POLISH SPACES OF BANACH SPACES. COMPLEXITY OF ISOMETRY CLASSES AND GENERIC PROPERTIES

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Abstract. We present and thoroughly study natural Polish spaces of separable Banach spaces. These spaces are defined as spaces of norms, resp. pseudonorms on the countable infinite-dimensional rational vector space. We provide an exhaustive comparison of these spaces with the admissible topologies recently introduced by Godefroy and Saint-Raymond and show that Borel complexities differ little with respect to these two different topological approaches.

We then focus mainly on the Borel complexities of isometry classes of the most classical Banach spaces. We prove that the infinite-dimensional Hilbert space is characterized as the unique separable infinite-dimensional Banach space whose isometry class is closed, and also as the unique separable infinite-dimensional Banach space whose isomorphism class is F_σ. For $p \in [1,2) \cup (2,\infty)$, we show that the isometry classes of $L_p[0,1]$ and ℓ_p are G_δ-complete and $F_{\sigma\delta}$-complete, respectively. Then we show that the isometry class of the Gurariĭ space is G_δ-complete and the isometry class of c_0 is $F_{\sigma\delta}$-complete. The isometry class of the former space is moreover proved to be dense G_δ. Additionally, we compute the complexities of many other natural classes of Banach spaces; for instance, $\mathcal{L}_{p,\lambda+}$-spaces, for $p, \lambda \geq 1$, are shown to be G_δ, superreflexive spaces are shown to be $F_{\sigma\delta}$, and spaces with local Π-basis structure are shown to be $\mathbf{\Sigma}^0_6$. The paper is concluded with many open problems and suggestions for a future research.

Contents

2010 *Mathematics Subject Classification.* 03E15, 46B04, 54E52 (primary), 46B20, 46B80 46C15 (secondary).

Key words and phrases. Banach spaces, descriptive set theory, Hilbert space, Lp spaces, Baire category.

M. Cúth was supported by Charles University Research program No. UNCE/SCI/023 and by the Czech Science Foundation, project no. GAČR 17-00941S. Research of M. Doležal was supported by the GAČR project 17-27844S and RVO: 67985840. M. Doucha was supported by the GAČR project 19-05271Y and RVO: 67985840. O. Kurka was supported by the Czech Science Foundation, project no. GAČR 17-00941S and RVO: 67985840.

Introduction

Banach spaces and descriptive set theory have a long history of mutual interactions. An explicit use of descriptive set theory to Banach space theory can be traced back at least to the seminal papers of Bourgain ([6], [5]), where it has become apparent that descriptive set theory is an indispensable tool for universality problems. That is a theme that has been investigated by researchers working with Banach spaces ever since (see e.g. [2] and [12] and references therein).

As it eventually turned out, 'Descriptive set theory of Banach spaces' is an interesting and rich subject of its own and it has received some considerable attention in the recent years. One of the starting points was the idea of Bossard of coding separable Banach spaces in [3], [4]. His approach was, which can be considered standard, to choose some universal separable Banach space X, e.g. $C(2^{\mathbb{N}})$, and consider the Effros-Borel space $F(X)$. Recall that this is the set of closed subsets of X equipped with a certain σ-algebra which makes $F(X)$ a standard Borel space, i.e. a measurable space which is isomorphic, as a measurable space, to a Polish space equipped with the σ-algebra of Borel sets. It is then not too difficult to show that a subset $SB(X) \subseteq F(X)$ consisting of all closed linear subspaces is a Borel subset, and therefore a standard Borel space itself.

A drawback of this idea is that there is no canonical or natural (Polish) topology on $SB(X)$. So although one can ask whether a given class of Banach spaces is Borel or not, the question about the exact complexity of that particular class is meaningless. A recent work [23] of Godefroy and Saint-Raymond addresses this issue. They still work with the space $SB(X)$, but among the many Polish topologies on $SB(X)$ giving the Effros-Borel structure, they select some particular subclass which is called *admissible topologies*. Although no particular admissible topology is canonical, the set of requirements put on this class guarantees that the exact Borel complexities vary little.

One of the aims of this paper is to present a concrete and natural Polish space (and some variants of it) of separable Banach spaces, which is convenient to work with and in which the computations of Borel complexities are usually as straightforward as they could be. Informally, it is the space of all norms, resp. pseudonorms, on the space of all finitely supported sequences of rational numbers - the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis. We note that this is, in spirit, similar to how (for instance) Vershik topologized the space of all Polish metric spaces ([48]), or how Grigorchuk topologized the space of all n-generated, resp. finitely generated, groups ([25]).

This space has already appeared in the previous work of the authors in [10] as a useful coding of Banach spaces. Here we investigate it further. Our goals are twofold:

(a) to show that the spaces of norms, resp. pseudonorms, have all the advantages of admissible topologies on $SB(X)$,
(b) to demonstrate the strength of this new approach by computing (in many cases, these computations are sharp) complexities of various classes of Banach spaces; this includes improving some estimates of Godefroy and Saint-Raymond and addressing some of their problems, as well as initiating the research of computing the complexities of isometry classes of Banach spaces.

The goal (a) is achieved mainly by the following theorem.

Theorem A. *There is a $\mathbf{\Sigma}^0_2$-measurable mapping from the space of norms to any admissible topology of Godefroy and Saint-Raymond that associates to each norm a space isometric to the space which the norm defines, and vice versa.*

Additionally, while the exact Borel complexities are more or less independent of the choice of the admissible topology, some finer topological properties such as being meager or comeager (this is mentioned below), or the description of the topological closures are not. We obtain a neat characterization of topological closures in the spaces of norms and pseudonorms in terms of finite representability, we refer the reader to Proposition 1.9

The rest of the paper focuses on goal (b). Let us describe some of the main results here.

First we focus on computing complexities of *isometry* classes of Banach spaces, i.e. how easy/difficult it is to define a concrete Banach space uniquely up to isometry. There is an active ongoing research whether for a particular Banach space its isomorphism class is Borel or not (see e.g. [21], [33], [17], [22]) while it is known that isometry classes of separable Banach spaces are always Borel (note that the linear isometry relation is Borel bi-reducible with an orbit equivalence relation [41] and orbit equivalence relations have Borel equivalence classes [29, Theorem 15.14]). Having a topology at our disposal we compute complexities of isometry classes of several classical Banach spaces.

Theorem B. *(1) The infinite-dimensional separable Hilbert space is characterized as the unique infinite-dimensional separable Banach space whose isometry class is closed (see Theorem 3.4).*
(2) For $p \in [1,2) \cup (2,\infty)$, the isometry class of $L_p([0,1],\lambda)$ is G_δ-complete. Moreover, for every $\lambda \geq 1$, the class of separable $\mathcal{L}_{p,\lambda+}$-spaces is G_δ and the class of separable $\mathcal{L}_p$-spaces is $G_{\delta\sigma}$, improving the estimate from [23] *(see Theorems 4.4 and 4.6, and Corollary 4.7).*
(3) For $p \in [1,2) \cup (2,\infty)$, the isometry class of ℓ_p is $F_{\sigma\delta}$-complete (see Theorem 5.1).
(4) The isometry class of c_0 is $F_{\sigma\delta}$-complete (see Theorem 5.1).
(5) The isometry class of the Gurariĭ space is G_δ-complete (see Corollary 4.2).

Regarding the Hilbert space, we also uniquely characterize it by the complexity of its isomorphism class.

Theorem C. *The infinite-dimensional separable Hilbert space is characterized as the unique, up to isomorphism, infinite-dimensional separable Banach space whose isomorphism class is F_σ (see Theorem 6.1).*

Let us also present here few sample results which involve complexities of more general classes of Banach spaces.

Theorem D. *(1) The class of all superreflexive spaces is $F_{\sigma\delta}$ (see Theorem 7.3).*
(2) The class of all spaces with local Π-basis structure is $\mathbf{\Sigma}^0_6$ (see Theorem 7.13).
(3) The class of spaces whose Szlenk index is bounded by ω^α is $\mathbf{\Pi}^0_{\omega^\alpha+1}$ (see Theorem 7.7).

Next we consider various 'genericity' or 'Baire category' problems. For the space of norms and pseudonorms we have a definitive solution.

Theorem E. *The isometry class of the Gurariĭ space is dense G_δ in the space of norms and pseudonorms, i.e. the Gurariĭ space is the generic separable Banach space (see Theorem 2.1).*

We then consider Baire category problems also for admissible topologies, thus addressing Problem 5.5 of Godefroy and Saint-Raymond from [23]. In particular, we confirm their suspicion that being meager is not independent of the choice of the admissible topology.

Theorem F. *For any universal Banach space X, any admissible topology τ on $SB(X)$ and infinite-dimensional Banach spaces Y and Z such that $Y \hookrightarrow Z$ and $Z \not\hookrightarrow Y \oplus F$ for every finite-dimensional space F, there exists a finer admissible topology $\tau' \supseteq \tau$ such that the class of Banach spaces isomorphic to Z is nowhere dense in $(SB(X), \tau')$. In particular, there exists an admissible topology in which the Gurariĭ space is meager (see Theorem 2.12).*

On the other hand, the Gurariĭ space is dense G_δ in the Wijsman topology (see Theorem 2.10).

The paper is organized as follows. Section 1 first introduces the Polish spaces of norms and pseudonorms and investigates their basic topological properties. The rest of the section is then devoted to the proof of Theorem A. This is a very technical part that is independent on the rest of the paper, so readers who are satisfied with our definitions of Polish spaces of norms/pseudonorms without objections can safely skip it and proceed immediately to Section 2. There we prove Theorems E and F. That part can be also read independently, so readers interested only in descriptive complexities can focus on Sections 3, 4, 5, 6, and 7, where Theorems B, C, and D are proved.

Final Section 8 presents several open problems and directions for a further research.

Let us conclude the introduction by setting up some notation that will be used throughout the paper.

Notation: Throughout the paper we usually denote the Borel classes of low complexity by the traditional notation such as F_σ and G_δ, or even $F_{\sigma\delta}$ (countable intersection of F_σ sets) and $G_{\delta\sigma}$ (countable union of G_δ sets). However whenever it is more convenient or necessary we use the notation $\mathbf{\Sigma}^0_\alpha$, resp. $\mathbf{\Pi}^0_\alpha$, where $\alpha < \omega_1$ (we refer to [29, Section 11] for this notation). We emphasize that open sets, resp. closed sets, are $\mathbf{\Sigma}^0_1$, resp. $\mathbf{\Pi}^0_1$, by this notation.

In a few occassions, for a Borel class $\mathbf{\Gamma}$ we will use the notion of $\mathbf{\Gamma}$-hard and $\mathbf{\Gamma}$-complete sets. We refer the reader to [29, Definition 22.9] for this notion. For a reader not familiar with this notion, let us emphasize that a set A being $\mathbf{\Gamma}$-hard, for a Borel class $\mathbf{\Gamma}$, in particular implies that A is not of lower complexity than $\mathbf{\Gamma}$. Thus results stating that some set is $\mathbf{\Sigma}^0_\alpha$-complete mean that the set is $\mathbf{\Sigma}^0_\alpha$ and not simpler.

Moreover, given a class $\mathbf{\Gamma}$ of sets in metrizable spaces, we say that $f : X \to Y$ is $\mathbf{\Gamma}$-measurable if $f^{-1}(U) \in \mathbf{\Gamma}$ for every open set $U \subseteq Y$.

Given Banach spaces X and Y, we denote by $X \equiv Y$ (resp. $X \simeq Y$) the fact that those two spaces are linearly isometric (resp. isomorphic). We denote by $X \hookrightarrow Y$ the fact that Y contains a subspace isomorphic to X. For $K \geq 1$, a K-*isomorphism* $T : X \to Y$ is a linear map with $K^{-1}\|x\| \leq \|Tx\| \leq K\|x\|$, $x \in X$. If $\{x_1, \ldots, x_n\}$ are linearly independent elements of X and $y_1, \ldots, y_n \in Y$, we write $(Y, y_1, \ldots, y_n) \overset{K}{\sim} (X, x_1, \ldots, x_n)$ if the linear operator $T : \operatorname{span}\{x_1, \ldots, x_n\} \to \operatorname{span}\{y_1, \ldots, y_n\}$ sending x_i to y_i satisfies $\max\{\|T\|, \|T^{-1}\|\} < K$. If X has a canonical basis $(x_1, \ldots, x_n)$ which is clear from the context, we just write $(Y, y_1, \ldots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \ldots, y_n) \overset{K}{\sim} (X, x_1, \ldots, x_n)$. Morevoer, if Y is clear from the context we write $(y_1, \ldots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \ldots, y_n) \overset{K}{\sim} X$.

Finally, in order to avoid any confusion, we emphasize that throughout the text ℓ_p^n denotes the n-dimensional ℓ_p-space, i.e. the upper index denotes dimension.

1. The Polish spaces of separable Banach spaces

We begin with formalizing the class of all separable (infinite-dimensional) Banach spaces as a Polish space. The main outcomes of this section are the following.

(1) In the first subsection, we introduce the main notions of this paper: the Polish spaces of pseudonorms $\mathcal{P}$ (and $\mathcal{P}_\infty$) representing separable (infinite-dimensional) Banach spaces, and we recall the space of norms $\mathcal{B}$ that has appeared in our previous work [10]. We show some interesting features of these space, e.g. the neat relation between finite representability and topological closures in these space; see Proposition 1.9 and its corollaries.
(2) In the second subsection, we recall the coding $SB(X)$ (and $SB_\infty(X)$) of separable (infinite-dimensional) Banach spaces. We recall the notion of an admissible topology introduced in [23], which is a Polish topology corresponding to the Effros-Borel structure of $SB(X)$. We explore some basic relations between codings $\mathcal{P}$, $\mathcal{P}_\infty$, $\mathcal{B}$, $SB(X)$ and $SB_\infty(X)$. We show there is a continuous reduction from $SB(X)$ to $\mathcal{P}$, a $\boldsymbol{\Sigma}_2^0$-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$, and a $\boldsymbol{\Sigma}_4^0$-measurable reduction from $\mathcal{P}$ to $SB(X)$, see Theorem 1.17, Proposition 1.20 and Theorem 1.24. Here by a 'reduction', we mean a Borel map Φ such that the code and its image are both codes of the same (up to isometry) Banach space.
(3) The third subsection is devoted to the proof of Theorem 1.25, by which there is a $\boldsymbol{\Sigma}_2^0$-measurable reduction from $\mathcal{B}$ to $SB_\infty(X)$. Further, we note that the developed techniques also lead to a $\boldsymbol{\Sigma}_3^0$-measurable reduction from $\mathcal{P}$ to $SB(X)$, which is an improvement of the result mentioned above.

