspace  $\mathcal{N}$  of  $\mathcal{M}$  such that  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p(\text{dens } \mathcal{M})$  is complemented in  $\mathcal{F}_p(\mathcal{M})$ . However, we do not know if this result holds in general.

#### 4. EMBEDDABILITY OF $\ell_p$ IN $\mathcal{F}_p(\mathcal{M})$ FOR $0 < p \leq 1$

The aim of this section is to exhibit a method (specifically tailored for  $p$ -metric spaces) to linearly embed  $\ell_p$  in  $\mathcal{F}_p(\mathcal{M})$  when  $\mathcal{M}$  is quasimetric and  $p \leq 1$ . Our main result here is Theorem 4.1, which will be obtained as a direct consequence of Theorems 3.1 and 4.3.

**Theorem 4.1.** *If  $\mathcal{M}$  is an infinite  $p$ -metric space,  $0 < p \leq 1$ , then  $\mathcal{F}_p(\mathcal{M})$  contains a subspace isomorphic to  $\ell_p$ .*

A quantitative estimate follows from Theorem 4.9, which is the analogue of James's  $\ell_1$  distortion theorem for  $1 < p \leq 1$ . We do not know whether for every nonseparable  $p$ -metric space  $\mathcal{M}$  we even have  $\ell_p(\text{dens } \mathcal{M}) \hookrightarrow \mathcal{F}_p(\mathcal{M})$ . It is worth it mentioning that Theorem 4.1 gives us the following application.

**Corollary 4.2.** *If  $\mathcal{M}$  is an infinite  $p$ -metric space,  $0 < p < 1$ , then  $\mathcal{F}_p(\mathcal{M})$  is not  $q$ -convex for any  $p < q \leq 1$ .*

As in the previous section, sometimes it is possible to have a more precise information about the copy of  $\ell_p$  in the space  $\mathcal{F}_p(\mathcal{M})$ . This is the content of the following result.

**Theorem 4.3.** *Let  $0 < p \leq 1$ . Suppose  $(\mathcal{M}, \rho)$  is a  $p$ -metric space that is either unbounded, or whose completion contains a limit point. Then for every  $t > 1$  there exists an infinite countable set  $\mathcal{N} \subset \mathcal{M}$  such that*

- (i) *the space  $\mathcal{F}_p(\mathcal{N})$  is  $2^{1/p-1}t$ -isomorphic to  $\ell_p$ , and*
- (ii) *the canonical map  $L_j: \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is an isomorphic embedding, where  $j: \mathcal{N} \rightarrow \mathcal{M}$  is the inclusion. Quantitatively,  $\|L_j^{-1}\| \leq 2^{1/p-1}t$ .*

Before giving a proof of Theorem 4.1, let us mention some applications.

**Corollary 4.4.** *Let  $0 < p \leq 1$ . Every infinite  $p$ -metric space  $\mathcal{M}$  has a subset  $\mathcal{N}$  with  $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p$ .*

*Proof.* By Theorem 4.3 it suffices to consider the case when  $\mathcal{M}$  is bounded and its completion is not compact. Then  $\mathcal{M}$  is not totally bounded, i.e.,  $\mathcal{M}$  contains a uniformly separated infinite set, but then we are done by [3, Theorem 4.14].  $\square$

It has been shown in [9, 13] that, despite the fact that the space  $\mathcal{F}(\mathcal{M})$  is isomorphic to  $\ell_1$  for any separable ultrametric space  $\mathcal{M}$ ,  $\mathcal{F}(\mathcal{M})$  is never isometric to  $\ell_1$ . In contrast, we have the following result.

**Proposition 4.5.** *For each  $\epsilon > 0$  there is an ultrametric space  $\mathcal{M}$  such that the Banach-Mazur distance between  $\mathcal{F}(\mathcal{M})$  and  $\ell_1$  is smaller than  $1 + \epsilon$ .*

*Proof.* By Theorem 4.3, in each ultrametric space that is unbounded or contains a limit point we may find a subset  $\mathcal{M}$  (which is also ultrametric) so that the Banach-Mazur distance between  $\mathcal{F}(\mathcal{M})$  and  $\ell_1$  is arbitrarily close to 1.  $\square$

In what remains of the section we will prove Theorems 4.1 and 4.3 and, for the sake of completeness, also the above-mentioned analogue of James's  $\ell_1$  distortion theorem for  $1 < p \leq 1$ . Given a map  $f: X \rightarrow Y$ , where  $X$  is a set and  $Y$  a vector space, we shall denote the set  $f^{-1}(Y \setminus \{0\})$  by  $\text{supp}_0(f)$ . The arguments in our proofs are inspired by [11].

