

# REMARKS ON WDC SETS

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**ABSTRACT.** We study WDC sets, which form a substantial generalization of sets with positive reach and still admit the definition of curvature measures. Main results concern WDC sets  $A \subset \mathbb{R}^2$ . We prove that, for such  $A$ , the distance function  $d_A = \text{dist}(\cdot, A)$  is a “DC aura” for  $A$ , which implies that each locally WDC set in  $\mathbb{R}^2$  is a WDC set. An another consequence is that compact WDC subsets of  $\mathbb{R}^2$  form a Borel subset of the space of all compact sets.

## 1. INTRODUCTION

In [10] (cf. also [8], [7] and [11]), the authors introduced the class of WDC sets which form a substantial generalization of sets with positive reach and still admit the definition of curvature measures. The following question naturally arises (see [8, Question 2, p. 829] and [7, 10.4.3]).

**Question.** Is the distance function  $d_A = \text{dist}(\cdot, A)$  of each WDC set  $A \subset \mathbb{R}^d$  a DC aura for  $F$  (see Definition 2.3)?

We answer this question positively in the case  $d = 2$  (Theorem 3.3 below); it remains open for  $d \geq 3$ . The proof is based on a characterization (proved in [11]) of locally WDC sets in  $\mathbb{R}^2$  and the main result of [12] which asserts that

$$(1.1) \quad d_A \text{ a DC function if } A \subset \mathbb{R}^2 \text{ is a graph of a DC function } g : \mathbb{R} \rightarrow \mathbb{R}.$$

Recall that a function is called DC, if it is the difference of two convex functions and note that each set  $A$  as in (1.1) is a WDC set.

Theorem 3.3 easily implies that each locally WDC set in  $\mathbb{R}^2$  is WDC. Further, we use Theorem 3.3 to prove that compact WDC subsets of  $\mathbb{R}^2$  form a Borel subset of the space of all compact sets of  $\mathbb{R}^2$  (Theorem 4.1 (i)). The importance of this result is the fact that it suggests that (at least in  $\mathbb{R}^2$ ) a theory of point processes on the space of compact WDC sets (analogous to the concept of point processes on the space of sets with positive reach introduced in [17]) can be build.

Concerning the compact WDC subsets of  $\mathbb{R}^d$  for  $d > 2$ , we are able to prove only a weaker fact that they form an analytic set (Theorem 4.1 (ii)) which is not probably sufficient for the above mentioned application.

## 2. PRELIMINARIES

**2.1. Basic definitions.** The symbol  $\mathbb{Q}$  denotes the set of all rational numbers. In any vector space  $V$ , we use the symbol  $0$  for the zero element. We denote by  $B(x, r)$  ( $U(x, r)$ ) the closed (open) ball with centre  $x$  and radius  $r$ . The boundary and the interior of a set  $M$  are denoted by  $\partial M$  and  $\text{int}M$ , respectively. A mapping is called  $K$ -Lipschitz if it is Lipschitz with a (not necessarily minimal) constant  $K \geq 0$ .

The metric space of all real-valued continuous functions on a compact  $K$  (equipped with the usual supremum metric  $\rho_{\text{sup}}$ ) will be denoted  $C(K)$ .

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In the Euclidean space  $\mathbb{R}^d$ , the norm is denoted by  $|\cdot|$  and the scalar product by  $\langle \cdot, \cdot \rangle$ . By  $S^{d-1}$  we denote the unit sphere in  $\mathbb{R}^d$ .

The distance function from a set  $A \subset \mathbb{R}^d$  is  $d_A := \text{dist}(\cdot, A)$  and the metric projection of  $z \in \mathbb{R}^d$  to  $A$  is  $\Pi_A(z) := \{a \in A : \text{dist}(z, A) = |z - a|\}$ .

**2.2. DC functions.** Let  $f$  be a real function defined on an open convex set  $C \subset \mathbb{R}^d$ . Then we say that  $f$  is a *DC function*, if it is the difference of two convex functions. Special DC functions are semiconvex and semiconcave functions. Namely,  $f$  is a *semiconvex* (resp. *semiconcave*) function, if there exist  $a > 0$  and a convex function  $g$  on  $C$  such that

$$f(x) = g(x) - a\|x\|^2 \quad (\text{resp. } f(x) = a\|x\|^2 - g(x)), \quad x \in C.$$

We will use the following well-known properties of DC functions.

**Lemma 2.1.** *Let  $C$  be an open convex subset of  $\mathbb{R}^d$ . Then the following assertions hold.*

- (i) *If  $f : C \rightarrow \mathbb{R}$  and  $g : C \rightarrow \mathbb{R}$  are DC, then (for each  $a \in \mathbb{R}, b \in \mathbb{R}$ ) the functions  $|f|, af + bg, \max(f, g)$  and  $\min(f, g)$  are DC.*
- (ii) *Each locally DC function  $f : C \rightarrow \mathbb{R}$  is DC.*
- (iii) *Each DC function  $f : C \rightarrow \mathbb{R}$  is Lipschitz on each compact convex set  $Z \subset C$ .*
- (iv) *Let  $f_i : C \rightarrow \mathbb{R}, i = 1, \dots, m$ , be DC functions. Let  $f : C \rightarrow \mathbb{R}$  be a continuous function such that  $f(x) \in \{f_1(x), \dots, f_m(x)\}$  for each  $x \in C$ . Then  $f$  is DC on  $C$ .*

*Proof.* Property (i) follows easily from definitions, see e.g. [15, p. 84]. Property (ii) was proved in [9]. Property (iii) easily follows from the local Lipschitzness of convex functions. Assertion (iv) is a special case of [16, Lemma 4.8.] (“Mixing lemma”).  $\square$