The meaning of the reductions above is that there is not a big difference between Borel ranks when considered in any of the Polish spaces mentioned above. However, it seems that considering $\mathcal{P}_\infty$ and $\mathcal{B}$ has some pleasant additional features, see e.g. Proposition 1.9 and its corollaries.

Let us emphasize that the existence of a Borel reduction from $\mathcal{B}$ to $SB_\infty(X)$ has been essentially proved in [32, Lemma 2.4]. Going through the proof of [32, Lemma 2.4], one may obtain a reduction which is $\boldsymbol{\Sigma}_3^0$-measurable; however, the proof does not seem to give a $\boldsymbol{\Sigma}_2^0$-measurable reduction (which is the optimal result). In order to obtain this improvement, see Theorem 1.25, we have to develop a whole machinery of new ideas in combination with very technical results, and this is the reason why we devote a whole subsection to the proof.

Since our reductions from $SB_\infty(X)$ to $\mathcal{P}_\infty$ and from $\mathcal{B}$ to $SB_\infty(X)$ are optimal, it seems to be a very interesting open problem of whether there exists a continuous reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$ or at least a $\boldsymbol{\Sigma}_2^0$-measurable reduction from $\mathcal{P}_\infty$ to $SB_\infty(X)$, see Question 1 and Question 2.

1.1. **Introduction of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$, and their topological features.** Let us start with the following idea of coding the class of separable (infinite-dimensional) Banach spaces. It is based on the idea presented already in our previous paper [10], where the space $\mathcal{B}$ was defined.

By V, we shall denote the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers; that is, the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis $(e_n)_{n\in\mathbb{N}}$.

Definition 1.1. Let us denote by $\mathcal{P}$ the space of all pseudonorms on the vector space V. Since $\mathcal{P}$ is a closed subset of $\mathbb{R}^V$, this gives $\mathcal{P}$ the Polish topology inherited from $\mathbb{R}^V$. The subbasis of this topology is given by sets of the form $U[v,I] := \{\mu \in \mathcal{P} :\ \mu(v) \in I\}$, where $v \in V$ and I an open interval.

We often identify $\mu \in \mathcal{P}$ with its extension to the pseudonorm on the space c_{00}, that is, the vector space over $\mathbb{R}$ of all finitely supported sequences of real numbers.

For every $\mu \in \mathcal{P}$ we denote by X_μ the Banach space given as the completion of the quotient space X/N, where $X = (c_{00}, \mu)$ and $N = \{x \in c_{00} : \ \mu(x) = 0\}$. In what follows we often consider V as a subspace of X_μ, that is, we identify every $v \in V$ with its equivalence class $[v]_N \in X_\mu$.

By $\mathcal{P}_\infty$ we denote the set of those $\mu \in \mathcal{P}$ for which X_μ is infinite-dimensional Banach space. As we did in [10], by $\mathcal{B}$ we denote the set of those $\mu \in \mathcal{P}_\infty$ for which the extension of μ to c_{00} is an actual norm, that is, the vectors $e_1, e_2, \ldots$ are linearly independent in X_μ.

We endow $\mathcal{P}_\infty$ and $\mathcal{B}$ with topologies inherited from $\mathcal{P}$.

Our first aim is to show that the topologies on $\mathcal{P}_\infty$ and $\mathcal{B}$ are Polish (see Corollary 1.5). This can be easily verified directly, here we obtain it as a corollary of the fact that the relation $\overset{K}{\sim}$ defined before is open in $\mathcal{P}$, a very useful fact that will prove important many times in the paper.

We need the following background first. Given a metric space (M, d), $\varepsilon > 0$ and $N, S \subseteq M$ we say that N is *ε-dense for S* if for every $x \in S$ there is $y \in N$ with $d(x, y) < \varepsilon$ (let us emphasize that we do not assume $N \subseteq S$). For further references, we recall the following well-known approximation lemma, for a proof see e.g. [1, Lemma 12.1.11].

Lemma 1.2. *There is a function $\phi_1 : [0, 1) \to [0, 1)$ continuous at zero with $\phi_1(0) = 0$ such that whenever $T : E \to X$ is a linear operator between Banach spaces, $\varepsilon \in (0, 1)$, $M \subseteq E$ is ε-dense for S_E and*

$$\forall m \in M : |\|Tm\| - 1| < \varepsilon,$$

then T is $(1 + \phi_1(\varepsilon))$-isomorphism between E and $T(E)$.

The following definition precises the notation $\overset{K}{\sim}$ defined in the introduction.

Definition 1.3. If $v_1, \ldots, v_n \in V$ are given, for $\mu \in \mathcal{P}$, instead of $(X_\mu, v_1, \ldots, v_n) \overset{K}{\sim} X$, we shall write $(\mu, v_1, \ldots, v_n) \overset{K}{\sim} X$.

Lemma 1.4. *Let X be a Banach space with $\{x_1, \ldots, x_n\} \subseteq X$ linearly independent and let $v_1, \ldots, v_n \in V$. Then for any $K > 1$ the set*

$$\mathcal{N}((x_i)_i, K, (v_i)_i) = \{\mu \in \mathcal{P} \colon (\mu, v_1, \ldots, v_n) \overset{K}{\sim} (X, x_1, \ldots, x_n)\}$$

is open in $\mathcal{P}$.

In particular, the set of those $\mu \in \mathcal{P}$ for which the set $\{v_1, \ldots, v_n\}$ is linearly independent in X_μ is open in $\mathcal{P}$.

Proof. Pick some $\mu \in \mathcal{N}((x_i)_i, K, (v_i)_i)$. By definition, the linear map T sending v_i to $x_i \in X$, $i \leq n$, is a linear isomorphism satisfying $\max\{\|T\|, \|T^{-1}\|\} < L$ for some $L < K$. Let ϕ_1 be the function provided by Lemma 1.2 and pick $\varepsilon > 0$ such that $L(1 + \phi_1(2\varepsilon)) < K$. Let $N \subseteq V$ be a finite ε-dense set for the sphere of $\operatorname{span}\{v_1, \ldots, v_n\} \subseteq X_\mu$ such that $\mu(v) \in (1 - \varepsilon, 1 + \varepsilon)$ for every $v \in N$. Then

$$U := \{\nu \in \mathcal{P} : \ \forall v \in N : \ |\nu(v) - \mu(v)| < \varepsilon\}$$

is an open neighborhood of μ and $U \subseteq \mathcal{N}((x_i)_i, K, (v_i)_i)$. Indeed, for any $\nu \in U$ we have that $id : (\operatorname{span}\{v_1, \ldots, v_n\}, \mu) \to (\operatorname{span}\{v_1, \ldots, v_n\}, \nu)$ is $(1 + \phi_1(2\varepsilon))$-isomorphism; hence, the linear map T considered as a map betweeen $(\operatorname{span}\{v_1, \ldots, v_n\}, \nu)$ and $\operatorname{span}\{x_1, \ldots, x_n\}$ satisfies $\|T\| < L(1+\phi_1(2\varepsilon)) < K$, and similarly $\|T^{-1}\| < K$; hence, $\nu \in \mathcal{N}((x_i)_i, K, (v_i)_i)$.

The "In particular" part easily follows, because $v_1, \ldots, v_n \in V$ are linearly independent if and only if there exists $K > 1$ with $(\mu, v_1, \ldots, v_n) \overset{K}{\sim} \ell_1^n$. $\square$

Corollary 1.5. *Both $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ sets in $\mathcal{P}$.*

Since we are interested mainly in subsets of $\mathcal{P}$ closed under isometries, we introduce the following notation.

Notation 1.6. Let Z be a separable Banach space and let $\mathcal{I}$ be a subset of $\mathcal{P}$. We put

$$\langle Z\rangle_{\equiv}^{\mathcal{I}} := \{\mu \in \mathcal{I} :\ X_\mu \equiv Z\} \quad \text{and} \quad \langle Z\rangle_{\simeq}^{\mathcal{I}} := \{\mu \in \mathcal{I} :\ X_\mu \simeq Z\}.$$

If $\mathcal{I}$ is clear from the context we write $\langle Z\rangle_{\equiv}$ and $\langle Z\rangle_{\simeq}$ instead of $\langle Z\rangle_{\equiv}^{\mathcal{I}}$ and $\langle Z\rangle_{\simeq}^{\mathcal{I}}$ respectively.

The important feature of the topology of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$ is that basic open neighborhoods are defined using finite data, i.e. finitely many vectors. That suggests that topological properties of the aforementioned spaces should be closely related to the local theory of Banach spaces. This is certainly a point that could be investigated further in a future research. Here we just observe how topological closures are related to finite representability, see Proposition 1.9. In order to formulate our results, let us consider the following generalization of the classical notion of finite representability.

Definition 1.7. Let $\mathcal{F}$ be a family of Banach spaces. We say that a Banach space X is finitely representable in $\mathcal{F}$ if given any finite-dimensional subspace E of X and $\varepsilon > 0$ there exists a finite-dimensional subspace F of some $Y \in \mathcal{F}$ which is $(1+\varepsilon)$-isomorphic to E.

If the family $\mathcal{F}$ consists of one Banach space Y only, we say rather that X is finitely representable in Y than in $\{Y\}$.

If $\mathcal{F} \subseteq \mathcal{P}$, by saying that X is finitely representable in $\mathcal{F}$ we mean it is finitely representable in $\{X_\mu :\ \mu \in \mathcal{F}\}$.

The following is an easy observation which we will use further, the proof follows e.g. immediately from [1, Lemma 12.1.7] in the case that $\mathcal{F}$ contains one Banach space only. For the more general situation the proof is analogous.

Lemma 1.8. *Let $\mathcal{F}$ be a family of infinite-dimensional Banach spaces and $\mu \in \mathcal{P}_\infty$. Let $\{k(n)\}_{n=1}^\infty$ be a sequence such that $\{e_{k(n)} :\ n \in \mathbb{N}\}$ is a linearly independent set in X_μ and $\overline{\operatorname{span}}\{e_{k(n)} :\ n \in \mathbb{N}\} = X_\mu$. Then X_μ is finitely representable in $\mathcal{F}$ if and only if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite dimensional subspace F of some $Y \in \mathcal{F}$ which is $(1+\varepsilon)$-isomorphic to $(\operatorname{span}\{e_{k(1)}, \dots, e_{k(n)}\}, \mu)$.*

Proposition 1.9. *Let $\mathcal{F} \subseteq \mathcal{B}$ be such that $\langle X_\mu\rangle_{\equiv}^{\mathcal{B}} \subseteq \mathcal{F}$ for every $\mu \in \mathcal{F}$. Then*

$$\{\nu \in \mathcal{B} :\ X_\nu \text{ is finitely representable in } \mathcal{F}\} = \overline{\mathcal{F}} \cap \mathcal{B}.$$

The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

In particular, if X is a separable infinite-dimensional Banach space, then

$$\{\nu \in \mathcal{B} :\ X_\nu \text{ is finitely representable in } X\} = \overline{\langle X\rangle_{\equiv}^{\mathcal{B}}} \cap \mathcal{B}$$

and similarly also if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

Proof. "$\subseteq$": Fix $\nu \in \mathcal{B}$ such that X_ν is finitely-representable in $\mathcal{F}$. Pick $v_1, \dots, v_n \in V$ and $\varepsilon > 0$. We shall show there is $\mu_0 \in \mathcal{F}$ with $|\mu_0(v_i) - \nu(v_i)| < \varepsilon$, $i \le n$. Let $m \in \mathbb{N}$ be such that $\{v_1, \dots, v_n\} \subseteq \operatorname{span}_{\mathbb{Q}}\{e_j :\ j \le m\}$. Put $C := \max\{\nu(v_i) :\ i = 1, \dots, n\}$ and $Z := \operatorname{span}\{e_1, \dots, e_m\} \subseteq X_\nu$. Since X_ν is finitely representable in $\mathcal{F}$, there is $\mu \in \mathcal{F}$ and an $(1 + \frac{\varepsilon}{2C})$-isomorphism $T : Z \to X_\mu$. Set $x_i := T(e_i)$, $i \le m$, and extend $x_1, \dots, x_m$ to a linearly independent sequence $(x_i)_{i=1}^\infty$ whose span is dense in X_μ. Consider $\mu_0 \in \mathcal{P}$ given by setting $\mu_0(\sum_{i\in I} \alpha_i e_i) = \mu(\sum_{i\in I} \alpha_i x_i)$, where $I \subseteq \mathbb{N}$ is finite and $(\alpha_i)_{i\in I} \subseteq \mathbb{Q}$. Clearly, $X_{\mu_0} \equiv X_\mu$ and $\mu_0 \in \mathcal{B}$, so $\mu_0 \in \mathcal{F}$.

Finally, for every $i \leq n$ we have $v_i = \sum_{j=1}^m \alpha_j e_j$ for some $(\alpha_j) \in \mathbb{R}^m$ and so we have

$$\mu_0(v_i) = \mu(\sum_{j=1}^m \alpha_j x_j) \leq (1 + \tfrac{\varepsilon}{2C})\nu(\sum_{j=1}^m \alpha_j e_j) = (1 + \tfrac{\varepsilon}{2C})\nu(v_i)$$

and similarly $\mu_0(v_i) \geq (1 + \frac{\varepsilon}{2C})^{-1}\nu(v_i) \geq (1 - \frac{\varepsilon}{2C})\nu(v_i)$. Thus, $|\mu_0(v_i) - \nu(v_i)| \leq \frac{\varepsilon}{2} < \varepsilon$ for every $i \leq n$.