**Lemma 4.6.** *Suppose that  $X$  is a  $p$ -normed space and that  $Y$  is a  $q$ -normed space,  $0 < q, p \leq 1$ . Let  $\Gamma$  a be set, and  $f_\gamma: X \rightarrow Y$  be an  $L$ -Lipschitz map for each  $\gamma \in \Gamma$ . Suppose that the sets in the family  $\{\text{supp}_0(f_\gamma)\}_{\gamma \in \Gamma}$  are disjoint. Then  $f = \sum_{\gamma \in \Gamma} f_\gamma$  is  $L2^{1/q-1}$ -Lipschitz.*

*Proof.* For each  $x \in X$  there is  $\gamma(x) \in \Gamma$  such that  $f_\gamma(x) = 0$  if  $\gamma \neq \gamma(x)$ . In particular, the sum defining  $f(x)$  is pointwise finite and so  $f$  is well-defined. Pick  $x, y \in X$  and set  $\alpha = \gamma(x)$  and  $\beta = \gamma(y)$ . Since the line segment  $[x, y]$  is connected and the sets  $\text{supp}_0(f_\alpha)$  and  $\text{supp}_0(f_\beta)$  are open and disjoint, there is  $z \in [x, y] \setminus (\text{supp}_0(f_\alpha) \cup \text{supp}_0(f_\beta))$ . Using the elementary fact that

$$\|y + z\|_Y \leq 2^{1/q-1}(\|y\|_Y + \|z\|_Y), \quad y, z \in Y,$$

we obtain

$$\begin{aligned} \|f(x) - f(y)\|_Y &= \|f_\alpha(x) - f_\beta(y)\|_Y \\ &= \|f_\alpha(x) - f_\alpha(z) + f_\beta(z) - f_\beta(y)\|_Y \\ &\leq 2^{1/q-1}(\|f_\alpha(x) - f_\alpha(z)\| + \|f_\beta(z) - f_\beta(y)\|_Y) \\ &\leq L2^{1/q-1}(\|x - z\|_X + \|z - y\|_X) \\ &= 2^{1/q-1}L\|x - y\|_X, \end{aligned}$$

from where the conclusion follows.  $\square$

**Lemma 4.7.** *Let  $(\mathcal{M}, d)$  be a pointed  $p$ -metric space. Assume that  $(x_n)_{n=0}^\infty$  in  $\mathcal{M}$  and  $(r_n)_{n=0}^\infty$  in  $[0, +\infty)$  are such that*

$$d^p(x_m, x_n) \geq r_m + r_n, \quad m, n \in \mathbb{N} \cup \{0\}, m \neq n. \quad (4.8)$$

*Then there is a sequence  $(f_n)_{n=0}^\infty$  of 1-Lipschitz maps from  $(\mathcal{M}, d)$  into  $L_p(\mathbb{R})$  such that  $f_n(x_n) = \chi_{(0, r_n]}$  for every  $n \in \mathbb{N} \cup \{0\}$  and  $(\text{supp}_0(f_n))_{n=0}^\infty$  is a pairwise disjoint sequence.*

*Proof.* For  $n \in \mathbb{N} \cup \{0\}$  define  $g_n: \mathcal{M} \rightarrow \mathbb{R}$  by

$$g_n(x) = \max\{r_n - d^p(x, x_n), 0\}, \quad x \in \mathcal{M}. \quad (4.9)$$

It is straightforward to check that the sets  $\{\text{supp}_0(g_n)\}_{n=0}^\infty$  are pairwise disjoint, that  $g(x_n) = r_n$ , and that  $g_n: (\mathcal{M}, \rho) \rightarrow (\mathbb{R}, |\cdot|^{1/p})$  is 1-Lipschitz. Then, if we consider the standard embedding

$$\Phi: (\mathbb{R}, |\cdot|^{1/p}) \rightarrow L_p(\mathbb{R}), \quad \Phi(x) = \chi_{(0, x]},$$

the sequence  $(\Phi \circ g_n)_{n=0}^\infty$  has the desired properties.  $\square$

The next proposition is the last ingredient that we need for proving the main results of this section.

**Proposition 4.8.** *Let  $0 < p \leq 1$ . Suppose  $(\mathcal{M}, d)$  is a  $p$ -metric space containing a sequence  $(x_n)_{n=0}^\infty$  such that for some monotone sequence  $(r_n)_{n=0}^\infty$  in  $[0, +\infty)$  and some  $t > 0$*

$$\frac{|r_n - r_{n-1}|}{d^p(x_n, x_{n-1})} \geq \frac{1}{t}, \quad n \in \mathbb{N}.$$

and (4.8) holds. Then, if  $\mathcal{N} = \{x_n : n \in \mathbb{N} \cup \{0\}\}$ ,

- (i) *The space  $\mathcal{F}_p(\mathcal{N})$  is  $2^{1/p-1}t$ -isomorphic to  $\ell_p$ , and*
- (ii) *The canonical map  $L_j : \mathcal{F}_p(\mathcal{N}) \rightarrow \mathcal{F}_p(\mathcal{M})$  is an isomorphic embedding, where  $j : \mathcal{N} \rightarrow \mathcal{M}$  is the inclusion. Quantitatively,  $\|L_j^{-1}\| \leq 2^{1/p-1}t$ .*