It is well-known (cf. [12]) that if  $\emptyset \neq A \subset \mathbb{R}^d$  is closed, then  $d_A$  need not be DC; however (see, e.g., [2, Proposition 2.2.2]),

$$(2.1) \quad d_A \text{ is locally semiconcave (and so locally DC) on } \mathbb{R}^d \setminus A.$$

**2.3. Clarke generalized gradient.** If  $U \subset \mathbb{R}^d$  is an open set,  $f : U \rightarrow \mathbb{R}$  is locally Lipschitz and  $x \in U$ , we denote by  $\partial_C f(x)$  the *generalized gradient of  $f$  at  $x$* , which can be defined as the closed convex hull of all limits  $\lim_{i \rightarrow \infty} f'(x_i)$  such that  $x_i \rightarrow x$  and  $f'(x_i)$  exists for all  $i \in \mathbb{N}$  (see [3, S1.1.2];  $\partial_C f(x)$  is also called *Clarke subdifferential of  $f$  at  $x$*  in the literature). Since we identify  $(\mathbb{R}^d)^*$  with  $\mathbb{R}^d$  in the standard way, we sometimes consider  $\partial_C f(x)$  as a subset of  $\mathbb{R}^d$ . We will repeatedly use the fact that the mapping  $x \mapsto \partial_C f(x)$  is upper semicontinuous and, hence (see [3, Theorem 2.1.5]),

$$(2.2) \quad v \in \partial_C f(x) \quad \text{whenever} \quad x_i \rightarrow x, \quad v_i \in \partial_C f(x_i) \quad \text{and} \quad v_i \rightarrow v.$$

We also use that  $|u| \leq K$  whenever  $u \in \partial_C f(x)$  and  $f$  is  $K$ -Lipschitz on a neighbourhood of  $x$ . Obviously,

$$(2.3) \quad \partial_C(\alpha f)(x) = \alpha \partial_C f(x).$$

Recall that

$$(2.4) \quad f^0(x, v) := \limsup_{y \rightarrow x, t \rightarrow 0+} \frac{f(y + tv) - f(y)}{t}$$

and (see [3])

$$(2.5) \quad f^0(x, v) = \sup\{\langle v, \nu \rangle : \nu \in \partial_C f(x)\}.$$

We will need the following simple lemma.

**Lemma 2.2.** *Let  $f$  be a Lipschitz function on an open set  $G \subset \mathbb{R}^d$ ,  $x \in G$  and  $\varepsilon > 0$ .*

(i) *If  $\text{dist}(0, \partial_C f(x)) \geq 2\varepsilon$ , then*

$$(2.6) \quad \exists v \in S^{d-1}, \rho > 0 \forall y \in U(x, \rho), 0 < \alpha < \rho: \frac{f(y + \alpha v) - f(y)}{\alpha} \leq -\varepsilon.$$

(ii) *If (2.6) holds, then  $\text{dist}(0, \partial_C f(x)) \geq \varepsilon$ .*

*Proof.* (i) Let  $\text{dist}(0, \partial_C f(x)) \geq 2\varepsilon$ . Since  $\partial_C f(x)$  is convex, there exists (see e.g. [4, Theorem 1.5.])  $v \in S^{d-1}$  such that

$$\text{dist}(0, \partial_C f(x)) = -\sup\{\langle v, \nu \rangle : \nu \in \partial_C f(x)\}.$$

So, by (2.5),  $f^0(x, v) \leq -2\varepsilon$  and thus (2.4) implies (2.6).

(ii) If (2.6) holds, choose corresponding  $v \in S^{d-1}$  and  $\rho > 0$ . Then  $f^0(x, v) \leq -\varepsilon$  by (2.4). Consequently, by (2.5),  $-|\nu| \leq \langle v, \nu \rangle \leq -\varepsilon$  for each  $\nu \in \partial_C f(x)$  and so  $\text{dist}(0, \partial_C f(x)) \geq \varepsilon$ .  $\square$

**2.4. WDC sets.** WDC sets (see the definition below) which provide a natural generalization of sets with positive reach were defined in [10] using Fu's notion of an "aura" of a set (see, e.g., [7] for more information). Note that the notion of a DC aura were defined in [10] and [8] by a formally different but equivalent way (cf. [11, Remark 2.12 (v)]).

**Definition 2.3** (cf. [11, Definitions 2.8, 2.10]). *Let  $U \subset \mathbb{R}^d$  be open and  $f : U \rightarrow \mathbb{R}$  be locally Lipschitz. A number  $c \in \mathbb{R}$  is called a weakly regular value of  $f$  if whenever  $x_i \rightarrow x$ ,  $f(x_i) > c = f(x)$  and  $u_i \in \partial_C f(x_i)$  for all  $i \in \mathbb{N}$  then  $\liminf_i |u_i| > 0$ .*

*A set  $A \subset \mathbb{R}^d$  is called WDC if there exists a DC function  $f : \mathbb{R}^d \rightarrow [0, \infty)$  such that  $A = f^{-1}\{0\}$  and 0 is a weakly regular value of  $f$ . In such a case, we call  $f$  a DC aura (for  $A$ ).*

*A set  $A \subset \mathbb{R}^d$  is called locally WDC if for any point  $a \in A$  there exists a WDC set  $A^* \subset \mathbb{R}^d$  that agrees with  $A$  on an open neighbourhood of  $a$ .*

(Note that a weakly regular value of  $f$  need not be in the range of  $f$ , and so  $\emptyset$  is clearly a WDC set by our definition.)