The case when we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$ is analogous, this time we only do not require $(x_i)_{i=1}^\infty$ to be linearly independent.

"$\supseteq$": Fix $\nu \in \overline{\mathcal{F}} \cap \mathcal{B}$. In order to see that X_ν is finitely representable in $\mathcal{F}$, we will use Lemma 1.8. Pick $n \in \mathbb{N}$ and $\varepsilon > 0$. Let ϕ_1 be the function from Lemma 1.2, let $\delta > 0$ be such that $\phi_1(2\delta) < \varepsilon$ and let $N \subseteq V$ be a finite set which is δ-dense for the sphere of $(\operatorname{span}\{e_1, \ldots, e_n\}, \nu)$ and $\nu(v) \in (1-\delta, 1+\delta)$ for every $v \in N$. Pick $\mu \in \mathcal{F}$ such that $|\mu(v) - \nu(v)| < \delta$, $v \in N$. Then $id : (\operatorname{span}\{e_1, \ldots, e_n\}, \nu) \to (\operatorname{span}\{e_1, \ldots, e_n\}, \mu)$ is $(1+\phi_1(2\delta))$-isomorphism. Thus, X_ν is finitely representable in $\mathcal{F}$. The case when we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$ is similar. ☐

This result will have many interesting consequences. Let us state one of them, of a general nature, here. Other will appear in appropriate sections when needed.

Corollary 1.10. *Let X be a separable infinite-dimensional Banach space. Then*

$$\{\mu \in \mathcal{B} :\ X_\mu \textit{ is finitely representable in } X\}$$

is a closed set in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

We also consider a kind of an opposite notion where instead of finitely representing one fixed Banach space in a class of Banach spaces, we represent a class of Banach spaces in one fixed Banach space.

Definition 1.11. Let $\mathcal{F}$ be a class of finite dimensional Banach spaces and let X be a Banach space. We say that $\mathcal{F}$ *is representable in* X if for every $F \in \mathcal{F}$ and $\varepsilon > 0$ there exists a subspace of X that is $(1+\varepsilon)$-isomorphic to F.

Analogously, we say that $\mathcal{F}$ *is crudely representable in* X if there is $\lambda \geq 1$ such that for every $F \in \mathcal{F}$ and $\varepsilon > 0$ there exists a subspace of X that is $(\lambda + \varepsilon)$-isomorphic to F. If the family $\mathcal{F}$ consists of all finite-dimensional subspaces of a (possibly infinite-dimensional) Banach space Y, we say that Y is crudely finitely representable in X.

Proposition 1.12. *Let $\mathcal{F}$ be an arbitrary class of finite dimensional Banach spaces. Then the set of those $\mu \in \mathcal{P}$ such that $\mathcal{F}$ is representable in X_μ is G_δ. Analogously, the set of those $\mu \in \mathcal{P}$ such that $\mathcal{F}$ is crudely representable in X_μ is $G_{\delta\sigma}$.*

In particular, for a fixed Banach space Y, the set of those $\mu \in \mathcal{P}$ such that Y is finitely representable, resp. crudely finitely representable, in X_μ is G_δ, resp. $G_{\delta\sigma}$.

Proof. For an arbitrary finite dimensional Banach space F (with a fixed basis $\mathfrak{b}_F$) and $K > 1$, we set

$$R(F, K) := \{\mu \in \mathcal{P} \colon \exists v_1, \ldots, v_n \in V\ (\mu, v_1, \ldots, v_n) \overset{K}{\sim} F\}.$$

By Lemma 1.4, $R(F, K)$ is open.

Now since for each n the space of all isometry classes of n-dimensional Banach spaces with the Banach-Mazur distance is compact, in particular it is separable, we can without loss of generality assume that the class $\mathcal{F}$ is countable. As a consequence, we get that the set of those $\mu \in \mathcal{P}$ such that $\mathcal{F}$ is representable in X_μ is equal to

$$\bigcap_{F \in \mathcal{F}} \bigcap_{n \in \mathbb{N}} R(F, 1 + 1/n),$$

which is G_δ.

Analogously, we get that the set of those $\mu \in \mathcal{P}$ such that $\mathcal{F}$ is crudely representable in X_μ is equal to

$$\bigcup_{\lambda} \bigcap_{F\in\mathcal{F}} \bigcap_{n\in\mathbb{N}} R(F, \lambda + 1/n),$$

which is $G_{\delta\sigma}$.

The 'In particular' part follows immediately. $\square$

We conclude this subsection by showing another nice features of the above topologies on examples. We can show that the natural maps $K \mapsto C(K)$ and $\lambda \mapsto L_p(\lambda)$, where K is a compact metrizable space and λ is a Borel probability measure on a fixed compact metric space, are continuous.

Example 1.13. (a) Let $\mathcal{K}([0,1]^{\mathbb{N}})$ denote the space of all compact subsets of the Hilbert cube $[0,1]^{\mathbb{N}}$ endowed with the Vietoris topology. Then there exists a continuous mapping $\rho\colon \mathcal{K}([0,1]^{\mathbb{N}}) \to \mathcal{P}$ such that $X_{\rho(K)} \equiv C(K)$ for every $K \in \mathcal{K}([0,1]^{\mathbb{N}})$.

(b) Let L be a compact metric space, let $p \in [1,\infty)$ be fixed and let $\mathcal{P}rob(L)$ denote the space of all Borel probability measures on L endowed with the weak* topology (generated by elements of the Banach space $C(L)$). Then there exists a continuous mapping $\sigma\colon \mathcal{P}rob(L) \to \mathcal{P}$ such that $X_{\sigma(\lambda)} \equiv L_p(\lambda)$ for every $\lambda \in \mathcal{P}rob(L)$.

Proof. (a) Let $\{f_i\colon i \in \mathbb{N}\}$ be a linearly dense subset of $C([0,1]^{\mathbb{N}})$. For every compact subset K of $[0,1]^{\mathbb{N}}$, we define $\rho(K) \in \mathcal{P}$ by

$$\rho(K)\left(\sum_{i=1}^n r_i e_i\right) = \sup_{x\in K}\left|\sum_{i=1}^n r_i f_i(x)\right|, \qquad \sum_{i=1}^n r_i e_i \in V.$$

It is clear that $X_{\rho(K)} \equiv C(K)$, so we only need to check the continuity of ρ. It is enough to show that $\rho^{-1}(U[v,I])$ is an open subset of $\mathcal{K}([0,1]^{\mathbb{N}})$ for every $v \in V$ and every open interval I (recall that $U[v,I] = \{\mu \in \mathcal{P}\colon \mu(v) \in I\}$). So let us fix $\widetilde{K} \in \rho^{-1}(U[v,I])$, and assume that $v = \sum_{i=1}^n r_i e_i$. Fix $x_0 \in \widetilde{K}$ such that

$$\left|\sum_{i=1}^n r_i f_i(x_0)\right| = \sup_{x\in\widetilde{K}}\left|\sum_{i=1}^n r_i f_i(x)\right|.$$

Fix also $\varepsilon > 0$ such that both numbers $|\sum_{i=1}^n r_i f_i(x_0)| \pm \varepsilon$ belong to I. Now find open subsets U, V of $[0,1]^{\mathbb{N}}$ such that $x_0 \in U$ and $\widetilde{K} \subseteq V$, and such that

$$\inf_{x\in U}\left|\sum_{i=1}^n r_i f_i(x)\right| > \left|\sum_{i=1}^n r_i f_i(x_0)\right| - \varepsilon$$

and

$$\sup_{x\in V}\left|\sum_{i=1}^n r_i f_i(x)\right| < \sup_{x\in\widetilde{K}}\left|\sum_{i=1}^n r_i f_i(x)\right| + \varepsilon.$$

Then

$$\mathcal{U} := \{K \in \mathcal{K}([0,1]^{\mathbb{N}})\colon K \cap U \neq \emptyset \text{ and } K \subseteq V\}$$

is an open neighborhood of $\widetilde{K}$ such that $\rho(\mathcal{U}) \subseteq U[v,I]$.

(b) This is similar to (a) but even easier. Let $\{g_i : i \in \mathbb{N}\}$ be a linearly dense subset of $C(L)$. For every Borel probability measure λ on L, we define $\sigma(\lambda) \in \mathcal{P}$ by

$$\sigma(\lambda)\left(\sum_{i=1}^{n} r_i e_i\right) = \left(\int_L \left|\sum_{i=1}^{n} r_i g_i\right|^p d\lambda\right)^{\frac{1}{p}}, \qquad \sum_{i=1}^{n} r_i e_i \in V.$$

It is clear that $X_{\sigma(\lambda)} \equiv L_p(\lambda)$, so we only need to check the continuity of σ. It is enough to show that $\sigma^{-1}(U[v, I])$ is an open subset of $\mathcal{P}rob(L)$ for every $v \in V$ and every open interval I. But this is clear as, for $v = \sum_{i=1}^{n} r_i e_i$, we have

$$\sigma^{-1}(U[v, I]) = \left\{\lambda \in \mathcal{P}rob(L) \colon \left(\int_L \left|\sum_{i=1}^{n} r_i g_i\right|^p d\lambda\right)^{\frac{1}{p}} \in I\right\}.$$

□

Remark 1.14. After the introduction of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$, and $\mathcal{B}$, one faces the question which of them is 'the right one', with which to work. For now, we leave the question undecided. Since we are mainly interested in infinite-dimensional Banach spaces, we prefer to work mainly with $\mathcal{P}_\infty$ and $\mathcal{B}$, and indeed the space $\mathcal{P}$ has a certain pathological property when computing the closed isometry classes (we refer to Remark 3.2). On the other hand, it turns out that at least as far as one wants to transfer some computations performed in the space of pseudonorms directly to the admissible topologies, the space $\mathcal{P}$ is useful: Theorem 1.17 below shows that whatever we compute in the space $\mathcal{P}$ holds true also in any admissible topology.

Regarding $\mathcal{P}_\infty$ and $\mathcal{B}$, in most of the arguments there is no difference whether we work with the former or the latter space. However, there are few exceptions when it seems to be convenient to work with the assumption that the sequence of vectors $\langle e_n : n \in \mathbb{N}\rangle \subseteq V$ is linearly independent, and then we work with $\mathcal{B}$.

1.2. **Relations between codings $\mathcal{P}$, $\mathcal{P}_\infty$, $\mathcal{B}$, $SB(X)$ and $SB_\infty(X)$.** Here we recall the approach to topologizing the class of all separable (infinite-dimensional) Banach spaces by Godefroy and Saint-Raymond from [23] which was a partial motivation for our research.

Definition 1.15. Let X be a Polish space and let us denote by $\mathcal{F}(X)$ the set of all the closed subsets of X. For an open set $U \subseteq X$ we put $E^+(U) = \{F \in \mathcal{F}(X) : U \cap F \neq \emptyset\}$. Following [23], we say that a Polish topology τ on the set $\mathcal{F}(X)$ is *admissible* if it satisfies the following two conditions:

(i) For every open subset U of X, the set $E^+(U)$ is τ-open.
(ii) There exists a subbasis of τ such that every set from this subbasis is a countable union of sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in X.[1]

We note that Godefroy and Saint-Raymond also suggest the following optional condition that is satisfied by many natural admissible topologies.

(iii) The set $\{(x, F) \in X \times \mathcal{F}(X) : x \in F\}$ is closed in $X \times \mathcal{F}(X)$.

If X is a separable Banach space, we denote by $SB(X) \subseteq \mathcal{F}(X)$ the set of closed vector subspaces of X. We denote by $SB_\infty(X)$ the subset of $SB(X)$ consisting of infinite-dimensional spaces. We say that a topology on $SB(X)$ or $SB_\infty(X)$ is

[1] note that the condition (ii) is different from what is mentioned in [23]; however, as the authors have confirmed, there is a typo in the condition from [23] which makes it wrong (otherwise no single one of the topologies mentioned in [23] would be admissible)

admissible if it induced by an admissible topology on $\mathcal{F}(X)$. Both $SB(X)$ and $SB_\infty(X)$ are Polish spaces when endowed with an admissible topology, see Remark 1.16.

If Z is a separable Banach space, we put, similarly as in Notation 1.6,

$$\langle Z\rangle_\equiv := \{F \in SB(X) : F \equiv Z\} \quad \text{and} \quad \langle Z\rangle_\simeq := \{F \in SB(X) : F \simeq Z\}.$$

It will be always clear from the context whether we work with subsets of $\mathcal{P}$, or $SB(X)$.

Remark 1.16. If τ is an admissible topology on a separable Banach space X then $SB(X)$ is a G_δ-subset of $(\mathcal{F}(X),\tau)$(see [23, Section 3]). Moreover, by [23, Corollary 4.2], $SB_\infty(X)$ is G_δ-subset of $(SB(X),\tau)$. (In fact, the paper [23] deals only with the case $X = C(2^\omega)$ but the generalization to any separable Banach space is easy.)

Certain connection between codings $SB(X)$ and $\mathcal{P}$ of separable Banach spaces might be deduced already from [23].

Theorem 1.17. *Let X be isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Then there is a continuous mapping $\Phi : (SB(X),\tau) \to \mathcal{P}$ such that for every $F \in SB(X)$ we have $F \equiv X_{\Phi(F)}$.*

Proof. By [23, Theorem 4.1], there are continuous functions $(f_n)_{n\in\mathbb{N}}$ on $SB(X)$ with values in X such that for each $F \in SB(X)$ we have $\overline{\{f_n(F) : n \in \mathbb{N}\}} = F$. Consider the mapping Φ given by $\Phi(F)(\sum_{n=1}^k a_n e_n) := \|\sum_{n=1}^k a_n f_n(F)\|_X$ for every $F \in SB(X)$ and $a_1,\ldots,a_k \in \mathbb{Q}$. Then it is easy to see that Φ is the mapping we need. $\square$

The following relation between various codings of Banach spaces as $SB(X)$ is easy.