*Proof.* We may assume that  $x_0$  is the base point of  $\mathcal{M}$ . For each  $n \in \mathbb{N}$  put

$$v_n = \frac{\delta_{\mathcal{N}}(x_{n-1}) - \delta_{\mathcal{N}}(x_n)}{d(x_{n-1}, x_n)} \in \mathcal{F}_p(\mathcal{N}),$$

$$b_n = \frac{\delta_{\mathcal{M}}(x_{n-1}) - \delta_{\mathcal{M}}(x_n)}{d(x_{n-1}, x_n)} \in \mathcal{F}_p(\mathcal{M}).$$

There is a norm-one linear map  $T : \ell_p \rightarrow \mathcal{F}_p(\mathcal{N})$  such that  $T(\mathbf{e}_n) = v_n$  for all  $n \in \mathbb{N}$ . Since  $L_j(v_n) = b_n$  for  $n \in \mathbb{N}$  and  $\overline{\text{span}}\{v_n : n \in \mathbb{N}\} = \mathcal{F}_p(\mathcal{N})$ , it suffices to prove that  $(b_n)_{n=1}^\infty$  is  $2^{1/p-1}t$ -dominated by the canonical basis of  $\ell_p$ .

We consider  $\mathcal{M}$  as a subset of  $X = \mathcal{F}_p(\mathcal{M})$ . Let  $(f_n)_{n=0}^\infty$  be the sequence of maps from  $X$  into  $L_p(\mathbb{R})$  provided by Lemma 4.7. Pick any  $a_1, \dots, a_N \in \mathbb{R}$  and consider  $x = \sum_{n=1}^N a_n b_n$ . Let  $g : X \rightarrow L_p(\mathbb{R})$  be defined for  $z \in \mathcal{F}_p(\mathcal{M})$  by

$$g(z) = \sum_{n=0}^N f_n(z).$$

Lemma 4.6 yields that  $g : X \rightarrow L_p(\mathbb{R})$  is  $2^{1/p-1}$ -Lipschitz. Hence,  $f = g|_{\mathcal{M}}$  is  $2^{1/p-1}$ -Lipschitz. Since  $f(x_n) = \chi_{(0, r_n]}$  we have

$$\begin{aligned} \|x\| &\geq \frac{2}{2^{1/p}} \left\| \sum_{n=1}^N a_n \frac{f(x_{n-1}) - f(x_n)}{d(x_{n-1}, x_n)} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left\| \sum_{n=1}^N a_n \frac{\chi_{(r_{n-1}, r_n]}}{d(x_{n-1}, x_n)} \right\|_{L_p} \\ &= \frac{2}{2^{1/p}} \left( \sum_{n=1}^N |a_n|^p \frac{|r_{n-1} - r_n|}{d^p(x_{n-1}, x_n)} \right)^{1/p} \end{aligned}$$

$$\geq \frac{2}{2^{1/p}} \frac{1}{t} \left( \sum_{n=1}^N |a_n|^p \right)^{1/p}. \quad \square$$

*Proof of Theorem 4.3.* It follows by combining Lemma 3.12 with Proposition 4.8.  $\square$

*Proof of Theorem 4.1.* By Theorem 3.8 we can assume that there is a limit point in the completion of  $\mathcal{M}$ . Then, the result follows from Theorem 4.3.  $\square$

As we advertised, there is a non-locally convex version of James's  $\ell_1$  distortion theorem. As far as we know, this is the first time that the validity of this result is explicitly stated and so we include its proof for further reference.

**Theorem 4.9** (James's  $\ell_p$  distortion theorem for  $0 < p \leq 1$ ). *Let  $(x_j)_{j=1}^\infty$  be a normalized basic sequence in a  $p$ -Banach space  $X$  which is equivalent to the canonical  $\ell_p$ -basis,  $0 < p \leq 1$ . Then given  $\epsilon > 0$  there is a normalized block basic sequence  $(y_k)_{k=1}^\infty$  of  $(x_j)_{j=1}^\infty$  such that*

$$\left\| \sum_{k=1}^\infty a_k y_k \right\| \geq (1 - \epsilon) \left( \sum_{k=1}^\infty |a_k|^p \right)^{1/p},$$

for any sequence of scalars  $(a_k)_{k=1}^\infty \in c_{00}$ .

*Proof.* By hypothesis, there is  $M_0 < \infty$  such that

$$\left( \sum_{j=1}^\infty |a_j|^p \right)^{1/p} \leq M_0 \left\| \sum_{j=1}^\infty a_j x_j \right\|$$

for every  $(a_j)_{j=1}^\infty \in c_{00}$ . Hence, for each integer  $n$  we can consider the least constant  $M_n$  so that if  $(a_j)_{j=1}^\infty \in c_{00}$  with  $a_j = 0$  for  $j \leq n$  then

$$\left( \sum_{j=1}^\infty |a_j|^p \right)^{1/p} \leq M_n \left\| \sum_{j=1}^\infty a_j x_j \right\|.$$