Note that a set  $A \subset \mathbb{R}^d$  has a positive reach at each point if and only there exists a DC aura for  $A$  which is even semiconvex ([1]).

### 3. DISTANCE FUNCTION OF A WDC SET IN $\mathbb{R}^2$ IS AN AURA

First we present (slightly formally rewritten) [11, Definition 7.9].

**Definition 3.1.** (i) *A set  $S \subset \mathbb{R}^2$  will be called a basic open DC sector (of radius  $r$ ) if  $S = U(0, r) \cap \{(u, v) \in \mathbb{R}^2 : u \in (-\omega, \omega), v > f(u)\}$ , where  $0 < r < \omega$  and  $f$  is a DC function on  $(-\omega, \omega)$  such that  $f(0) = 0$ ,  $R(u) := \sqrt{u^2 + f^2(u)}$  is strictly increasing on  $[0, \omega)$  and strictly decreasing on  $(-\omega, 0]$ .*

*By an open DC sector (of radius  $r$ ) we mean an image  $\gamma(S)$  of a basic open DC sector  $S$  (of radius  $r$ ) under a rotation around the origin  $\gamma$ .*

(ii) *A set of the form  $\gamma(\{(u, v) \in \mathbb{R}^2 : u \in [0, \omega), g(u) \leq v \leq f(u)\}) \cap U(0, r)$ , where  $\gamma$  is a rotation around the origin,  $0 < r < \omega$  and  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are DC functions such that  $g \leq f$  on  $[0, \omega)$ ,  $f(0) = g(0) = f'_+(0) = g'_+(0) = 0$  and the functions  $R_f(u) := \sqrt{u^2 + f^2(u)}$ ,  $R_g(u) := \sqrt{u^2 + g^2(u)}$  are strictly increasing on  $[0, \omega)$ , will be called a degenerated closed DC sector (of radius  $r$ ).*

We will use the following complete characterization of locally WDC sets in  $\mathbb{R}^2$  ([11, Theorem 8.14]).

**Theorem PRZ.** *Let  $M$  be a closed subset of  $\mathbb{R}^2$ . Then  $M$  is a locally WDC set if and only if for each  $x \in \partial M$  there is  $\rho > 0$  such that one of the following conditions holds:*

- (i)  $M \cap U(x, \rho) = \{x\}$ ,
- (ii) there is a degenerated closed DC sector  $C$  of radius  $\rho$  such that

$$M \cap U(x, \rho) = x + C,$$

- (iii) there are pairwise disjoint open DC sectors  $C_1, \dots, C_k$  of radius  $\rho$  such that

$$(3.1) \quad U(x, \rho) \setminus M = \bigcup_{i=1}^k (x + C_i).$$

**Lemma 3.2.** *Let  $f$  be an  $L$ -Lipschitz function on  $\mathbb{R}$ . Denote  $d := \text{dist}(\cdot, \text{graph } f)$ . Then  $|\xi_2| \geq \frac{1}{\sqrt{L^2+1}}$  whenever  $\xi = (\xi_1, \xi_2) \in \partial_C d(x)$  and  $x \in \mathbb{R}^2 \setminus \text{graph } f$ .*

*Proof.* Pick  $x \in \mathbb{R}^2 \setminus \text{graph } f$ . Without any loss of generality we can assume that  $x = 0$ . We will assume that  $f(0) < 0$ ; the case  $f(0) > 0$  is quite analogous. Denote  $r := d(0)$  and  $P := \Pi_{\text{graph } f}(0)$ . Set  $g(u) := -\sqrt{r^2 - u^2}$ ,  $u \in [-r, r]$ . Clearly  $f \leq g$  on  $[-r, r]$  and  $(u, v) \in P$  if and only if  $f(u) = g(u) = v$ . We will show that

$$(3.2) \quad |u| \leq \frac{Lr}{\sqrt{1+L^2}} \text{ whenever } (u, v) \in P.$$

To this end, suppose  $(u, v) \in P$ . If  $u > 0$ , then

$$L \geq \frac{f(t) - f(u)}{t - u} \geq \frac{g(t) - g(u)}{t - u} \text{ for each } 0 < t < u,$$

and consequently  $L \geq g'_-(u)$ . Therefore  $u < r$  and  $L \geq u(r^2 - u^2)^{-1/2}$ . Analogously considering  $g'_+(u)$ , we obtain for  $u < 0$  that  $u > -r$  and  $u(r^2 - u^2)^{-1/2} \geq -L$ . In both cases we have  $L \geq |u|(r^2 - u^2)^{-1/2}$  and an elementary computation gives (3.2).

Using (3.2) we obtain that if  $(u, v) \in P$  then

$$(3.3) \quad v = g(u) \leq -\sqrt{r^2 - \left(\frac{Lr}{\sqrt{1+L^2}}\right)^2} = -\frac{r}{\sqrt{1+L^2}}.$$

By [6, Lemma 4.2] and (3.3) we obtain

$$\partial_C d(0) = \text{conv} \left\{ \frac{1}{r}(-u, -v) : (u, v) \in P \right\} \subset \mathbb{R} \times \left[ \frac{1}{\sqrt{L^2+1}}, \infty \right)$$

and the assertion of the lemma follows.  $\square$

**Theorem 3.3.** *Let  $M \neq \emptyset$  be a locally WDC set in  $\mathbb{R}^2$ . Then the distance function  $d_M$  is a DC aura for  $M$ . In particular,  $M$  is a WDC set.*

*Proof.* Denote  $d := d_M$ . For each  $x \in \partial M$  choose  $\rho = \rho(x)$  by Theorem PRZ. We will prove that

- (a)  $d$  is DC on  $U(x, \frac{\rho}{3})$ ,
- (b) there is  $\varepsilon = \varepsilon(x) > 0$  such that  $|\xi| \geq \varepsilon$  whenever  $y \in U(0, \frac{\rho}{3}) \setminus M$  and  $\xi \in \partial_C d(y)$ .