Observation 1.18. *Let X, Y be isometrically universal separable Banach spaces and let τ_1 and τ_2 be admissible topologies on $SB(X)$ and $SB(Y)$, respectively. Then there is a $\mathbf{\Sigma}^0_2$-measurable mapping $f : (SB(X),\tau_1) \to (SB(Y),\tau_2)$ such that for every $F \in SB(X)$ we have $F \equiv f(F)$. Moreover, f can be chosen such that for every open set $U \subseteq Y$ there is an open set $V \subseteq X$ such that $f^{-1}(E^+(U)) = E^+(V)$.*

Proof. Let $j : X \to Y$ be an isometry (not necessarily surjective). Then the mapping f given by $f(F) := j(F)$, $F \in SB(X)$, does the job, because $f^{-1}(E^+(U)) = E^+(j^{-1}(U))$ for every open set $U \subseteq Y$. $\square$

Let us note the following easy fact which we record here for a later reference. The proof is easy and so it is omitted.

Lemma 1.19. *Let X be isometrically universal separable Banach space, τ be an admissible topology on $SB(X)$, Y be a Polish space, $f : Y \to SB(X)$ be a mapping and $n \in \mathbb{N}$, $n \geq 2$, be such that $f^{-1}(E^+(U))$ is a $\mathbf{\Delta}^0_n$ set in Y for every open set $U \subseteq X$. Then f is $\mathbf{\Sigma}^0_n$-measurable.*

A straightforward idea leads to the following relation between $\mathcal{P}_\infty$ and $\mathcal{B}$.

Proposition 1.20. *There is a $\mathbf{\Sigma}^0_2$-measurable mapping $\Phi : \mathcal{P}_\infty \to \mathcal{B}$ such that for every $\mu \in \mathcal{P}_\infty$ we have $X_\mu \equiv X_{\Phi(\mu)}$.*

Moreover, Φ can be chosen such that $\Phi^{-1}(U[v,I]) \in \Delta^0_2(\mathcal{P}_\infty)$ for each $v \in V$ and open interval I.

Proof. For each $\mu \in \mathcal{P}_\infty$ let us inductively define natural numbers $(n_k(\mu))_{k\in\mathbb{N}}$ by

$$n_1(\mu) := \min\{n \in \mathbb{N} : \mu(e_n) \neq 0\},$$
$$n_{k+1}(\mu) := \min\{n \in \mathbb{N} : e_{n_1(\mu)},\ldots,e_{n_k(\mu)},e_n \text{ are linearly independent}\}.$$

Consider the mapping Φ given by $\Phi(\mu)(\sum_{i=1}^k a_n e_n) := \mu(\sum_{i=1}^k a_i e_{n_i(\mu)})$ for every $\mu \in \mathcal{P}_\infty$ and $a_1, \ldots, a_k \in \mathbb{Q}$. It is easy to see that $\Phi(\mu) \in \mathcal{B}$ and that X_μ is isometric to $X_{\Phi(\mu)}$ for each $\mu \in \mathcal{P}_\infty$.

For all natural numbers $N_1 < \ldots < N_k$ the set $\{\mu \in \mathcal{P}_\infty : \; n_1(\mu) = N_1, \ldots, n_k(\mu) = N_k\}$ is a Δ_2^0 set in $\mathcal{P}_\infty$. Indeed, we may prove it by induction on k because for each $k \in \mathbb{N}$ and each $\mu \in \mathcal{P}_\infty$ we have that $n_1(\mu) = N_1, \ldots, n_{k+1}(\mu) = N_{k+1}$ iff

$$\begin{aligned} & n_1(\mu) = N_1, \ldots, n_k(\mu) = N_k \quad \& \\ & \qquad \forall n = N_k + 1, \ldots, N_{k+1} - 1 : e_{N_1}, \ldots, e_{N_k}, e_n \text{ are linearly dependent} \\ & \qquad \& \quad e_{N_1}, \ldots, e_{N_{k+1}} \text{ are linearly independent,} \end{aligned}$$

which is an intersection of a Δ_2^0-condition (by the inductive assumption) with a closed and an open condition (by Lemma 1.4).

Let us pick $v = \sum_{i=1}^k a_n e_n \in V$ and an open interval I. Then

$$\Phi^{-1}(U[v, I]) = \{\mu \in \mathcal{P}_\infty : \; \mu(\sum_{i=1}^k a_i e_{n_i(\mu)}) \in I\},$$

which is a $\mathbf{\Delta}_2^0$ set in $\mathcal{P}_\infty$. Indeed, on one hand we have $\mu \in \Phi^{-1}(U[v, I])$ iff there are natural numbers $N_1 < N_2 < \ldots < N_k$ such that $n_1(\mu) = N_1, \ldots, n_k(\mu) = N_k$ and $\mu(\sum_{i=1}^k a_i e_{N_i}) \in I$, which witnesses that $\Phi^{-1}(U[v, I]) \in \mathbf{\Sigma}_2^0(\mathcal{P}_\infty)$ as it is a countable union of Δ_2^0 sets. On the other hand, we have that $\mu \in \Phi^{-1}(U[v, I])$ iff for each $l \in \mathbb{N}$ we have that either $n_k(\mu) > l$ or there are natural numbers $N_1 < N_2 < \ldots < N_k \leq l$ such that $n_1(\mu) = N_1, \ldots, n_k(\mu) = N_k$ and $\mu(\sum_{i=1}^k a_i e_{N_i}) \in I$, which witnesses that $\Phi^{-1}(U[v, I]) \in \mathbf{\Pi}_2^0(\mathcal{P}_\infty)$ as it is a countable intersection of Δ_2^0 sets.

This proves the "Moreover" part from which it easily follows that Φ is $\mathbf{\Sigma}_2^0$-measurable. $\square$

Remark 1.21. For $d \in \mathbb{N}$, let us consider the sets $\mathcal{P}_d := \{\mu \in \mathcal{P} : \; \dim X_\mu = d\}$ and

$$\mathcal{B}_d := \{\mu \in \mathcal{P} : \; e_1, \ldots, e_d \text{ is a basis of } X_\mu \text{ and } \mu(e_i) = 0 \text{ for every } i > d\}.$$

A similar argument as in Proposition 1.20 shows that for every $d \in \mathbb{N}$ there is a $\mathbf{\Sigma}_2^0$-measurable mapping $\Phi : \mathcal{P}_d \to \mathcal{B}_d$ such that for every $\mu \in \mathcal{P}_d$ we have $X_\mu \equiv X_{\Phi(\mu)}$.

Finally let us consider the reduction from $\mathcal{P}$ to $SB(X)$. An optimal result would be to have a $\mathbf{\Sigma}_2^0$-reduction. This is because, as was already observed in [23], the identity map between two admissible topologies is only Σ_2^0-measurable in general. Using the ideas of the proof of [32, Lemma 2.4] we obtain Theorem 1.24. This result is improved in the next subsection, see Theorem 1.28, but since some steps remain the same, let us give a sketch of the argument (we will be a bit sketchy at the places which will be modified later).

Lemma 1.22. *Let $n \in \mathbb{N}$, X be isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Let there exist $\mathbf{\Sigma}_n^0$-measurable mappings $\chi_k : \mathcal{B} \to X$, $k \in \mathbb{N}$, such that $X_\mu \equiv \overline{\operatorname{span}}\{\chi_k(\mu) : \; k \in \mathbb{N}\}$ for every $\mu \in \mathcal{B}$.*

Then there exists a $\mathbf{\Sigma}_{n+1}^0$-measurable mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{B}$ we have $X_\mu \equiv \Phi(\mu)$.

Proof. Consider the mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ defined as

$$\Phi(\nu) := \overline{\operatorname{span}}\{\chi_k(\nu) : \; k \in \mathbb{N}\}, \quad \nu \in \mathcal{B}.$$

We have $X_\nu \equiv \Phi(\nu)$. For every open set $U \subseteq X$, using the $\mathbf{\Sigma}_n^0$-measurability of χ_k's, it is easy to see that $\Phi^{-1}(E^+(U))$ is $\mathbf{\Sigma}_n^0$ in $\mathcal{B}$. Thus, by Lemma 1.19, the mapping Φ is $\mathbf{\Sigma}_{n+1}^0$-measurable. $\square$

Remark 1.23. Similarly as in Remark 1.21, an analogous approach leads to a similar statement valid for any $\mathcal{B}_d$, $d \in \mathbb{N}$, instead of $\mathcal{B}$.

Theorem 1.24. *Let X be isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Then there is a $\mathbf{\Sigma}^0_4$-measurable mapping $\Phi : \mathcal{P} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}$ we have $X_\mu \equiv \Phi(\mu)$.*

Sketch of the proof. By Remark 1.21, it suffices to find, for every $d \in \mathbb{N} \cup \{\infty\}$, a $\mathbf{\Sigma}^0_3$-measurable reduction from $\mathcal{B}_d$ to $SB(X)$, where $\mathcal{B}_\infty = \mathcal{B}$. This is done for every $d \in \mathbb{N} \cup \{\infty\}$ in a similar way. Let us concentrate further only on the case of $d = \infty$, the other cases are similar. From the proof of [32, Lemma 2.4], it follows that there are Borel measurable mappings $\chi_k : \mathcal{B} \to X$, $k \in \mathbb{N}$, such that $X_\mu \equiv \overline{\operatorname{span}}\{\chi_k(\mu) : k \in \mathbb{N}\}$ for every $\mu \in \mathcal{B}$. A careful inspection of the proof actually shows that the mappings χ_k are $\mathbf{\Sigma}^0_2$-measurable (since this part is improved in the next subsection, see Proposition 1.26, we do not give any more details here). Thus, an application of Lemma 1.22 finishes the proof. $\square$

1.3. **An optimal reduction from $\mathcal{B}$ to $SB(X)$.** This subsection is devoted mainly to the proof of the following result.

Theorem 1.25. *Let X be isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Then there is a $\mathbf{\Sigma}^0_2$-measurable mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{B}$ we have $X_\mu \equiv \Phi(\mu)$.*

The main ingredient of the proof is the following.

Proposition 1.26. *For any isometrically universal separable Banach space X, there exist continuous mappings $\chi_k : \mathcal{B} \to X, k \in \mathbb{N}$, such that*

$$\Big\| \sum_{k=1}^n a_k \chi_k(\nu) \Big\| = \nu\Big(\sum_{k=1}^n a_k e_k \Big)$$

for every $\sum_{k=1}^n a_k e_k \in c_{00}$ and every $\nu \in \mathcal{B}$.

Remark 1.27. Similarly as in Remark 1.21, we may easily obtain a variant of Proposition 1.26 for $\mathcal{B}_d$, $d \in \mathbb{N}$. Indeed, let $d \in \mathbb{N}$ be given. For $\nu \in \mathcal{B}_d$, let us define $\widetilde{\nu} \in \mathcal{B}$ by

$$\widetilde{\nu}\Big(\sum_{i=1}^\infty a_i e_i\Big) := \nu\Big(\sum_{i=1}^d a_i e_i\Big) + \sum_{i=d+1}^\infty |a_i|, \quad \sum_{i=1}^\infty a_i e_i \in c_{00}.$$

If χ_k, $k \in \mathbb{N}$, are as in Proposition 1.26, then we may consider mappings $\widetilde{\chi}_k : \mathcal{B}_d \to X$, $k \leq d$, defined by $\widetilde{\chi}_k(\nu) = \chi_k(\widetilde{\nu})$, $\nu \in \mathcal{B}_d$.

We postpone the proof of Proposition 1.26 to the very end of this subsection.

Proof of Theorem 1.25. Follows immediately from Lemma 1.22 and Proposition 1.26. $\square$

Similarly as above, we obtain also the following.

Theorem 1.28. *Let X be isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Then there is a $\mathbf{\Sigma}^0_3$-measurable mapping $\Phi : \mathcal{P} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}$ we have $X_\mu \equiv \Phi(\mu)$.*

Proof. This is similar to the proof of Theorem 1.24, the only modification is that we use Proposition 1.26 and Remark 1.27 instead of the reference to the proof of [32, Lemma 2.4]. $\square$

The aim of the remainder of this subsection is now to prove Proposition 1.26.

Proposition 1.29. *For each $\mathcal{A} \subseteq \mathcal{P}$, the following conditions are equivalent:*

(1) There are a separable Banach space U and continuous mappings $\chi_k : \mathcal{A} \to U, k \in \mathbb{N}$, such that

$$\Big\| \sum_{k=1}^{n} a_k \chi_k(\nu) \Big\| = \nu\Big(\sum_{k=1}^{n} a_k e_k \Big)$$

for every $\sum_{k=1}^{n} a_k e_k \in c_{00}$ and every $\nu \in \mathcal{A}$.

(2) There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0, \infty), k \in \mathbb{N}$, such that

$$\alpha_k(\nu, \nu) = 0$$

for every ν and k and the following property is satisfied: If $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : \mathcal{A} \to (c_{00})^\#$ such that $\Gamma(\nu) = z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in \mathcal{A}$ and $x \in c_{00}$, and

$$|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \alpha_k(\mu, \lambda)$$

for every $\mu, \lambda \in \mathcal{A}$ and every k.

Moreover, if $\mathcal{A}$ consists only of pseudonorms ν with $\nu(e_k) \leq 1$ for every k, then these conditions are equivalent with:

(2') For every $\eta \in [0, 1)$, there are continuous functions $\beta_k : \mathcal{A}^2 \to [0, \infty), k \in \mathbb{N}$, such that

$$\beta_k(\nu, \nu) = 0$$

for every ν and k and the following property is satisfied: If $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : \mathcal{A} \to (c_{00})^\#$ such that $\Gamma(\nu) = \eta \cdot z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in \mathcal{A}$ and $x \in c_{00}$, and

$$|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \beta_k(\mu, \lambda)$$

for every $\mu, \lambda \in \mathcal{A}$ and every k.