The sequence  $(M_n)_{n=1}^\infty$  is decreasing, and the inequality

$$\left\| \sum_{j=1}^\infty a_j x_j \right\| \leq \left( \sum_{j=1}^\infty |a_j|^p \right)^{1/p}$$

yields  $M_n \geq 1$ . Let  $M = \lim_{n \rightarrow \infty} M_n \geq 1$ . We recursively construct an increasing sequence  $(n_k)_{k=0}^\infty$  of positive integers and a sequence  $(y_k)_{k=1}^\infty$  in  $X$ . We start by choosing  $n_0 \in \mathbb{N}$  such that

$$M_{n_0} \leq (1 - \epsilon)^{-1/2} M.$$

Let  $k \in \mathbb{N}$  and assume that  $n_i$  and  $y_i$  are constructed for  $i < k$ . Since  $(1 - \epsilon)^{1/2}M < M_{n_{k-1}}$  there is  $n_k > n_{k-1}$  and a norm-one vector  $y_k = \sum_{j=1+n_{k-1}}^{n_k} b_j x_j \in X$  such that

$$\left( \sum_{j=1+n_{k-1}}^{n_k} |b_j|^p \right)^{1/p} \geq (1 - \epsilon)^{1/2}M.$$

The normalized block basic sequence  $(y_n)_{n=1}^{\infty}$  satisfies the desired property. Indeed, for any  $(a_j)_{j=1}^{\infty} \in c_{00}$  we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} a_k y_k \right\| &= \left\| \sum_{k=1}^{\infty} \sum_{j=1+n_{k-1}}^{n_k} a_k b_j x_j \right\| \\ &\geq M_{n_0}^{-1} \left( \sum_{k=1}^{\infty} \sum_{j=1+n_{k-1}}^{n_k} |a_k|^p |b_j|^p \right)^{1/p} \\ &\geq (1 - \epsilon)^{1/2} M M_{n_0}^{-1} \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} \\ &\geq (1 - \epsilon) \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}. \quad \square \end{aligned}$$

In [14] the following improvement of James's  $\ell_1$  distortion theorem is obtained: whenever a Banach space  $X$  has a complemented subspace isomorphic to  $\ell_1$ , then  $X$  has for each  $\epsilon > 0$  a  $(1 + \epsilon)$ -complemented subspace  $(1 + \epsilon)$ -isomorphic to  $\ell_1$ . Since the proof of this fact relies on duality techniques we wonder whether there is an analogue for  $p$ -Banach spaces.

*Question 4.10.* Does there exist  $C \geq 1$  such that whenever a  $p$ -Banach space  $X$  has a complemented subspace isomorphic to  $\ell_p$ , then it has a  $C$ -complemented subspace  $C$ -isomorphic to  $\ell_p$ ?

## 5. BASES IN $\mathcal{F}_p(\mathbb{N})$ AND $\mathcal{F}_p([0, 1])$

Given  $0 < p \leq 1$ ,  $n \in \mathbb{N}$ , and  $K \subseteq \mathbb{R}^n$  let us denote by  $\mathcal{F}_p(K)$  the Lipschitz free  $p$ -space over  $K$  equipped with the Euclidean metric. In this section we address the study of Schauder bases for the Lipschitz-free  $p$ -spaces  $\mathcal{F}_p(\mathbb{N})$  and  $\mathcal{F}_p([0, 1])$ . If  $p = 1$  those spaces are isometric to  $\ell_1$  and  $L_1$ , respectively. However, for  $p < 1$  we obtain an interesting family of non-classical  $p$ -Banach spaces. One of the most important results of this section is that  $\mathcal{F}_p([0, 1])$  has a Schauder basis for  $0 <$



$p < 1$ . This appears to provide the first-known examples of  $p$ -Banach spaces which have a basis but cannot have an unconditional basis. Indeed, by [3, Proposition 4.20], the Banach envelope of  $\mathcal{F}_p([0, 1])$  is isometrically isomorphic to  $\mathcal{F}([0, 1]) \simeq L_1[0, 1]$ , which does not have an unconditional basis (see, e.g., [5, Theorem 6.3.3]).

First, we deal with  $\mathcal{F}_p(\mathbb{N}_*)$ , where  $\mathbb{N}_* := \mathbb{N} \cup \{0\}$ . Of course,  $\mathbb{N}_*$  and  $\mathbb{N}$  are isometric as metric spaces, but we prefer to work with the former and choose 0 as its base point.

**Lemma 5.1.** *Let  $(a_n)_{n=1}^\infty$  be an eventually null sequence of scalars. Then*

$$\left\| \sum_{n=1}^{\infty} a_n \delta(n) \right\|_{\mathcal{F}_p(\mathbb{N}_*)} \geq \left( \sum_{\substack{n \in \mathbb{N} \\ a_n \geq 0}} a_n^p \right)^{1/p}.$$

*Proof.* Let  $\alpha$  be the  $\{0, 1\}$ -metric on  $\mathbb{N}_*$ . Since  $\alpha(n, m) \leq |n - m|$  for every  $n, m \in \mathbb{N}_*$ , we have

$$\|f\|_{\mathcal{F}_p(\mathbb{N}_*, \alpha)} \leq \|f\|_{\mathcal{F}_p(\mathbb{N}_*)}$$

for every  $f \in \mathcal{P}(\mathbb{N}_*)$ . An appeal to [3, Proposition 4.16] completes the proof.  $\square$

By  $\mathbb{Z}[k, m]$  we denote the set  $\{n \in \mathbb{Z} : k \leq n \leq m\}$  whenever  $k, m \in \mathbb{Z}$ . Let us see a preliminary result that we will need.