Without any loss of generality we can assume that  $x = 0$ .

If case (i) from Theorem PRZ holds, then  $d(y) = |y|$ ,  $y \in U(0, \frac{\rho}{3})$ , and so  $d$  is convex and therefore DC on  $U(0, \frac{\rho}{3})$ . Similarly, condition (b) holds as well, since if  $y \in U(0, \frac{\rho}{3}) \setminus M$  and  $\xi \in \partial_C d(y)$  then  $\xi = \frac{y}{|y|}$  and so  $|\xi| = 1$ .

If case (ii) from Theorem PRZ holds, we know that  $M \cap U(0, \rho)$  is a degenerated closed DC sector  $C$  of radius  $\rho$ . Let  $\gamma, f, g$  and  $\omega$  be as in Definition 3.1. Without any loss of generality we may assume that  $\gamma$  is the identity map.

By Lemma 2.1 (iii) we can choose  $L > 0$  such that both  $f$  and  $g$  are  $L$ -Lipschitz on  $[0, \rho]$  and define

$$\tilde{f}(u) := \begin{cases} f(u) & \text{if } 0 \leq u \leq \rho, \\ f(\rho) & \text{if } \rho < u, \\ 2Ly & \text{if } u < 0. \end{cases} \quad \text{and} \quad \tilde{g}(u) := \begin{cases} g(u) & \text{if } 0 \leq u \leq \rho, \\ g(\rho) & \text{if } \rho < u, \\ -2Lu & \text{if } u < 0. \end{cases}$$

It is easy to see that both  $\tilde{f}$  and  $\tilde{g}$  are  $2L$ -Lipschitz and they are DC by Lemma 2.1 (iv).

Put

$$M_0 := \{(u, v) \in \mathbb{R}^2 : u \geq 0, \tilde{g}(u) \leq v \leq \tilde{f}(u)\},$$

$$M_1 := \{(u, v) \in \mathbb{R}^2 : u \geq 0, \tilde{f}(u) < v\} \cup \{(u, v) \in \mathbb{R}^2 : u < 0, -\frac{u}{2L} < v\},$$

$$M_2 := \{(u, v) \in \mathbb{R}^2 : u \geq 0, \tilde{g}(u) > v\} \cup \{(u, v) \in \mathbb{R}^2 : u < 0, \frac{u}{2L} > v\}$$

and

$$M_3 := \{(u, v) \in \mathbb{R}^2 : \frac{u}{L} < v < -\frac{u}{L}\}.$$

Clearly  $\mathbb{R}^2 = M_0 \cup M_1 \cup M_2 \cup M_3$  and  $M_1, M_2, M_3$  are open.

Set  $\tilde{d} := \text{dist}(\cdot, M_0)$  and, for each  $y \in \mathbb{R}^2$ , define

$$d_0(y) = 0, \quad d_1(y) := \text{dist}(y, \text{graph } \tilde{f}), \quad d_2(y) := \text{dist}(y, \text{graph } \tilde{g}), \quad d_3(y) := |y|.$$

Functions  $d_1$  and  $d_2$  are DC on  $\mathbb{R}^2$  by (1.1),  $d_0$  and  $d_3$  are convex and therefore DC on  $\mathbb{R}^2$ .

Using (for  $K = 1/L, -1/L, 1/(2L), -1/(2L)$ ) the facts that the lines with the slopes  $K$  and  $-1/K$  are orthogonal and  $M_0 \subset \{(u, v) : u \geq 0, -Lu \leq v \leq Lu\}$ , easy geometrical observations show that

$$(3.4) \quad \tilde{d}(y) = d_i(y) \quad \text{if } y \in M_i, \quad 0 \leq i \leq 3,$$

and so Lemma 2.1 (iv) implies that  $\tilde{d}$  is DC.

Now pick an arbitrary  $y \in \mathbb{R}^2 \setminus M_0 = M_1 \cup M_2 \cup M_3$  and choose  $\xi = (\xi_1, \xi_2) \in \partial_C \tilde{d}(y)$ . Using (3.4), we obtain that if  $y \in M_3$  then  $\xi = \frac{y}{|y|}$  and so  $|\xi| = 1$ . Using Lemma 3.2, we obtain that if  $y \in M_1 \cup M_2$ , then  $|\xi| \geq |\xi_1| \geq \frac{1}{\sqrt{4L^2+1}}$ .

Now, since  $d = \tilde{d}$  on  $U(0, \frac{\rho}{3})$  both (a) and (b) follow.

It remains to prove (a) and (b) if case (iii) from Theorem PRZ holds. Let  $C_i$ ,  $i = 1, \dots, k$ , be the open DC sectors as in (iii). Denote  $A_i := \mathbb{R}^2 \setminus C_i$  and define  $\delta_i := \text{dist}(\cdot, A_i)$ ,  $i = 1, \dots, k$ .