Remark 1.30. The conditions (1) and (2) from Proposition 1.29 are equivalent also with the following one:

(3) There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0, \infty), k \in \mathbb{N}$, such that

$$\alpha_k(\nu, \nu) = 0$$

for every ν and k and

$$\sum_{\mu,\lambda,k} |a_{\mu,\lambda,k}| \alpha_k(\mu, \lambda) \geq \nu\Big(\sum_{\lambda,k} (a_{\nu,\lambda,k} - a_{\lambda,\nu,k}) e_k \Big) - \sum_{\mu \neq \nu} \mu\Big(\sum_{\lambda,k} (a_{\mu,\lambda,k} - a_{\lambda,\mu,k}) e_k \Big)$$

for every $\nu \in \mathcal{A}$ and every system $(a_{\mu,\lambda,k})_{\mu,\lambda \in \mathcal{A}, k \in \mathbb{N}}$ of real numbers with finite support.

The proof is similar to the proof of (1) $\Leftrightarrow$ (2) below (for (1) $\Rightarrow$ (3) the choice of α_k's is the same as in the proof of (1) $\Rightarrow$ (2), for (3) $\Rightarrow$ (1) the construction of the space U is the same as in (2) $\Rightarrow$ (1)). We omit the full proof, because the details are technical and we do not use the condition (3) any further. Let us note that even though we tried to find an application of the condition (3), we did not find it and this is basically the reason why we had to develop conditions (2) and (2′).

Proof of Proposition 1.29. (1) $\Rightarrow$ (2): Given such U and $\chi_k : \mathcal{A} \to U, k \in \mathbb{N}$, we put

$$\alpha_k(\nu, \mu) = \|\chi_k(\nu) - \chi_k(\mu)\|, \quad \nu, \mu \in \mathcal{A}, k \in \mathbb{N}.$$

Denote by $I_\mu : (c_{00}, \mu) \to U$ the isometry given by $e_k \mapsto \chi_k(\mu)$. Let $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfying $|z^*(x)| \leq \nu(x)$ be given. For $x, y \in c_{00}$ with $I_\nu x = I_\nu y$, we have $|z^*(y - x)| \leq \nu(y - x) = \|I_\nu(y - x)\| = 0$, and so $z^*(x) = z^*(y)$. Thus, the formula

$$u^*(I_\nu x) = z^*(x), \quad x \in c_{00},$$

defines a functional on $I_\nu(c_{00})$ such that $|u^*(I_\nu x)| = |z^*(x)| \leq \nu(x) = \|I_\nu x\|$. By the Hahn-Banach theorem, we can extend u^* to the whole U in the way that

$$|u^*(u)| \leq \|u\|, \quad u \in U.$$

For every $\mu \in \mathcal{A}$, let us put

$$\Gamma(\mu)(x) = u^*(I_\mu x), \quad x \in c_{00}.$$

We obtain $|\Gamma(\mu)(x)| = |u^*(I_\mu x)| \leq \|I_\mu x\| = \mu(x)$ and $|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| = |u^*(I_\mu e_k) - u^*(I_\lambda e_k)| = |u^*(\chi_k(\mu) - \chi_k(\lambda))| \leq \|\chi_k(\mu) - \chi_k(\lambda)\| = \alpha_k(\mu, \lambda)$ for $\mu, \lambda \in \mathcal{A}$ and $k \in \mathbb{N}$.

(2) $\Rightarrow$ (1): Given such $\alpha_k : \mathcal{A}^2 \to [0, \infty), k \in \mathbb{N}$, we define a subset of $c_{00}(\mathcal{A} \times \mathbb{N})$ by

$$\Omega = \mathrm{co}\bigg(\bigcup_\mu \Big\{\sum_k a_k e_{\mu,k} : \mu\Big(\sum_k a_k e_k\Big) \leq 1\Big\} \cup \bigcup_{\mu,\lambda,k} \big\{c \cdot (e_{\mu,k} - e_{\lambda,k}) : |c| \cdot \alpha_k(\mu, \lambda) \leq 1\big\}\bigg),$$

and denote the corresponding Minkowski functional by ϱ. Let U be the completion of the quotient space X/N, where $X = (c_{00}(\mathcal{A} \times \mathbb{N}), \varrho)$ and $N = \{x \in c_{00}(\mathcal{A} \times \mathbb{N}) : \rho(x) = 0\}$. In what follows, we identify every $x \in c_{00}(\mathcal{A} \times \mathbb{N})$ with its equivalence class $[x]_N \in U$. Let us define

$$\chi_k : \mathcal{A} \to U, \quad \nu \mapsto e_{\nu,k}.$$

As $c \cdot (e_{\mu,k} - e_{\lambda,k}) \in \Omega$ whenever $|c| \cdot \alpha_k(\mu, \lambda) \leq 1$, we obtain $\varrho(e_{\mu,k} - e_{\lambda,k}) \leq \alpha_k(\mu, \lambda)$, that is, $\varrho(\chi_k(\mu) - \chi_k(\lambda)) \leq \alpha_k(\mu, \lambda)$. For a fixed μ, we have $\alpha_k(\mu, \lambda) \to \alpha_k(\mu, \mu) = 0$ as $\lambda \to \mu$, and consequently $\varrho(\chi_k(\mu) - \chi_k(\lambda)) \to 0$ as $\lambda \to \mu$. Therefore, χ_k is continuous on $\mathcal{A}$. It follows that the image of χ_k is separable. As these images contain all basic vectors $e_{\nu,k}$, the space U is separable.

We need to show that

$$\nu(x) = \varrho(\overline{x})$$

for fixed $\nu \in \mathcal{A}$, $x = \sum_{k \in \mathbb{N}} a_k e_k \in c_{00}$ and its image $\overline{x} = \sum_{k \in \mathbb{N}} a_k e_{\nu,k}$. The inequality $\nu(x) \geq \varrho(\overline{x})$ follows immediately from the definition of Ω (for any $c \geq \nu(x)$ with $c > 0$, we have $\nu(\frac{1}{c}x) \leq 1$, and so $\frac{1}{c}\overline{x} \in \Omega$, hence $\varrho(\frac{1}{c}\overline{x}) \leq 1$ and $\varrho(\overline{x}) \leq c$). Let us show the opposite inequality $\nu(x) \leq \varrho(\overline{x})$. Using the Hahn-Banach theorem, we can pick $z^* \in (c_{00})^\#$ satisfying $z^*(x) = \nu(x)$ and $|z^*(y)| \leq \nu(y)$ for every $y \in c_{00}$. Let $\Gamma : \mathcal{A} \to (c_{00})^\#$ be the mapping provided for ν and z^*, and let $u^* \in (c_{00}(\mathcal{A} \times \mathbb{N}))^\#$ be given by

$$u^*(e_{\mu,k}) = \Gamma(\mu)(e_k), \quad \mu \in \mathcal{A}, k \in \mathbb{N}.$$

Then $u^*(\overline{x}) = u^*(\sum_k a_k e_{\nu,k}) = \sum_k a_k u^*(e_{\nu,k}) = \sum_k a_k \Gamma(\nu)(e_k) = \Gamma(\nu)(\sum_k a_k e_k) = z^*(x) = \nu(x)$. It is sufficient to show that $u^* \leq 1$ on Ω (equivalently $|u^*(y)| \leq \varrho(y)$ for every $y \in c_{00}(\mathcal{A} \times \mathbb{N})$), since it follows that $\nu(x) = u^*(\overline{x}) \leq \varrho(\overline{x})$.

To show that $u^* \leq 1$ on Ω, we need to check that

$$\mu\Big(\sum_k b_k e_k\Big) \leq 1 \quad \Rightarrow \quad u^*\Big(\sum_k b_k e_{\mu,k}\Big) \leq 1$$

and

$$|c| \cdot \alpha_k(\mu, \lambda) \leq 1 \quad \Rightarrow \quad u^*\big(c \cdot (e_{\mu,k} - e_{\lambda,k})\big) \leq 1.$$

Concerning the first implication, we compute $u^*(\sum_k b_k e_{\mu,k}) = \sum_k b_k u^*(e_{\mu,k}) = \sum_k b_k \Gamma(\mu)(e_k) = \Gamma(\mu)(\sum_k b_k e_k) \leq \mu(\sum_k b_k e_k) \leq 1$. Concerning the second implication, we compute $u^*(c \cdot (e_{\mu,k} - e_{\lambda,k})) = cu^*(e_{\mu,k}) - cu^*(e_{\lambda,k}) = c\Gamma(\mu)(e_k) - c\Gamma(\lambda)(e_k) \leq |c|\alpha_k(\mu, \lambda) \leq 1$.

(2) $\Rightarrow$ (2'): The choice $\beta_k = \alpha_k$ works. Indeed, if Γ is provided by (2), we can take $\eta \cdot \Gamma$.

(2') $\Rightarrow$ (2): For every $n \in \mathbb{N}$, let $\beta_k^n : \mathcal{A}^2 \to [0, \infty), k \in \mathbb{N}$, be provided by (2') for $\eta = (1 - 2^{-n})$. We can assume that each β_k^n is a pseudometric. Indeed, instead of β_k^n, we can take the maximal minorizing pseudometric $\widetilde{\beta}_k^n$ (in such a case, $\widetilde{\beta}_k^n$ is continuous and, since the function $(\mu,\lambda)\mapsto |\Gamma(\mu)(e_k)-\Gamma(\lambda)(e_k)|$ is a pseudometric, if it minorizes β_k^n, then it minorizes $\widetilde{\beta}_k^n$ as well). Moreover, we can assume that β_1^1 is a metric (it is possible to add a compatible metric on $\mathcal{A}$ to β_1^1).

Let us define

$$\alpha(\mu,\lambda)=\max_{n,k}\min\{\beta_k^n(\mu,\lambda),2^{-\max\{n,k\}}\},\quad \mu,\lambda\in\mathcal{A}.$$

It is easy to check that α is continuous. Due to our additional assumptions, α is a metric on $\mathcal{A}$. We want to show that there are some constants c_k such that the choice $\alpha_k=c_k\cdot\alpha$ works.

Let $\nu\in\mathcal{A}$ and $z^*\in(c_{00})^\#$ satisfy $|z^*(x)|\le\nu(x)$ for every $x\in c_{00}$. For every $n\in\mathbb{N}$, there is a mapping $\Gamma^n:\mathcal{A}\to(c_{00})^\#$ such that

$$\Gamma^n(\nu)=(1-2^{-n})\cdot z^*,$$

$$|\Gamma^n(\mu)(x)|\le\mu(x),\quad \mu\in\mathcal{A},\ x\in c_{00},$$

and, if we denote

$$\gamma_k^n(\mu)=\Gamma^n(\mu)(e_k),$$

then

$$|\gamma_k^n(\mu)-\gamma_k^n(\lambda)|\le\beta_k^n(\mu,\lambda)$$

for every $\mu,\lambda\in\mathcal{A}$ and every k. Let us note that

$$|\gamma_k^n(\mu)|\le 1,$$

as $|\gamma_k^n(\mu)|=|\Gamma^n(\mu)(e_k)|\le\mu(e_k)\le 1$ by the assumption on $\mathcal{A}$.

Now, we define the desired mapping Γ. For practical purposes, we first define

$$\gamma_k^n(\mu)=0\quad\text{for } n\in\mathbb{Z},n\le 0.$$

For every $n\in\mathbb{Z}$, let f_n denote the piecewise linear function supported by $[2^{-n-3},2^{-n-1}]$ which is linear on $[2^{-n-3},2^{-n-2}]$ and $[2^{-n-2},2^{-n-1}]$, and for which $f_n(2^{-n-2})=1$. In this way, we have $\sum_{n\in\mathbb{Z}}f_n=1$ on $(0,\infty)$. We define

$$\gamma_k(\nu)=z^*(e_k)$$

and

$$\gamma_k(\mu)=\sum_{n\in\mathbb{Z}}f_n(\alpha(\mu,\nu))\gamma_k^n(\mu),\quad \mu\ne\nu.$$

Finally, we put $\Gamma(\mu)(e_k)=\gamma_k(\mu)$, so $\Gamma(\nu)=z^*$ and $\Gamma(\mu)=\sum_{n\in\mathbb{N}}f_n(\alpha(\mu,\nu))\Gamma^n(\mu)$ for $\mu\ne\nu$. In both cases $\mu=\nu$ and $\mu\ne\nu$, it follows that $|\Gamma(\mu)(x)|\le\mu(x)$ for every $x\in c_{00}$. It remains to prove the inequality

$$|\gamma_k(\mu)-\gamma_k(\lambda)|\le c_k\cdot\alpha(\mu,\lambda)$$

for some suitable constants c_k.

Let us show that the implication

$$\text{(1)}\qquad \alpha(\mu,\lambda)<2^{-n}\quad\Rightarrow\quad|\gamma_k^n(\mu)-\gamma_k^n(\lambda)|\le 2^{k+1}\alpha(\mu,\lambda)$$

holds. Clearly, we can suppose that $n\ge 1$. If $\alpha(\mu,\lambda)\ge 2^{-k}$, then $2^{k+1}\alpha(\mu,\lambda)\ge 2\ge|\gamma_k^n(\mu)-\gamma_k^n(\lambda)|$. So, let us assume that $\alpha(\mu,\lambda)<2^{-k}$. Since $\min\{\beta_k^n(\mu,\lambda),2^{-\max\{n,k\}}\}\le\alpha(\mu,\lambda)<2^{-\max\{n,k\}}$, we have $\min\{\beta_k^n(\mu,\lambda),2^{-\max\{n,k\}}\}=\beta_k^n(\mu,\lambda)$, and so $|\gamma_k^n(\mu)-\gamma_k^n(\lambda)|\le\beta_k^n(\mu,\lambda)=\min\{\beta_k^n(\mu,\lambda),2^{-\max\{n,k\}}\}\le\alpha(\mu,\lambda)\le 2^{k+1}\alpha(\mu,\lambda)$.