**Theorem 5.2.** *Let  $0 < p \leq 1$  and  $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$  be the sequence in  $\mathcal{F}_p(\mathbb{N}_*)$  defined by  $\mathbf{x}_n = \delta(n) - \delta(n-1)$ . Then:*

- (a)  $\mathcal{B}$  is a normalized bi-monotone basis of  $\mathcal{F}_p(\mathbb{N}_*)$ .
- (b) For every  $m \in \mathbb{N}$ ,  $\text{span}(\{\mathbf{x}_n : 1 \leq n \leq m\}) \simeq \mathcal{F}_p(\mathbb{Z}[0, m])$  isometrically.
- (c) The subbases  $(\mathbf{x}_{2k-1})_{n=1}^\infty$  and  $(\mathbf{x}_{2k})_{n=1}^\infty$  are isometrically equivalent to the unit vector basis of  $\ell_p$ ,
- (d) If  $1 \leq k \leq m$ , then  $\|\sum_{n=k+1}^m \mathbf{x}_n\| = m - k$ ,
- (e) If  $p < 1$  then  $\mathcal{B}$  is a conditional basis.
- (f) The sequence  $\mathcal{B}$ , regarded as a basis of the Banach envelope of  $\mathcal{F}_p(\mathbb{N}_*)$ , is isometrically equivalent to the unit vector basis of  $\ell_1$ .

*Proof.* Let  $0 \leq k < m$ . The map

$$r[k, m]: \mathbb{N}_* \rightarrow \mathbb{N}_* : n \mapsto \max\{k, \min\{n, m\}\}$$

is a 1-Lipschitz retraction from  $\mathbb{N}_*$  onto  $\mathbb{Z}[k, m]$ . Then there is a norm-one linear projection  $P[k, m]$  from  $\mathcal{F}_p(\mathbb{N}_*)$  to itself such that

$$P[k, m](\delta(n)) = \begin{cases} 0 & \text{if } n \leq k, \\ \delta(m) - \delta(k) & \text{if } n \geq m, \\ \delta(n) - \delta(k) & \text{if } k \leq n \leq m, \end{cases}$$

and an isometric linear embedding  $L[k, m]: \mathcal{F}_p(\mathbb{Z}[k, m]) \rightarrow \mathcal{F}_p(\mathbb{N}_*)$  such that, if we choose  $k$  as the base point of  $\mathbb{Z}[k, m]$ ,

$$L[k, m](\delta(n)) = \delta(n) - \delta(k).$$

We deduce from the case  $k = 0$  that (b) holds. Moreover, if we put  $P_m = P[0, m]$  and  $V_m = P_m(\mathcal{F}_p(\mathbb{N}_*))$ , we have

$$V_m = \text{span}(\{\delta(n) : 1 \leq n \leq m\}) = \text{span}(\{\mathbf{x}_n : 1 \leq n \leq m\}).$$

Consequently,  $\dim V_m = m$  and  $\cup_{m=1}^{\infty} V_m$  is dense in  $\mathcal{F}_p(\mathbb{N}_*)$ . It is clear that  $P_m \circ P_j = P_{\min\{j, m\}}$  and that  $P_m(\mathbf{x}_n) = \mathbf{x}_n$  if  $m > n$ . Hence,  $\mathcal{B}$  is a Schauder basis with partial-sum projections  $(P_m)_{m=1}^{\infty}$ . As  $P_m - P_k = P[k+1, m]$ , the basis  $\mathcal{B}$  is bi-monotone. Since  $\sum_{n=k+1}^m \mathbf{x}_n = \delta(m) - \delta(k)$ , we have  $\|\sum_{n=k+1}^m \mathbf{x}_n\| = m - k$  for every  $0 \leq k < m$ . In particular,  $\|\mathbf{x}_n\|_{\mathcal{F}_p(\mathbb{N}_*)} = 1$  for every  $n \in \mathbb{N}$ . Summarizing, we have proved that (a) and (d) hold.