Note that, for  $y \in U(0, \frac{\rho}{3})$ , one has

$$d(y) = \begin{cases} \delta_i(y) & \text{if } y \in C_i, \\ 0 & \text{if } y \in M. \end{cases}$$

Therefore (by Lemma 2.1 (iv)) it is enough to prove that (a) and (b) hold with  $d$  and  $M$  being replaced by  $\delta_i$  and  $A_i$ , respectively ( $i = 1, \dots, k$ ). Fix some  $i \in \{1, \dots, k\}$ . Without any loss of generality we can assume that  $C_i$  is an basic open DC sector of radius  $\rho$  with corresponding  $f_i$  and  $\omega_i$ . Now define

$$\tilde{f}_i(u) := \begin{cases} f_i(u) & \text{if } u \in [-\rho, \rho], \\ f_i(-\rho) & \text{if } u < -\rho, \\ f_i(\rho) & \text{if } u > \rho. \end{cases}$$

Then  $\tilde{f}_i$  is Lipschitz and DC on  $\mathbb{R}$ . Put  $\tilde{d}_i(y) = \text{dist}(y, \text{graph } \tilde{f}_i)$ . Then  $\tilde{d}_i$  is DC by (1.1) and 0 is a weakly regular value of  $\tilde{d}_i$  by Lemma 3.2. And since  $d_i = \tilde{d}_i$  on  $U(0, \frac{\rho}{3})$  we are done.

Since  $d$  is locally DC on  $\mathbb{R}^2 \setminus M$  by (2.1) and on the interior of  $M$  (trivially), (a) implies that  $d$  is locally DC and so DC by Lemma 2.1 (ii). Further, (b) immediately implies that 0 is a weakly regular value of  $d$  and thus  $d = d_M$  is an aura for  $M$ .  $\square$

**Remark 3.4.** *By Theorem 3.3, in  $\mathbb{R}^2$  locally WDC sets and WDC sets coincide. This gives a partial answer to the part of [10, Problem 10.2] which asks whether the same is true in each  $\mathbb{R}^d$ .*

#### 4. COMPLEXITY OF THE SYSTEM OF WDC SETS

In the following, we will work in each moment in an  $\mathbb{R}^d$  with a fixed  $d$ , and so for simplicity we will use the notation, in which the dependence on  $d$  is usually omitted.

The space of all nonempty compact subsets of  $\mathbb{R}^d$  equipped with the usual Hausdorff metric  $\rho_H$  is denoted by  $\mathcal{K}$ . It is well-known (see, e.g., [14, Proposition 2.4.15 and Corollary 2.4.16]) that  $\mathcal{K}$  is a separable complete metric space. For a closed set  $M \subset \mathbb{R}^d$ , we set  $\mathcal{K}(M) := \{K \in \mathcal{K} : K \subset M\}$ , which is clearly a closed subspace of  $\mathcal{K}$ . The set of all nonempty compact WDC subsets of  $M \subset \mathbb{R}^d$  will be denoted by  $WDC(M)$ .

In this section, we will prove the following theorem.

**Theorem 4.1.** *(i)  $WDC(\mathbb{R}^2)$  is an  $F_{\sigma\delta\sigma}$  subset of  $\mathcal{K}(\mathbb{R}^2)$ .  
(ii)  $WDC(\mathbb{R}^d)$  is an analytic subset of  $\mathcal{K}(\mathbb{R}^d)$  for each  $d \in \mathbb{N}$ .*

Before the proof of this theorem, we introduce some spaces, make a number of observations, and prove a technical lemma.

First observe that  $WDC(\mathbb{R}^d) = \bigcup_{n=1}^{\infty} WDC(B(0, n))$  and so, to prove Theorem 4.1, it is sufficient to prove that, for each  $r > 0$ ,

$$(4.1) \quad \begin{aligned} & \text{for } d = 2 \text{ (resp. } d \in \mathbb{N}\text{), } WDC(B(0, r)) \\ & \text{is an } F_{\sigma\delta\sigma} \text{ (resp. analytic) subset of } \mathcal{K}(B(0, r)). \end{aligned}$$

Further observe that it is sufficient to prove (4.1) for  $r = 1$ . Indeed, denoting  $H(x) := x/r$ ,  $x \in \mathbb{R}^d$ , it is obvious that  $H^* : K \mapsto H(K)$  gives a homeomorphism of  $\mathcal{K}(B(0, r))$  onto  $\mathcal{K}(B(0, 1))$  and  $H^*(WDC(B(0, r))) = WDC(B(0, 1))$  (clearly  $f$  is an aura for  $K$  if and only if  $f \circ H^{-1}$  is an aura for  $H^*(K)$ ).

To prove (4.1) for  $r = 1$ , we will consider the space  $X$  of all 1-Lipschitz functions  $f : B(0, 4) \rightarrow [0, 4]$  such that  $f \geq 1$  on  $B(0, 4) \setminus U(0, 3)$ , equipped with the supremum metric  $\rho_{\text{sup}}$ . Obviously,  $X$  is a closed subspace of  $C(B(0, 4))$  and so it is a separable complete metric space.

The motivation for introducing  $X$  is the fact that

$$(4.2) \quad \text{if } K \in \mathcal{K}(B(0, 1)), \text{ then } f_K := d_K \upharpoonright_{B(0, 4)} \in X.$$

Since we are interested in  $K \in WDC(B(0, 1))$ , we define also two subspaces of  $X$ :

$$A := \{f \in X : 0 \text{ is a weakly regular value of } f|_{U(0, 4)}\},$$

$$D := \{f \in X : f = g - h \text{ for some convex Lipschitz functions } g, h \text{ on } B(0, 4)\}.$$

Their complexity is closely related to the complexity of  $WDC(B(0, 1))$ , as the following lemma indicate.