Next, we show that

$$\text{(2)}\qquad 2^{-n-4}\le\alpha(\mu,\nu)<2^{-n}\quad\Rightarrow\quad|\gamma_k^n(\mu)-\gamma_k(\nu)|\le(2^{k+1}+16)\alpha(\mu,\nu).$$

If $n \geq 1$, then $\gamma_k(\nu) - \gamma_k^n(\nu) = z^*(e_k) - (1-2^{-n})z^*(e_k) = 2^{-n}z^*(e_k)$. If $n \leq 0$, then $\gamma_k(\nu) - \gamma_k^n(\nu) = z^*(e_k)$. In both cases, $|\gamma_k(\nu) - \gamma_k^n(\nu)| \leq 2^{-n}|z^*(e_k)| \leq 2^{-n}\nu(e_k) \leq 2^{-n} \leq 2^4\alpha(\mu,\nu)$. Using (1), we can compute

$$|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq |\gamma_k^n(\mu) - \gamma_k^n(\nu)| + |\gamma_k^n(\nu) - \gamma_k(\nu)| \leq (2^{k+1} + 2^4)\alpha(\mu,\nu).$$

Further, it follows from (2) that

$$|\gamma_k(\mu) - \gamma_k(\nu)| \leq (2^{k+1} + 16)\alpha(\mu,\nu). \tag{3}$$

Indeed, since f_n is supported by $[2^{-n-3}, 2^{-n-1}]$, we have always

$$f_n(\alpha(\mu,\nu))|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq f_n(\alpha(\mu,\nu))(2^{k+1} + 16)\alpha(\mu,\nu),$$

and it is sufficient to use that $\gamma_k(\mu) - \gamma_k(\nu) = \sum_{n\in\mathbb{Z}} f_n(\alpha(\mu,\nu))(\gamma_k^n(\mu) - \gamma_k(\nu))$ for $\mu \neq \nu$.

Now, we are going to investigate the value $|\gamma_k(\mu) - \gamma_k(\lambda)|$. First, we have

$$\alpha(\lambda,\nu) \geq 2\alpha(\mu,\nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq 3 \cdot (2^{k+1} + 16)\alpha(\mu,\lambda). \tag{4}$$

Indeed, as $\alpha(\mu,\lambda) \geq \alpha(\lambda,\nu) - \alpha(\mu,\nu) \geq 2\alpha(\mu,\nu) - \alpha(\mu,\nu) = \alpha(\mu,\nu)$, we can apply (3) and write

$$\begin{aligned}
|\gamma_k(\mu) - \gamma_k(\lambda)| &\leq |\gamma_k(\mu) - \gamma_k(\nu)| + |\gamma_k(\lambda) - \gamma_k(\nu)| \\
&\leq (2^{k+1} + 16)(\alpha(\mu,\nu) + \alpha(\lambda,\nu)) \\
&= (2^{k+1} + 16)(\alpha(\lambda,\nu) - \alpha(\mu,\nu) + 2\alpha(\mu,\nu)) \\
&\leq (2^{k+1} + 16)(1 + 2)\alpha(\mu,\lambda).
\end{aligned}$$

Now, we prove the last but the most challenging implication

(5)

$$\alpha(\mu,\nu) \leq \alpha(\lambda,\nu) < 2\alpha(\mu,\nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq [12(2^{k+1} + 16) + 2^{k+1}]\alpha(\mu,\lambda).$$

Let us compute

$$\begin{aligned}
&\gamma_k(\mu) - \gamma_k(\lambda) = \sum_{n\in\mathbb{Z}} \big[f_n(\alpha(\mu,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\lambda)\big] \\
&= \sum_{n\in\mathbb{Z}} \big[f_n(\alpha(\mu,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\mu) + f_n(\alpha(\lambda,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\lambda)\big] \\
&= \sum_{n\in\mathbb{Z}} [f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))]\gamma_k^n(\mu) + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)] \\
&= \sum_{n\in\mathbb{Z}} [f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))](\gamma_k^n(\mu) - \gamma_k(\nu)) + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)].
\end{aligned}$$

Hence, $|\gamma_k(\mu) - \gamma_k(\lambda)|$ is less than or equal to

$$\sum_{n\in\mathbb{Z}} |f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))||\gamma_k^n(\mu) - \gamma_k(\nu)| + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))|\gamma_k^n(\mu) - \gamma_k^n(\lambda)|.$$

Let us notice that

- $f_n(\alpha(\mu,\nu)) \neq 0$ iff $2^{-n-3} < \alpha(\mu,\nu) < 2^{-n-1}$,
- $f_n(\alpha(\lambda,\nu)) \neq 0$ iff $2^{-n-3} < \alpha(\lambda,\nu) < 2^{-n-1}$, and $2^{-n-4} < \alpha(\mu,\nu) < 2^{-n-1}$ in this case,
- the function f_n is Lipschitz with the constant 2^{n+3}.

So, if $f_n(\alpha(\mu,\nu)) \neq 0$ or $f_n(\alpha(\lambda,\nu)) \neq 0$, then $2^{-n-4} < \alpha(\mu,\nu) < 2^{-n-1}$, and (2) can be applied. We obtain for the first sum that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} &|f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))||\gamma_k^n(\mu) - \gamma_k(\nu)| \\ &\leq \sum_{2^{-n-4}<\alpha(\mu,\nu)<2^{-n-1}} 2^{n+3}|\alpha(\mu,\nu) - \alpha(\lambda,\nu)|(2^{k+1}+16)\alpha(\mu,\nu) \\ &\leq \sum_{2^{-n-4}<\alpha(\mu,\nu)<2^{-n-1}} 2^{n+3}\alpha(\mu,\lambda)(2^{k+1}+16)2^{-n-1} \\ &\leq 3\cdot 2^2(2^{k+1}+16)\alpha(\mu,\lambda).\end{aligned}$$

Concerning the second sum, we notice that if $f_n(\alpha(\lambda,\nu)) \neq 0$, then $\alpha(\lambda,\nu) < 2^{-n-1}$, and so $\alpha(\mu,\lambda) \leq \alpha(\mu,\nu) + \alpha(\lambda,\nu) \leq 2\alpha(\lambda,\nu) < 2^{-n}$. Applying (1), we obtain

$$\sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))|\gamma_k^n(\mu) - \gamma_k^n(\lambda)| \leq \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))2^{k+1}\alpha(\mu,\lambda) = 2^{k+1}\alpha(\mu,\lambda),$$

and (5) follows.

Finally, we finish the proof with the observation that (4) and (5) provide

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \leq [12(2^{k+1}+16) + 2^{k+1}]\alpha(\mu,\lambda).$$

We can suppose that $\alpha(\mu,\nu) \leq \alpha(\lambda,\nu)$. If $\alpha(\lambda,\nu) \geq 2\alpha(\mu,\nu)$, we use (4), and if $\alpha(\lambda,\nu) < 2\alpha(\mu,\nu)$, we use (5). □

Definition 1.31. Let $\mathcal{B}_{(1)}$ denote the set of all norms $\nu \in \mathcal{B}$ such that $\nu(e_k) = 1$ for each $k \in \mathbb{N}$.

Lemma 1.32. *The condition (2') from Proposition 1.29 is valid for $\mathcal{A} = \mathcal{B}_{(1)}$.*

Proof. Let $\eta \in [0,1)$ be given. We fix numbers $\kappa_k < 1$ such that $\eta \leq \kappa_1 < \kappa_2 < \kappa_3 < \dots$. For every $\mu,\lambda \in \mathcal{B}_{(1)}$, we define recursively

$$\beta_1(\mu,\lambda) = 0$$

and

$$\begin{aligned}\beta_{k+1}(\mu,\lambda) = \sup\Big\{ &\Big|\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big)\Big| + \sum_{i=1}^k |a_i|\beta_i(\mu,\lambda) : \\ &a_1,\dots,a_k \in \mathbb{R},\ \min\Big\{\mu\Big(\sum_{i=1}^k a_ie_i\Big), \lambda\Big(\sum_{i=1}^k a_ie_i\Big)\Big\} < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}\Big\}.\end{aligned}$$

Clearly, $\beta_k(\nu,\nu) = 0$ for every $\nu \in \mathcal{B}_{(1)}$. Let us sketch a proof of continuity of the functions β_k. The function $\beta_1 = 0$ is obviously continuous. Assuming that β_i is continuous for every $i \leq k$, we consider for $\delta > 0$ the set

$$\mathcal{U}_\delta^{k+1}(\mu) = \Big\{\mu' \in \mathcal{B}_{(1)} : \Big(\forall x \in \operatorname{span}\{e_1,\dots,e_{k+1}\}\setminus\{0\} : (1+\delta)^{-1} < \frac{\mu'(x)}{\mu(x)} < 1+\delta\Big)\Big\}.$$

Using Lemma 1.2, it is easy to see that $\mathcal{U}_\delta^{k+1}(\mu)$ is an open neighborhood of μ in $\mathcal{B}_{(1)}$. Given $\varepsilon > 0$, we can find $\delta > 0$ such that $|\beta_{k+1}(\mu',\lambda') - \beta_{k+1}(\mu,\lambda)| < \varepsilon$ for every $(\mu',\lambda') \in \mathcal{U}_\delta^{k+1}(\mu) \times \mathcal{U}_\delta^{k+1}(\lambda)$. The details are left to the reader.

Let us prove that the functions β_k work. Given $\nu \in \mathcal{B}_{(1)}$ and $z^* \in (c_{00})^\#$ satisfying $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, we define first

$$\gamma_1(\mu) = \eta\cdot z^*(e_1), \quad \mu \in \mathcal{B}_{(1)}.$$

Recursively, we define for every $k \in \mathbb{N}$ functions

$$u_{k+1}(\mu) = \sup_{a_1,\dots,a_k} \Big[-\kappa_{k+1}\mu\Big(-e_{k+1} + \sum_{i=1}^k a_i e_i \Big) + \sum_{i=1}^k a_i \gamma_i(\mu) \Big],$$

$$v_{k+1}(\mu) = \inf_{a_1,\dots,a_k} \Big[\kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^k a_i e_i \Big) - \sum_{i=1}^k a_i \gamma_i(\mu) \Big]$$

and

$$\gamma_{k+1}(\mu) = p_{k+1} u_{k+1}(\mu) + q_{k+1} v_{k+1}(\mu),$$

where numbers $p_{k+1} \geq 0, q_{k+1} \geq 0$ with $p_{k+1} + q_{k+1} = 1$ are chosen in the way that

$$\gamma_{k+1}(\nu) = \eta \cdot z^*(e_{k+1}). \tag{6}$$

Let us check that it is possible to choose such numbers. Note first that $\gamma_1(\nu) = \eta \cdot z^*(e_1)$. Assuming that the functions γ_i are already defined and satisfy $\gamma_i(\nu) = \eta \cdot z^*(e_i)$ for $i \leq k$, we notice that, for every $a_1, \dots, a_k \in \mathbb{R}$,

$$\pm\eta \cdot z^*(e_{k+1}) + \sum_{i=1}^k a_i \gamma_i(\nu) = \eta \cdot z^*\Big(\pm e_{k+1} + \sum_{i=1}^k a_i e_i \Big) \leq \kappa_{k+1}\nu\Big(\pm e_{k+1} + \sum_{i=1}^k a_i e_i \Big),$$

and consequently

$$\eta \cdot z^*(e_{k+1}) \geq -\kappa_{k+1}\nu\Big(-e_{k+1} + \sum_{i=1}^k a_i e_i \Big) + \sum_{i=1}^k a_i \gamma_i(\nu),$$

$$\eta \cdot z^*(e_{k+1}) \leq \kappa_{k+1}\nu\Big(e_{k+1} + \sum_{i=1}^k a_i e_i \Big) - \sum_{i=1}^k a_i \gamma_i(\nu).$$

This gives

$$u_{k+1}(\nu) \leq \eta \cdot z^*(e_{k+1}) \leq v_{k+1}(\nu),$$

and it follows that suitable p_{k+1} and q_{k+1} do exist.

Let us prove that

$$\sum_{i=1}^k a_i \gamma_i(\mu) \leq \kappa_k \mu\Big(\sum_{i=1}^k a_i e_i \Big) \tag{7}$$

for every $\mu \in \mathcal{B}_{(1)}$, $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{R}$. For $k = 1$, we just write $a_1\gamma_1(\mu) = a_1\eta \cdot z^*(e_1) \leq |a_1|\kappa_1\nu(e_1) = |a_1|\kappa_1 = |a_1|\kappa_1\mu(e_1) = \kappa_1\mu(a_1e_1)$. Assume that (7) is valid for k. We show first that

$$u_{k+1}(\mu) \leq \gamma_{k+1}(\mu) \leq v_{k+1}(\mu).$$

Clearly, it is sufficient to show just that $u_{k+1}(\mu) \leq v_{k+1}(\mu)$. Given $b_1, \dots, b_k$ and $c_1, \dots, c_k$, we need to check that

$$-\kappa_{k+1}\mu\Big(-e_{k+1} + \sum_{i=1}^k b_i e_i \Big) + \sum_{i=1}^k b_i \gamma_i(\mu) \leq \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^k c_i e_i \Big) - \sum_{i=1}^k c_i \gamma_i(\mu).$$

But this is easy, as

$$\begin{aligned} \sum_{i=1}^k b_i \gamma_i(\mu) + \sum_{i=1}^k c_i \gamma_i(\mu) &= \sum_{i=1}^k (b_i + c_i)\gamma_i(\mu) \leq \kappa_k \mu\Big(\sum_{i=1}^k (b_i + c_i) e_i \Big) \\ &\leq \kappa_{k+1}\Big[\mu\Big(-e_{k+1} + \sum_{i=1}^k b_i e_i \Big) + \mu\Big(e_{k+1} + \sum_{i=1}^k c_i e_i \Big) \Big]. \end{aligned}$$

Now, let us verify (7) for $k+1$. We can suppose that $a_{k+1} = \pm 1$. For $a_{k+1} = 1$, it is enough to use

$$\gamma_{k+1}(\mu) \leq v_{k+1}(\mu) \leq \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^{k} a_i e_i\Big) - \sum_{i=1}^{k} a_i \gamma_i(\mu),$$

and for $a_{k+1} = -1$, it is enough to use

$$\gamma_{k+1}(\mu) \geq u_{k+1}(\mu) \geq -\kappa_{k+1}\mu\Big(-e_{k+1} + \sum_{i=1}^{k} a_i e_i\Big) + \sum_{i=1}^{k} a_i \gamma_i(\mu).$$