Let  $(a_k)_{k=1}^{\infty}$  be eventually zero and consider  $A = \{k : a_k > 0\}$  and  $B = \{k : a_k < 0\}$ . Using Lemma 5.1 and  $p$ -convexity yields

$$\begin{aligned} \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} &\geq \left\| \sum_{k=1}^{\infty} a_k \mathbf{x}_{2k} \right\|_{\mathcal{F}_p(\mathbb{N}_*)} \\ &= \left\| \sum_{k=1}^{\infty} a_k \delta(2k) - \sum_{k=1}^{\infty} a_k \delta(2k-1) \right\|_{\mathcal{F}_p(\mathbb{N}_*)} \\ &\geq \left( \sum_{k \in A} a_k^p + \sum_{k \in B} (-a_k)^p \right)^{1/p} \\ &= \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}. \end{aligned}$$

For the odd terms, we proceed analogously. Since  $\mathcal{B}$  is normalized, (c) holds. Part (e) is a consequence of the previous ones. Indeed, if  $\mathcal{B}$  were unconditional we would have

$$\left\| \sum_{k=1}^{\infty} a_n \mathbf{x}_n \right\|_{\mathcal{F}_p(\mathbb{N}_*)}^p \approx \left\| \sum_{k=1}^{\infty} a_{2k-1} \mathbf{x}_{2k-1} \right\|_{\mathcal{F}_p(\mathbb{N}_*)}^p + \left\| \sum_{k=1}^{\infty} a_{2k} \mathbf{x}_{2k} \right\|_{\mathcal{F}_p(\mathbb{N}_*)}^p$$

$$= \sum_{n=1}^{\infty} |a_n|^p$$

for all sequences of scalars  $(a_n)_{n=1}^{\infty}$  eventually zero. In particular, by (d) we would have  $m \approx m^p$  for  $m \in \mathbb{N}$ , which is false unless  $p = 1$ .

By [3, Proposition 4.20],  $\mathcal{B}$  is, when regarded in the Banach envelope, the sequence  $(\delta(n) - \delta(n-1))_{n=1}^{\infty}$  of the Banach space  $\mathcal{F}(\mathbb{N}_*)$ . Hence, (f) is known and follows, e.g., from the more general [16, Proposition 2.3].  $\square$

As the attentive reader should have noticed, a result similar to Theorem 5.2 holds for  $\mathcal{F}_p(\mathbb{Z})$ . Let us point out that the existence of a Schauder basis for  $\mathcal{F}_p(\mathbb{Z})$  can also be deduced from the following general result.

**Theorem 5.3.** *Let  $\mathcal{M}$  be either a net on  $c_0$  or a net on  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ . Then  $\mathcal{F}_p(\mathcal{M})$  has a Schauder basis.*

*Proof.* The proof can be carried out exactly as in [19, Corollaries 16 and 18], with the exception that instead of [19, Proposition 5] we use Proposition 3.6 and that we need to prove [19, Theorem 13] also for  $p < 1$ , which is easy (actually, the proof of [19, Theorem 13] uses the universal property of Lipschitz free spaces only, so the same proof works even for  $p < 1$ ).  $\square$

Now, let us mention a preliminary result which concerns the structure of  $\mathcal{F}_p([0, 1])$ .

**Lemma 5.4.** *For each pair  $(K_1, K_2)$  with  $\{0, 1\} \subseteq K_2 \subseteq K_1 \subseteq [0, 1]$  there is a linear operator  $P_{K_1, K_2}: \mathcal{F}_p(K_1) \rightarrow \mathcal{F}_p(K_2)$  such that, if  $L_{K_1, K_2}$  denotes the canonical linear map from  $\mathcal{F}_p(K_2)$  into  $\mathcal{F}_p(K_1)$  and  $\{0, 1\} \subseteq K_3 \subseteq K_2 \subseteq K_1 \subseteq [0, 1]$ ,*

- (i)  $\|P_{K_1, K_2}\| \leq 3^{1/p-1}$  and  $P_{K_1, K_2} \circ L_{K_1, K_2} = \text{Id}_{\mathcal{F}_p(K_2)}$ ;
- (ii)  $P_{K_2, K_3} \circ P_{K_1, K_2} = P_{K_1, K_3}$ ; and
- (iii)  $P_{K_1, K_3} \circ L_{K_1, K_2} = P_{K_2, K_3}$  and  $P_{K_1, K_2} \circ L_{K_1, K_3} = L_{K_2, K_3}$ .

Moreover, if  $a < x < b$  are such that  $[a, b] \cap K_2 = \{a, b\}$  and  $x \in K_1$ , we have

$$P_{K_1, K_2}(\delta_{K_1}(x)) = \frac{b-x}{b-a} \delta_{K_2}(a) + \frac{x-a}{b-a} \delta_{K_2}(b).$$

*Proof.* By [3, Proposition 4.17] we can assume that  $K_i$  is closed for  $i \in \{1, 2, 3\}$ . Since for  $0 \leq a \leq b \leq 1$  the mapping

$$x \mapsto \max\{a, \min\{x, b\}\}$$

is a 1-Lipschitz retraction from  $[0, 1]$  onto  $[a, b]$  we can assume that  $\{0, 1\} \subseteq K_3$ . Given  $x \in [0, 1] \setminus K_2$  there are  $a = a[x, K_2]$ , and  $b =$