**Lemma 4.2.** *Let  $\emptyset \neq K \subset B(0, 1) \subset \mathbb{R}^d$  be compact. Then:*

- (i)  *$K$  is WDC if and only if there is a function  $g \in D \cap A$  such that  $K = g^{-1}(0)$ .*
- (ii) *If  $d = 2$ , then  $K$  is WDC if and only if  $f_K := d_K \upharpoonright_{B(0, 4)} \in D \cap A$ .*

*Proof.* (i) Suppose first that  $K$  is WDC and  $f$  is an aura for  $K$ . Using Lemma 2.1(iii), we can choose  $\alpha > 0$  so small that the function  $\alpha f$  is 1-Lipschitz on  $B(0, 4)$  and  $0 \leq \alpha f(x) \leq 4$  for  $x \in B(0, 4)$ . Set

$$h(x) := \max(|x| - 2, \alpha f(x)), \quad x \in \mathbb{R}^d, \quad \text{and} \quad g := f \lfloor_{B(0,4)}.$$

Then clearly  $K = g^{-1}(0)$ . Since  $h$  is DC on  $\mathbb{R}^d$  by Lemma 2.1(i), we obtain  $g \in D$  by Lemma 2.1(iii). Finally,  $g \in A$ , since  $g = \alpha f$  on  $U(0, 2)$ .

Conversely, suppose that  $K = g^{-1}(0)$  for some  $g \in A \cap D$  and set

$$f(x) := \begin{cases} \min(g(x), 1), & \text{if } x \in U(0, 4), \\ 1 & \text{otherwise.} \end{cases}$$

Since  $f$  is DC on  $U(0, 4)$  by Lemma 2.1 (i) and  $f = 1$  on  $\mathbb{R}^d \setminus B(0, 3)$ , we see that  $f$  is locally DC and so DC by Lemma 2.1 (ii). Since 0 is clearly a weakly regular value of  $f$ , we obtain that  $f$  is an aura for  $K$ .

(ii) If  $K$  is WDC, first note that  $f_K \in X$  (see (4.2)). Since  $d_K$  is an aura for  $K$  by Theorem 3.3, we obtain immediately that  $f_K \in A$ , and also  $f_K \in D$  by Lemma 2.1 (iii).

If  $f_K \in A \cap D$ , then  $K$  is WDC by (i).  $\square$

For the application of Lemma 4.2(ii) we need the simple fact that

$$(4.3) \quad \Psi : K \mapsto f_K, \quad K \in \mathcal{K}(B(0, 1)), \quad \text{is a continuous mapping into } X.$$

Indeed, if  $K_1, K_2 \in \mathcal{K}(B(0, 1))$  with  $\rho_H(K_1, K_2) < \varepsilon$  and  $x \in B(0, 4)$ , then clearly  $d_{K_1}(x) < d_{K_2}(x) + \varepsilon$ ,  $d_{K_2}(x) < d_{K_1}(x) + \varepsilon$ , and consequently  $\rho_{\text{sup}}(f_{K_1}, f_{K_2}) \leq \varepsilon$ .

Further observe that

$$(4.4) \quad D \text{ is an } F_\sigma \text{ subset of } X.$$

To prove it, for each  $n \in \mathbb{N}$  set

$$C_n := \{g \in C(B(0, 4)) : g \text{ is convex } n\text{-Lipschitz and } |g(x)| \leq 4n + 4, x \in B(0, 4)\}.$$

Now observe that if  $f \in D$  then we can choose  $n \in \mathbb{N}$  and convex  $n$ -Lipschitz functions  $g, h$  such that  $f = g - h$ ,  $g(0) = 0$  and consequently  $\|g\| \leq 4n$ ,  $\|h\| \leq 4n + 4$ , and so  $g, h \in C_n$ . Consequently,  $D = X \cap \bigcup_{n=1}^{\infty} (C_n - C_n)$ . Each  $C_n$  is clearly closed in  $C(B(0, 4))$  and so it is compact in  $C(B(0, 4))$  by the Arzelà-Ascoli theorem. Consequently also  $C_n - C_n = \sigma(C_n \times C_n)$ , where  $\sigma$  is the continuous mapping  $\sigma : (g, h) \mapsto g - h$ , is compact, and (4.4) follows.

The most technical part of the proof of Theorem 4.1 is to show that  $A$  is an  $F_{\sigma\delta\sigma}$  subset of  $X$ . To prove it, we need some lemmas.

**Lemma 4.3.** *Let  $f \in X$ . Then  $f \in A$  if and only*

$$(4.5) \quad \exists 0 < \varepsilon \forall x \in f^{-1}(0, \varepsilon), \nu \in \partial_C f(x) : |\nu| \geq \varepsilon.$$

*Proof.* If (4.5) holds, then we easily obtain  $f \in A$  directly from the definition of a weakly regular value.

To prove the opposite implication, suppose that  $f \in A$  and (4.5) does not hold. Then there exist points  $x_n \in f^{-1}(0, 1/n)$ ,  $n \in \mathbb{N}$ , and  $\nu_n \in \partial_C f(x_n)$  such that  $|\nu_n| < 1/n$ . Choose a subsequence  $x_{n_k} \rightarrow x \in B(0, 4)$ . Since  $0 \leq f(x_{n_k}) < 1/n_k$ , we have  $f(x_{n_k}) \rightarrow f(x) = 0$ , and consequently  $x \in U(0, 4)$ . Since  $\nu_{n_k} \rightarrow 0$ , we obtain that 0 is not a weakly regular value of  $f|_{U(0,4)}$ , which contradicts  $f \in A$ .  $\square$

Denote  $\mathbb{Q}^* := \mathbb{Q} \cap (0, 1)$  and for every  $\varepsilon \in \mathbb{Q}^*$  and  $d \in \mathbb{N}$  pick a finite set  $\mathfrak{S}_\varepsilon^d \subset S^{d-1}$  such that for every  $v \in S^{d-1}$  there is some  $\nu \in \mathfrak{S}_\varepsilon^d$  satisfying  $|v - \nu| < \varepsilon$ .