Next, let us prove that

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \leq \beta_k(\mu, \lambda) \tag{8}$$

for every $\mu, \lambda \in \mathcal{B}_{(1)}$ and $k \in \mathbb{N}$. This is clear for $k = 1$, as γ_1 is constant. Assume that (8) is valid for $i \leq k$. To prove it for $k+1$, it is sufficient to show the inequalities

$$|u_{k+1}(\mu) - u_{k+1}(\lambda)| \leq \beta_{k+1}(\mu, \lambda) \quad \text{and} \quad |v_{k+1}(\mu) - v_{k+1}(\lambda)| \leq \beta_{k+1}(\mu, \lambda).$$

We consider the function v_{k+1} only, since the inequality for u_{k+1} can be shown in the same way. Let us note first that, in the definition of $v_{k+1}(\mu)$, it is possible to take the infimum only over k-tuples with

$$\mu\Big(\sum_{i=1}^{k} a_i e_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1} - \kappa_k}.$$

Indeed, for $a_1, \dots, a_k$ which do not satisfy this condition, using (7), we obtain

$$\begin{aligned}
\kappa_{k+1}\mu\Big(e_{k+1} + &\sum_{i=1}^{k} a_i e_i\Big) - \sum_{i=1}^{k} a_i \gamma_i(\mu) \\
&\geq \kappa_{k+1}\mu\Big(\sum_{i=1}^{k} a_i e_i\Big) - \kappa_{k+1}\mu(e_{k+1}) - \kappa_k\mu\Big(\sum_{i=1}^{k} a_i e_i\Big) \\
&= (\kappa_{k+1} - \kappa_k)\mu\Big(\sum_{i=1}^{k} a_i e_i\Big) - \kappa_{k+1} \\
&\geq 2\kappa_{k+1} - \kappa_{k+1} = \kappa_{k+1} = \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^{k} 0 \cdot e_i\Big) - \sum_{i=1}^{k} 0 \cdot \gamma_i(\mu).
\end{aligned}$$

Now, (8) is provided by the following computation, in which every sup/inf is meant over k-tuples with $\mu\Big(\sum_{i=1}^k a_ie_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}$ or $\lambda\Big(\sum_{i=1}^k a_ie_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}$:

$$
\begin{aligned}
&|v_{k+1}(\mu) - v_{k+1}(\lambda)| \\
&= \Big| \inf \Big[\kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \sum_{i=1}^k a_i\gamma_i(\mu)\Big] \\
&\qquad\qquad - \inf \Big[\kappa_{k+1}\lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \sum_{i=1}^k a_i\gamma_i(\lambda)\Big]\Big| \\
&\le \sup \Big| \Big[\kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \sum_{i=1}^k a_i\gamma_i(\mu)\Big] \\
&\qquad\qquad - \Big[\kappa_{k+1}\lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \sum_{i=1}^k a_i\gamma_i(\lambda)\Big]\Big| \\
&= \sup \Big|\kappa_{k+1}\Big[\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big)\Big] - \sum_{i=1}^k a_i\big(\gamma_i(\mu) - \gamma_i(\lambda)\big)\Big| \\
&\le \sup \Big[\Big|\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big)\Big| + \sum_{i=1}^k |a_i|\beta_i(\mu,\lambda)\Big] = \beta_{k+1}(\mu,\lambda).
\end{aligned}
$$

Finally, as usual, we put $\Gamma(\mu)(e_k) = \gamma_k(\mu)$. The required properties of Γ follow now from (6), (7) and (8). Thus, the functions β_k work, and the proof of the lemma is completed. □

Proof of Proposition 1.26. Let X be an isometrically universal separable Banach space. By Lemma 1.32, the condition (2') from Proposition 1.29 is valid for $\mathcal{A} = \mathcal{B}_{(1)}$. Hence, the condition (1) from this proposition is valid for $\mathcal{A} = \mathcal{B}_{(1)}$ as well. There are a separable Banach space U and continuous mappings $\chi_k : \mathcal{B}_{(1)} \to U, k \in \mathbb{N}$, such that

$$\Big\|\sum_{k=1}^n a_k\chi_k(\nu)\Big\| = \nu\Big(\sum_{k=1}^n a_ke_k\Big)$$

for every $\sum_{k=1}^n a_ke_k \in c_{00}$ and every $\nu \in \mathcal{B}_{(1)}$. Since X contains an isometric copy of U, we can suppose that $U \subseteq X$.

Let us consider the continuous mapping $\Psi : \mathcal{B} \to \mathcal{B}_{(1)}$ given by

$$\Psi(\mu)\Big(\sum_{k=1}^n a_ke_k\Big) = \mu\Big(\sum_{k=1}^n \frac{a_k}{\mu(e_k)}e_k\Big).$$

If we define

$$\widetilde{\chi}_k(\mu) = \mu(e_k)\cdot\chi_k(\Psi(\mu)), \quad \mu \in \mathcal{B},$$

for each $k \in \mathbb{N}$, then we get

$$\Big\|\sum_{k=1}^n b_k\widetilde{\chi}_k(\mu)\Big\| = \Big\|\sum_{k=1}^n b_k\mu(e_k)\chi_k(\Psi(\mu))\Big\| = \Psi(\mu)\Big(\sum_{k=1}^n b_k\mu(e_k)e_k\Big) = \mu\Big(\sum_{k=1}^n b_ke_k\Big)$$

for every $\sum_{k=1}^n b_ke_k \in c_{00}$ and every $\mu \in \mathcal{B}$. □

2. Generic properties

As soon as one has a Polish space, or more generally a Baire space, of some objects it is natural and often useful to find properties (of these objects) that are generic; that is, the corresponding subset of the space is comeager. In the case of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$ we resolve this problem completely, see Theorem 2.1.

In the case of the spaces $SB(X)$ with the admissible topology, this is the content of Problem 5.5 from [23]. We show that in that case the situation is more complicated. In particular, we confirm the suspicion of Godefroy and Saint-Raymond that being meager in $SB(X)$ is not independent of the chosen admissible topology.

2.1. **Generic objects in $\mathcal{P}$.** The main result of this subsection is the following.

Theorem 2.1. *Let $\mathbb{G}$ be the Gurariĭ space. The set $\langle \mathbb{G} \rangle^{\mathcal{I}}_{\equiv}$ is dense G_δ in $\mathcal{I}$ for any $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.*

Let us recall what the Gurariĭ space is. One of the characterizations of the Gurariĭ space is the following, for more details we refer the interested reader e.g. to [8] (the characterization below is provided by [8, Lemma 2.2]).

Definition 2.2. The Gurariĭ space is the unique (up to isometry) separable Banach space such that for every $\varepsilon > 0$ and every isometric embedding $g : A \to B$, where B is a finite-dimensional Banach space and A is a subspace of $\mathbb{G}$, there is a $(1+\varepsilon)$-isomorphism $f : B \to \mathbb{G}$ such that $\|f \circ g - id_A\| \leq \varepsilon$.

In the remainder of this subsection we prove Theorem 2.1. Let us start with the most technical part, namely that $\langle \mathbb{G} \rangle^{\mathcal{P}_\infty}_{\equiv}$ is G_δ in $\mathcal{P}_\infty$.

We need two technical lemmas first.

Lemma 2.3. *For every $\mu \in \mathcal{P}$, finite set $A \subseteq V$ and $\varepsilon > 0$ there exists $\nu \in \mathcal{B}$ with $|\mu(x) - \nu(x)| < \varepsilon$ and $\nu(x) \in \mathbb{Q}$ for every $x \in A$.*

Proof. It suffices to define such norm ν on $\operatorname{span} A$ since then we can easily find some extension to the whole V. We assume that $0 \notin A$ and moreover we can assume that no two elements of A lie in the same one-dimensional subspace, i.e. are scalar multiples of each other. Indeed, otherwise we would find a subset $A' \subseteq A$ where no elements are scalar multiples of each other and every element of A is a scalar multiple, necessarily rational scalar multiple, of some element from A'. Then proving the fact for A' for sufficiently small δ automatically proves it for A and ε.

We enumerate A as $\{a_1, \ldots, a_n\}$ and so that the first k elements $a_1, \ldots, a_k$, for some $k \leq n$, are linearly independent and form a basis of $\operatorname{span} A$.

Claim. By perturbing μ on A by an arbitrarily small $\delta > 0$ we can without loss of generality assume that for every $i \leq n$, $\mu(a_i) < K_i := \inf\{\sum_{j \in J} \mu(\alpha_j a_j) \colon i \notin J \subseteq \{1, \ldots, n\}, a_i = \sum_{j \in J} \alpha_j a_j\}$.

Suppose the claim is proved. Then for every $i \leq n$ we set $\nu'(a_i)$ to be an arbitrary positive rational number in the interval $[\mu(a_i), \min\{K_i, \mu(a_i) + \varepsilon\})$. From the assumption it is now clear that for all $i \leq n$, we have

$$\nu'(a_i) \leq \inf\{\sum_{j=1}^{n} |\alpha_j| \nu'(a_j) \colon a_i = \sum_{j=1}^{n} \alpha_j a_j\}.$$

We extend ν' to a norm ν on $\operatorname{span} A$ by the formula

$$\nu(v) := \inf\{\sum_{i=1}^{n} |\alpha_i| \nu'(a_i) \colon v = \sum_{i=1}^{n} \alpha_i a_i\},$$

for $v \in \operatorname{span} A$. From the previous assumption, it follows that $\nu(a_i) = \nu'(a_i)$, for all $i \leq n$. Moreover, ν is indeed a norm since $\nu(a_i) > 0$ for all $i \leq n$, and the infimum in the definition of ν is, by compactness, always attained.

It remains to prove the claim. Let $\|\cdot\|_2$ be the ℓ_2 norm on $\operatorname{span} A$ with $a_1, \ldots, a_k$ the orthonormal basis. For each $m \in \mathbb{N}$ set $\mu_m := \mu + \frac{\|\cdot\|_2}{m}$. Clearly $\mu_m \to \mu$, so it suffices to show that each μ_m satisfies the condition from the claim. Suppose that for some $m \in \mathbb{N}$ and $i \leq n$ we have

$$\mu_m(a_i) = \inf\{\sum_{j\in J} \mu_m(\alpha_j a_j) \colon i \notin J \subseteq \{1, \dots, n\}, a_i = \sum_{j\in J} \alpha_j a_j\}.$$

By compactness, the infimum is attained, i.e. there exists $(\alpha_j)_{j\leq n}$, with $\alpha_i = 0$, $a_i = \sum_{j\leq n} \alpha_j a_j$ and $\mu_m(a_i) = \sum_{j\leq n} \mu_m(\alpha_j a_j)$. Indeed, if the infimum is approximated by a sequence $(\alpha_1^l, \dots, \alpha_n^l)_{l\in\mathbb{N}} \subseteq \mathbb{R}^n$, then since each coordinate is bounded (because up to finitely many l's we have $\sum_{j=1}^n \mu_m(\alpha_j^l a_j) \leq 2\mu_m(a_i)$), we may pass to a converging subsequence and attain the infimum at the limit. The ℓ_2 norm $\|\cdot\|_2$ is strictly convex, so $\|a_i\|_2 < \sum_{j\leq n} \|\alpha_j a_j\|_2$, while μ by triangle inequality satisfies $\mu(a_i) \leq \sum_{j\leq n} \mu(\alpha_j a_j)$. Since μ_m is the sum of μ and a positive multiple of the ℓ_2 norm, we must have $\mu_m(a_i) < \sum_{j\leq n} \mu_m(\alpha_j a_j)$, a contradiction. □

Let us emphasize that if we write that a mapping is an "isometry" or an "isomorphism", we do not mean it is surjective if this is not explicitly mentioned.

Lemma 2.4. *(i) Given a basis $\mathfrak{b}_E = \{e_1, \dots, e_n\}$ of a finite-dimensional Banach space E, there is $C > 0$ and a function $\phi_2^{\mathfrak{b}_E} : [0, C) \to [0, \infty)$ continuous at zero with $\phi_2^{\mathfrak{b}_E}(0) = 0$ such that whenever X is a Banach space with $E \subseteq X$ and $\{x_i : i \leq n\} \subseteq X$ are such that $\|x_i - e_i\| < \varepsilon$, $i \leq n$, for some $\varepsilon < C$, then the linear operator $T : E \to X$ given by $T(e_i) := x_i$ is $(1 + \phi_2^{\mathfrak{b}_E}(\varepsilon))$-isomorphism and $\|T - Id_E\| \leq \phi_2^{\mathfrak{b}_E}(\varepsilon)$.*

(ii) Let $\varepsilon \in (0, 1)$, $T : X \to Y$ be a surjective $(1 + \varepsilon)$-isomorphism between Banach spaces X and Y, N be ε-dense for S_X. Then $T(N)$ is 3ε-dense for S_Y.

Proof. (i): Pick $C > 0$ such that $C \sum_{i=1}^n |\lambda_i| \leq \|\sum_{i=1}^n \lambda_i e_i\|$ for every $(\lambda_i)_{i=1}^n \in \mathbb{R}^n$. Then for any $x = \sum_{i=1}^n \lambda_i e_i$ we have

$$\|Tx - x\| \leq \sum_{i=1}^n |\lambda_i| \|x_i - e_i\| < \frac{\varepsilon}{C} \|x\|.$$

Thus, $\|T - Id_E\| < \frac{\varepsilon}{C}$, $\|T\| \leq 1 + \frac{\varepsilon}{C}$ and $\|Tx\| \geq (1 - \frac{\varepsilon}{C})\|x\| = (1 + \frac{\varepsilon}{C-\varepsilon})^{-1}\|x\|$. Thus, we may put $\phi_2^{\mathfrak{b}_E}(\varepsilon) := \frac{\varepsilon}{C-\varepsilon}$ for $\varepsilon \in [0, C)$.