$b[x, K_2]$  such that  $\{a, b\} \subseteq K_2$  and  $x \in (a, b) \cap K_2 = \emptyset$ . Define  $f = f_{K_1, K_2}: K_1 \rightarrow \mathcal{F}_p(K_2)$  by

$$f_{K_1, K_2}(x) = \begin{cases} \frac{b-x}{b-a}\delta_{K_2}(a) + \frac{x-a}{b-a}\delta_{K_2}(b) & \text{if } x \in K_1 \setminus K_2, \\ \delta_{K_2}(x) & \text{if } x \in K_2. \end{cases} \quad (5.10)$$

Let  $x, y \in K_1$  with  $x < y$ . If  $x, y \in [a, b]$  with  $a, b \in K_2$  and  $(a, b) \cap K_2 = \emptyset$  we have

$$\|f(x) - f(y)\| = \left\| \frac{x-y}{b-a}(\delta_{K_2}(b) - \delta_{K_2}(a)) \right\| = |x-y|.$$

In general, there are  $a, b, c, d \in K_2$  such that  $a \leq x \leq b \leq c \leq y \leq d$ . Then

$$\begin{aligned} \|f(x) - f(y)\|^p &= \|f(x) - f(b)\|^p + \|f(b) - f(c)\|^p + \|f(c) - f(y)\|^p \\ &\leq |x-b|^p + |b-c|^p + |c-y|^p \\ &\leq 3^{1-p}|y-x|^p. \end{aligned}$$

Hence, by [3, Theorem 4.5], there is  $P_{K_1, K_2}: \mathcal{F}_p(K_1) \rightarrow \mathcal{F}_p(K_2)$  such that  $\|P_{K_1, K_2}\| \leq 3^{1/p-1}$  and

$$P_{K_1, K_2} \circ \delta_{K_1} = f_{K_1, K_2}. \quad (5.11)$$

If  $x \in K_2$  we have

$$P_{K_1, K_2}(L_{K_1, K_2}(\delta_{K_2}(x))) = f_{K_1, K_2}(x) = \delta_{K_2}(x).$$

and so (i) holds. In order to prove (ii), we pick  $x \in K_1$ . In the case when  $x \in K_2$  it is clear from (5.10) and (5.11) that

$$P_{K_2, K_3}(P_{K_1, K_2}(\delta_{K_1}(x))) = P_{K_2, K_3}(\delta_{K_2}(x)) = f_{K_1, K_3}(x).$$

Assume that  $x \notin K_2$  and set  $a_2 = a[x, K_2]$ ,  $b_2 = b[x, K_2]$ ,  $a_3 = a[x, K_3]$  and  $b_3 = b[x, K_3]$ . We have

$$\begin{aligned} P_{K_2, K_3}(P_{K_1, K_2}(\delta_{K_1}(x))) &= \frac{b_2-x}{b_2-a_2} \left( \frac{b_3-a_2}{b_3-a_3}\delta_{K_3}(a_3) + \frac{a_2-a_3}{b_3-a_3}\delta_{K_3}(b_3) \right) \\ &\quad + \frac{x-a_2}{b_2-a_2} \left( \frac{b_3-b_2}{b_3-a_3}\delta_{K_3}(a_3) + \frac{b_2-a_3}{b_3-a_3}\delta_{K_3}(b_3) \right) \\ &= \frac{b_3-x}{b_3-a_3}\delta_{K_3}(a_3) + \frac{x-a_3}{b_3-a_3}\delta_{K_3}(b_3) \\ &= f_{K_1, K_3}(x). \end{aligned}$$

Thus, (ii) holds.

(iii) is a straightforward consequence of (i) and (ii).  $\square$

The following result is a version (and a generalization) of the fact that conditional expectations define bounded operators in  $L_1$ .

**Theorem 5.5.** *For any  $K \subseteq [0, 1]$ , the space  $\mathcal{F}_p(K)$  is complemented in  $\mathcal{F}_p([0, 1])$ . To be precise, there is a linear map  $P: \mathcal{F}_p([0, 1]) \rightarrow \mathcal{F}_p(K)$  such that  $\|P\| \leq 3^{1/p-1}$  and, if  $j: K \rightarrow [0, 1]$  is the inclusion map,  $P \circ L_j = \text{Id}_{\mathcal{F}_p(K)}$ .*

*Proof.* Since, for  $0 \leq a \leq b \leq 1$ , the mapping

$$x \mapsto \max\{a, \min\{x, b\}\}$$

is a 1-Lipschitz retraction from  $[0, 1]$  onto  $[a, b]$  we can assume that  $\{0, 1\} \subseteq K$  (see, e.g., [3, Lemma 4.19]). Now the result is immediate from Lemma 5.4.  $\square$

**Theorem 5.6.** *The space  $\mathcal{F}_p([0, 1])$  is finitely crudely representable in  $\mathcal{F}_p(\mathbb{N}_*)$ , and the space  $\mathcal{F}_p(\mathbb{N}_*)$  is finitely crudely representable in  $\mathcal{F}_p([0, 1])$ . That is, the finite dimensional subspace structures of  $\mathcal{F}_p([0, 1])$  and  $\mathcal{F}_p(\mathbb{N}_*)$  coincide.*