**Lemma 4.4.** *Let  $f$  be a function from  $X$ . Then  $f \in A$  if and only if*

$$(4.6) \quad \begin{aligned} & \exists \varepsilon \in \mathbb{Q}^* \forall p, q \in \mathbb{Q}^*, 0 < p < q < \varepsilon \exists \rho \in \mathbb{Q}^* \forall x \in U(0, 4) : \\ & (f(x) \notin (p, q) \vee \exists \nu \in \mathfrak{S}_\varepsilon^d \forall y \in U(x, \rho), 0 < \alpha < \rho : f(y + \alpha\nu) - f(y) \leq -\varepsilon\alpha). \end{aligned}$$

*Proof.* First suppose that (4.6) holds and choose  $\varepsilon \in \mathbb{Q}^*$  by (4.6). We will show that

$$(4.7) \quad \forall x \in f^{-1}(0, \varepsilon), \nu \in \partial_C f(x) : |\nu| \geq \varepsilon.$$

To this end, consider an arbitrary  $x \in f^{-1}(0, \varepsilon)$  and choose  $p, q \in \mathbb{Q}^*$  such that  $0 < p < q < \varepsilon$  and  $f(x) \in (p, q)$ . Choose  $\rho \in \mathbb{Q}^*$  which exists for  $\varepsilon, p, q$  by (4.6). So there exists  $\nu \in \mathfrak{S}_\varepsilon^d$  such that

$$\forall y \in U(x, \rho), 0 < \alpha < \rho : f(y + \alpha\nu) - f(y) \leq -\varepsilon\alpha.$$

Therefore Lemma 2.2 (ii) gives that  $|\nu| \geq \varepsilon$  for each  $\nu \in \partial_C f(x)$ . Thus (4.7) holds and so  $f \in A$  by Lemma 4.3.

Now suppose  $f \in A$ . Using (4.5), we can choose  $\varepsilon \in \mathbb{Q}^*$  such that

$$(4.8) \quad \forall x \in f^{-1}(0, \varepsilon), \nu \in \partial_C f(x) : |\nu| \geq 4\varepsilon.$$

To prove (4.6), consider arbitrary  $p, q \in \mathbb{Q}^*$ ,  $0 < p < q < \varepsilon$ . Using Lemma 2.2 (i), we easily obtain that for each  $z \in K := f^{-1}([p, q])$  there exist  $\rho(z) > 0$  and  $v(z) \in S^{d-1}$  such that

$$(4.9) \quad \forall y \in U(z, \rho(z)), 0 < \alpha < \rho(z) : f(y + \alpha v(z)) - f(y) \leq -2\varepsilon\alpha.$$

Choose  $\rho \in \mathbb{Q}^*$  as a Lebesgue number (see [5]) of the open covering  $\{U(z, \rho(z))\}_{z \in K}$  of the compact  $K$ . For an arbitrary  $x \in U(0, 4)$ , either  $f(x) \notin (p, q)$  or  $x \in K$ . In the second case, by the definition of Lebesgue number, there exists  $z \in K$  such that  $U(x, \rho) \subset U(z, \rho(z))$ . Then clearly  $\rho < \rho(z)$  and so (4.9) implies

$$(4.10) \quad \forall y \in U(x, \rho), 0 < \alpha < \rho : f(y + \alpha v(z)) - f(y) \leq -2\varepsilon\alpha.$$

By the choice of  $\mathfrak{S}_\varepsilon^d$  there is some  $\nu \in \mathfrak{S}_\varepsilon^d$  such that  $|v(z) - \nu| < \varepsilon$ . By (4.10), for each  $y \in U(x, \rho)$  and  $0 < \alpha < \rho$ ,

$$f(y + \alpha v(z)) - f(y) \leq -2\varepsilon\alpha.$$

Consequently, using 1-Lipschitzness of  $f \in X$ , we obtain

$$\begin{aligned} f(y + \alpha\nu) - f(y) & \leq f(y + \alpha v(z)) - f(y) + |f(y + \alpha\nu) - f(y + \alpha v(z))| \\ & \leq f(y + \alpha v(z)) - f(y) + |\nu - v(z)|\alpha \leq -2\varepsilon\alpha + \varepsilon\alpha = -\varepsilon\alpha, \end{aligned}$$

and so (4.6) holds.  $\square$

**Corollary 4.5.** *The set  $A$  is an  $F_{\sigma\delta\sigma}$  subset of  $X$ .*

*Proof.* For each quadruple  $y \in \mathbb{R}^d$ ,  $\nu \in S^{d-1}$ ,  $\alpha > 0$ ,  $\varepsilon > 0$  we set

$$C(y, \nu, \alpha, \varepsilon) := \{f \in X : f(y + \alpha\nu) - f(y) \leq -\varepsilon\alpha\}.$$