*Proof.* Let  $K_n = \{i2^{-n} : 0 \leq i \leq 2^n\}$ ,  $X_n := \text{span}(\{\delta_{[0,1]}(x) : x \in K_n\})$ ,  $M_n = \mathbb{Z}[0, 2^n]$  and  $Y_n := \text{span}(\{\delta_{\mathbb{N}_*}(x) : x \in M_n\})$ . By Theorem 5.5,  $X_n$  is uniformly isomorphic to  $\mathcal{F}_p(K_n)$ , and by Theorem 5.2,  $Y_n$  is uniformly isomorphic to  $\mathcal{F}_p(M_n)$ . Moreover  $\cup_{n=1}^{\infty} X_n$  is dense in  $\mathcal{F}_p([0, 1])$  and  $\cup_{n=1}^{\infty} Y_n$  is dense in  $\mathcal{F}_p(\mathbb{N}_*)$ . Since  $K_n$  is Lipschitz-isomorphic to  $M_n$  with distortion one, we are done.  $\square$

Next, we generalize the fact that the Haar system is a Schauder basis of  $L_1([0, 1])$ . Given an interval  $J = [a, b] \subseteq [0, 1]$  we define the Haar molecule  $h_J$  of the interval  $J$  by

$$\begin{aligned} h_J &= - \left( \delta_{[0,1]} \left( \frac{a+b}{2} \right) - \delta_{[0,1]}(a) \right) + \left( \delta_{[0,1]}(b) - \delta_{[0,1]} \left( \frac{a+b}{2} \right) \right) \\ &= \delta_{[0,1]}(a) + \delta_{[0,1]}(b) - 2\delta_{[0,1]} \left( \frac{a+b}{2} \right). \end{aligned}$$

Denote also  $h_0 = \delta_{[0,1]}(1) - \delta_{[0,1]}(0)$ . The Haar system of  $\mathcal{F}_p(\mathcal{M})$  is the family  $\mathcal{H} = (h_J)_{\{0\} \cup \mathcal{D}}$ , where  $\mathcal{D}$  is the set of dyadic intervals contained in  $[0, 1]$ .

**Theorem 5.7.** *The Haar system, arranged in such a way that  $h_0$  is its first term and Haar molecules of bigger intervals appear before, is a Schauder basis of  $\mathcal{F}_p([0, 1])$  with basis constant not bigger than  $3^{1/p-1}$ .*

*Proof.* Let  $(\mathbf{x}_n)_{n=0}^{\infty}$  be such an rearrangement of the Haar system of  $\mathcal{F}_p(\mathcal{M})$ . Let  $(K_n)_{n=0}^{\infty}$  be the sequence of subsets of  $[0, 1]$  constructed

recursively as follows:  $K_0 = \{0, 1\}$  and, if  $\mathbf{x}_n = h_J$  and  $c$  is the middle point of  $J$ ,  $K_n = K_{n-1} \cup \{c\}$ . By induction we see that

$$\mathbf{x}_k \in X_n := \text{span}(\{\delta(x) : x \in K_n\}), \quad 0 \leq k \leq n.$$

Since the dyadic points are dense in  $[0, 1]$ ,  $\cup_{n=0}^{\infty} X_n$  is dense in  $\mathcal{F}_p([0, 1])$  by [3, Proposition 4.17]. Using the notation of Lemma 5.4, we set

$$P_n = L_{[0,1],K_n} \circ P_{[0,1],K_n}, \quad n \geq 0.$$

It is clear that  $P_n(\mathcal{F}_p([0, 1])) = X_n$ . By Lemma 5.4 (i),  $\|P_n\| \leq 3^{1/p-1}$  and by Lemma 5.4 (ii) and (iii),  $P_n \circ P_m = P_{\min\{n,m\}}$ . Moreover, if  $\mathbf{x}_{n+1} = h_J$  and  $J = [a, b]$ ,

$$\begin{aligned} P_n(\mathbf{x}_{n+1}) &= P_n \left( \delta(a) + \delta(b) - 2\delta \left( \frac{a+b}{2} \right) \right) \\ &= \delta(a) + \delta(b) - 2 \left( \frac{1}{2}\delta(a) + \frac{1}{2}\delta(b) \right) = 0. \end{aligned}$$

Thus  $(\mathbf{x}_n)_{n=0}^{\infty}$  is a Schauder basis for  $\mathcal{F}_p([0, 1])$  with partial-sum projections  $(P_n)_{n=0}^{\infty}$ .  $\square$

Let us note here that by Theorem 5.3,  $\mathcal{F}_p(\mathbb{Z}^n)$  has a Schauder basis for each  $n \in \mathbb{N}$ . It seems to be an interesting problem whether  $\mathcal{F}_p([0, 1]^n)$  has a Schauder basis for each  $p \in (0, 1]$  (for  $p = 1$  this is true, see [20, 23]).

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