(Of course, we have  $C(y, \nu, \alpha, \varepsilon) = \emptyset$  if  $y \notin U(0, 4)$  or  $y + \alpha\nu \notin U(0, 4)$ .) Further, for each triple  $x \in U(0, 4)$ ,  $0 < p < q$ , we set

$$D(x, p, q) := \{f \in X : f(x) \notin (p, q)\}.$$

It is easy to see that both  $C(y, \nu, \alpha, \varepsilon)$  and  $D(x, p, q)$  are always closed subsets of  $X$ . It is easy to see that Lemma 4.4 is equivalent to

$$A = \bigcup_{\varepsilon \in \mathbb{Q}^*} \bigcap_{\substack{p, q \in \mathbb{Q}^*, \\ 0 < p < q < \varepsilon}} \bigcup_{\rho \in \mathbb{Q}^*} \bigcap_{x \in U(0, 4)} \left( D(x, p, q) \cup \bigcup_{\nu \in \mathfrak{S}_\varepsilon^d} \bigcap_{\substack{y \in U(x, \rho), \\ 0 < \alpha < \rho}} C(y, \nu, \alpha, \varepsilon) \right).$$



Therefore, since  $\mathbb{Q}^*$  is countable and each  $\mathfrak{S}_\varepsilon^d$  is finite, we obtain that  $A$  is an  $F_{\sigma\delta\sigma}$  subset of  $X$ .  $\square$

**The proof of Theorem 4.1.** We know that it is sufficient to prove (4.1) for  $r = 1$ .

Suppose  $d = 2$ . Then Lemma 4.2(ii) gives that  $WDC(B(0, 1)) = \psi^{-1}(A \cap D)$ , where  $\psi : \mathcal{K}(B(0, 4)) \rightarrow X$  is the continuous mapping from (4.3). Since  $A \cap D$  is an  $F_{\sigma\delta\sigma}$  subset of  $X$  by Corollary 4.5 and (4.4), we obtain (4.1) for  $r = 1$  and  $d = 2$ , and thus also assertion (i) of Theorem 4.1.

To prove assertion (ii) of Theorem 4.1, it is sufficient to prove that (in each  $\mathbb{R}^d$ )  $WDC(B(0, 1))$  is an analytic subset of  $\mathcal{K}(B(0, 1))$ . To this end, consider the following subset  $S$  of  $\mathcal{K}(B(0, 1)) \times X$ :

$$S := \{(K, f) \in \mathcal{K}(B(0, 1)) \times X : f^{-1}(0) = K, f \in A \cap D\}.$$

By Lemma 4.2(i),  $WDC(B(0, 1)) = \pi_1(S)$  (where  $\pi_1(K, f) := K$ ) and so it is sufficient to prove that  $S$  is Borel. Denoting

$$Z := \{(K, f) \in \mathcal{K}(B(0, 1)) \times X : K = f^{-1}(0), f \in X\},$$

we have  $S = Z \cap (\mathcal{K}(B(0, 1)) \times (A \cap D))$ . So, since  $A \cap D$  is Borel by Corollary 4.5 and (4.4), to prove that  $S$  is Borel, it is sufficient to show that  $Z$  is Borel in  $\mathcal{K}(B(0, 1)) \times X$ . To this end, denote for each  $n \in \mathbb{N}$

$$P_n := \{(K, f) \in \mathcal{K}(B(0, 1)) \times X : \exists x \in K : f(x) \geq 1/n\},$$

$$Q_n := \{(K, f) \in \mathcal{K}(B(0, 1)) \times X : \exists x \in B(0, 4) : \text{dist}(x, K) \geq 1/n, f(x) = 0\}.$$

Since clearly

$$Z = (\mathcal{K}(B(0, 1)) \times X) \setminus \left( \bigcup_{n=1}^{\infty} P_n \cup \bigcup_{n=1}^{\infty} Q_n \right),$$

it is sufficient to prove that all  $P_n$  and  $Q_n$  are closed.

So suppose that  $(K_i, f_i) \in \mathcal{K}(B(0, 1)) \times X$ ,  $i = 1, 2, \dots$ ,  $(K, f) \in \mathcal{K}(B(0, 1)) \times X$ ,  $\rho_H(K_i, K) \rightarrow 0$  and  $\rho_{\text{sup}}(f_i, f) \rightarrow 0$ .

First suppose that  $n \in \mathbb{N}$  and all  $(K_i, f_i) \in P_n$ . Choose  $x_i \in K_i$  with  $f_i(x_i) \geq 1/n$ . Choose a convergent subsequence  $x_{i_j} \rightarrow x \in \mathbb{R}^d$ . It is easy to see that  $x \in K$ . Since  $|f_{i_j}(x_{i_j}) - f(x_{i_j})| \rightarrow 0$  and  $f(x_{i_j}) \rightarrow f(x)$ , we obtain  $f_{i_j}(x_{i_j}) \rightarrow f(x)$ , and consequently  $f(x) \geq 1/n$ . Thus  $(K, f) \in P_n$  and therefore  $P_n$  is closed.

Second, suppose that  $n \in \mathbb{N}$  and all  $(K_i, f_i) \in P_n$ . Choose  $x_i \in B(0, 4)$  such that  $\text{dist}(x_i, K_i) \geq 1/n$  and  $f_i(x_i) = 0$ . Choose a convergent subsequence  $x_{i_j} \rightarrow x \in B(0, 4)$ . Since  $|f_{i_j}(x_{i_j}) - f(x_{i_j})| \rightarrow 0$  and  $f(x_{i_j}) \rightarrow f(x)$ , we obtain  $f(x) = 0$ . Now consider an arbitrary  $y \in K$  and choose a sequence  $y_j \in K_{i_j}$  with  $y_j \rightarrow y$ . Since  $|x_{i_j} - y_j| \geq 1/n$  and  $x_{i_j} \rightarrow x$ , we obtain that  $|y - x| \geq 1/n$  and consequently  $\text{dist}(x, K) \geq 1/n$ . Thus  $(K, f) \in Q_n$  and therefore  $Q_n$  is closed.

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