(ii): Let $\varepsilon > 0$, $T : X \to Y$ and N be as in the assumptions. Then for every $y \in S_Y$ there is $x \in N$ with $\|x - \frac{T^{-1}(y)}{\|T^{-1}(y)\|}\| < \varepsilon$. Thus, we have

$$\begin{aligned}\|y - Tx\| &\leq \Big\|y - \frac{y}{\|T^{-1}(y)\|}\Big\| + \|T\| \cdot \Big\|x - \frac{T^{-1}(y)}{\|T^{-1}(y)\|}\Big\| \\ &< \Big|1 - \frac{1}{\|T^{-1}y\|}\Big| + (1 + \varepsilon)\varepsilon \leq \varepsilon + 2\varepsilon = 3\varepsilon.\end{aligned}$$

□

Notation 2.5. For a finite set $A \subseteq V$ and P, P' partial functions on V (i.e. functions whose domains are subsets of V) with $A \subseteq \mathrm{dom}(P), \mathrm{dom}(P')$, we put $d_A(P, P') := \max_{a\in A} |P(a) - P'(a)|$.

Let T be the countable set of tuples (n, n', P, P', g) such that:

(a) $n, n' \in \mathbb{N}$;
(b) $P \in \mathbb{Q}^{\mathrm{dom}(P)}$, $P' \in \mathbb{Q}^{\mathrm{dom}(P')}$ where $\mathrm{dom}(P)$ and $\mathrm{dom}(P')$ are finite subsets of V;
(c) there exists $\nu \in \mathcal{B}$ such that $P' = \nu|_{\mathrm{dom}(P')}$;
(d) $g : \mathrm{dom}(P) \to \mathrm{dom}(P')$ is a one-to-one mapping;
(e) $P = P' \circ g$;

(f) Whenever $\mu' \in \mathcal{P}$, $\nu' \in \mathcal{B}$ are such that $P' \subseteq \nu'$, $d_{\mathrm{dom}(P)}(P, \mu') < \frac{1}{n}$ and μ' restricted to $\mathrm{span}(\mathrm{dom}(P)) \subseteq c_{00}$ is a norm then there exists $T_g : (\mathrm{span}(\mathrm{dom}(P)), \mu') \to (\mathrm{span}(\mathrm{dom}(P')), \nu')$ which is $(1 + \frac{1}{n'})$-isomorphism and $T_g \supseteq g$.

For $(n, n', P, P', g) \in T$, we let $G(n, n', P, P', g)$ be the set of $\mu \in \mathcal{P}_\infty$ such that

- whenever $d_{\mathrm{dom}(P)}(P, \mu) < \frac{1}{n}$ and μ restricted to $\mathrm{span}(\mathrm{dom}(P)) \subseteq c_{00}$ is a norm, there is a $\mathbb{Q}$-linear mapping $\Phi : V \cap \mathrm{span}(\mathrm{dom}(P')) \to V$ such that $\mu(\Phi(gx) - x) < \frac{2}{n'}\mu(x)$ for every $x \in \mathrm{dom}(P)$ and $|P'(x) - \mu(\Phi(x))| < \frac{1}{n'}P'(x)$ for every $x \in \mathrm{dom}(P')$.

Proposition 2.6. *Let $\mu \in \mathcal{P}_\infty$. Then X_μ is isometric to the Gurariĭ space if and only if $\mu \in G(n, n', P, P', g)$ for every $(n, n', P, P', g) \in T$.*

Proof. In order to prove the first implication, let $\mu \in \mathcal{P}_\infty$ be such that X_μ is isometric to the Gurariĭ space and let $(n, n', P, P', g) \in T$ be such that $d_{\mathrm{dom}(P)}(P, \mu) < \frac{1}{n}$ and μ restricted to $\mathrm{span}(\mathrm{dom}(P)) \subseteq c_{00}$ is a norm. Consider the finite dimensional space $A := (\mathrm{span}(\mathrm{dom}(P)), \mu)$. Let $\nu \in \mathcal{B}$ be as in (c). Put $B = (\mathrm{span}(\mathrm{dom}\, P'), \nu)$ and pick a basis $\mathfrak{b} \subseteq V$ of B. By (f) there exists $T_g : A \to B$, which is $(1 + \frac{1}{n'})$-isomorphism and $T_g \supseteq g$. By [31, Lemma 2.2], there is a $(1 + \frac{1}{3n'})$-isomorphism $S : B \to X_\mu$ such that $\|ST_g - Id_A\| < \frac{1}{n'}$. By Lemma 2.4(i), we may for every $b \in \mathfrak{b}$ find $x_b \in V$ such that the linear mapping $Q : S(B) \to X_\mu$ given by $Q(S(b)) = x_b$, $b \in \mathfrak{b}$, is $(1 + \frac{1}{3n'})$-isomorphism with $\|Q - Id\| < \frac{1}{3n'}$. Consider $\Phi = QS|_{V \cap \mathrm{span}(\mathrm{dom}\, P')}$. This is indeed a $\mathbb{Q}$-linear map and since QS is $(1 + \frac{1}{3n'})^2$-isomorphism and $(1 + \frac{1}{3n'})^2 < 1 + \frac{1}{n'}$, we have $|\mu(\Phi(x)) - \nu(x)| < \frac{1}{n'}\nu(x)$ for $x \in \mathrm{dom}(P')$. Moreover, for every $x \in \mathrm{dom}(P)$ we have

$$\begin{aligned} \mu(\Phi(gx) - x) &= \mu(QST_g x - x) \le \mu(QST_g x - ST_g x) + \mu(ST_g x - x) \\ &< \tfrac{1}{3n'}\|ST_g\|\mu(x) + \tfrac{1}{n'}\mu(x) \le \tfrac{2}{n'}\mu(x). \end{aligned}$$

This shows that $\mu \in G(n, n', P, P', g)$.

In order to prove the second implication, let $\mu \in \mathcal{P}_\infty$ be such that $\mu \in G(n, n', P, P', g)$ whenever $(n, n', P, P', g) \in T$. In what follows for $x \in c_{00}$ we denote by $[x] \in X_\mu$ the equivalence class corresponding to x. Pick a finite dimensional space $A \subseteq X_\mu$, and an isometry $G : A \to B$, where B is a finite dimensional Banach space, we may without loss of generality assume $B \subseteq X_{\mu_B}$ for some $\mu_B \in \mathcal{B}$. Let $\mathfrak{b}_A := \{a_1, \dots, a_j\}$ be a normalized basis of A and extend $G(\mathfrak{b}_A) = \{G(a_1), \dots, G(a_j)\}$ to a normalized basis $\mathfrak{b}_B = \{b_1, \dots, b_k\}$ of B. Fix $\eta > 0$. It suffices to find $(1 + \eta)$-isomorphism $\Psi : B \to X_\mu$ with $\|\Psi G - I_A\| \le \eta$. Consider the functions ϕ_1 and $\phi_2^{\mathfrak{b}_A}$ from Lemma 1.2 and Lemma 2.4(i). Pick $\delta \in (0, 1)$ such that $\max\{\phi_1(t), \phi_2^{\mathfrak{b}_A}(t)\} < \eta$ whenever $t < \delta$ and $\varepsilon \in (0, \frac{1}{20})$ such that $\phi_1(5\varepsilon) < \frac{1}{20}$ and $\varepsilon + 72\max\{\varepsilon, \phi_1(5\varepsilon)\} < \delta$.

Claim 1. There are $M, N \subset V$ finite sets such that μ restricted to $\mathrm{span}\, N \subset c_{00}$ is a norm and surjective $(1 + \varepsilon)$-isomorphisms $T_A : A \to (\mathrm{span}\, N, \mu)$, $T_B : B \to (\mathrm{span}\, M, \mu_B)$ such that:

- N and M are ε-dense sets for $S_{T_A(A)}$ and $S_{T_B(B)}$ respectively.
- We have $\|[T_A a_i] - a_i\|_{X_\mu} < \varepsilon$ for every $a_i \in \mathfrak{b}_A$ and $\|(T_A)^{-1}x - [x]\|_{X_\mu} < \varepsilon$, $|\mu(x) - 1| < \varepsilon$ for every $x \in N$.
- $(T_B)^{-1}(M)$ is $\frac{\varepsilon}{3}$-dense for S_B and $\max\{|\mu_B((T_B)^{-1}x) - 1|, |\mu_B(x) - 1|\} < \frac{\varepsilon}{2}$ for every $x \in M$.
- $\big(T_B G (T_A)^{-1}\big)(N) \subseteq M$.

Proof of Claim 1. By Lemma 2.4(i), we may pick $\{f_1, \dots, f_n\} \subseteq V$ such that the linear operator $T_A : A \to X_\mu$ given by $T_A(a_i) = [f_i]$, $i \le j$, is $(1 + \frac{\varepsilon}{6})$-isomorphism and $\|[T_A x] - x\|_{X_\mu} < \frac{\varepsilon}{6}\|x\|_{X_\mu}$, $x \in A$. This implies that μ restricted to $\operatorname{span}\{f_1,\dots,f_j\}$ is a norm and since $T_A(A)$ is isometric to $(\operatorname{span}\{f_1,\dots,f_n\},\mu)$ we consider T_A as a $(1+\frac{\varepsilon}{6})$-isomorphism between A and $(\operatorname{span}\{f_1,\dots,f_n\},\mu)$. Now, pick $N'\subseteq A$ a finite $\frac{\varepsilon}{6}$-dense set for S_A consisting of rational linear combinations of points from $\mathfrak{b}_A$ with $\mathfrak{b}_A\subseteq N'$ such that $|\|x\|_{X_\mu}-1|<\frac{\varepsilon}{6}$ for every $x\in N'$. Then $\|[T_Ax]-x\|_{X_\mu}<\frac{\varepsilon}{6}\|x\|_{X_\mu}<\frac{\varepsilon}{6}(1+\frac{\varepsilon}{6})<\frac{\varepsilon}{5}$ for every $x\in N'$. Put $N:=T_A(N')\subseteq V$. Then we easily obtain $|\mu(x)-1|<\frac{\varepsilon}{2}$ for every $x\in N$ and, by Lemma 2.4(ii), N is $\frac{\varepsilon}{2}$-dense in $S_{T_A(A)}$. Similarly as above, we may pick $\{g_1,\dots,g_k\}\subseteq V$ such that the linear operator $T_B: B\to X_{\mu_B}$ given by $T_B(b_i)=g_i$, $i\le k$, is $(1+\frac{\varepsilon}{6})$-isomorphism and we find $M'\subseteq B$ a finite $\frac{\varepsilon}{6}$-dense set for S_B consisting of rational linear combinations of points from $\mathfrak{b}_B$ with $M'\supseteq\{G(x):\ x\in N'\}\cup\mathfrak{b}_B$ and $|\mu_B(x)-1|<\frac{\varepsilon}{6}$ for $x\in M'$. Put $M:=T_B(M')$, then similarly as above $|\mu_B(x)-1|<\frac{\varepsilon}{2}$ for every $x\in M$ and M is $\frac{\varepsilon}{2}$-dense in $S_{T_B(B)}$. Finally, we obviously have $\big(T_BG(T_A)^{-1}\big)(N)=T_B(G(N'))\subseteq T_B(M')=M$. □

By Lemma 2.3, there is $\nu\in\mathcal{B}$ having rational values on M with $d_M(\nu,\mu_B\circ(T_B)^{-1})<\frac{\varepsilon}{2}$. Put $P'=\nu|_M$, consider the one-to-one map $g: N\to M$ given by $g:=T_BG(T_A)^{-1}|_N$ and put $P=P'\circ g$. Let $n\in\mathbb{N}$ be the integer part of $\frac{2}{3\varepsilon}$ and $n'\in\mathbb{N}$ be the integer part of $\frac{1}{9\max\{\varepsilon,\phi_1(5\varepsilon)\}}$. Easy computations show that $\frac{3}{2}\varepsilon\le\frac{1}{n}<2\varepsilon$ and $9\max\{\varepsilon,\phi_1(5\varepsilon)\}\le\frac{1}{n'}<18\max\{\varepsilon,\phi_1(5\varepsilon)\}$ (in the last inequality we are using that $\max\{\varepsilon,\phi_1(5\varepsilon)\}<\frac{1}{20}$).

Note that for every $x\in M$ we have

$$\max\{|\nu(T_Bx)-1|,|\nu(x)-1|\}\le\tfrac{\varepsilon}{2}+\max\{|\mu_B(x)-1|,|\mu_B((T_B)^{-1}x)-1|\}<\varepsilon. \tag{9}$$

Claim 2. We have $(n,n',P,P',g)\in T$ and $d_N(P,\mu)<\frac{1}{n}$.

Proof of Claim 2. In order to see that $d_N(P,\mu)<\frac{1}{n}$, pick $x\in N$. Then

$$|P(x)-\mu(x)|\le\tfrac{\varepsilon}{2}+|\mu_B(G(T_A)^{-1}(x))-\mu(x)|=\tfrac{\varepsilon}{2}+|\|(T_A)^{-1}(x)\|_{X_\mu}-\|[x]\|_{X_\mu}|<\tfrac{3}{2}\varepsilon.$$

In order to see that $(n,n',P,P',g)\in T$, let us verify the condition (f). Let $\mu'\in\mathcal{P}$, $\nu'\in\mathcal{B}$ be such that $P'\subseteq\nu'$, $d_N(P,\mu')<\frac{1}{n}<2\varepsilon$ and μ' restricted to $\operatorname{span}N\subseteq c_{00}$ is a norm. Note that $|\mu'(x)-1|<5\varepsilon$ for every $x\in N$ and so, since N is ε-dense for the sphere of $T_A(A)=(\operatorname{span}N,\mu)$, the mapping $id:(\operatorname{span}N,\mu)\to(\operatorname{span}N,\mu')$ is $(1+\phi_1(5\varepsilon))$-isomorphism. Further, $|\nu'(x)-1|=|\nu(x)-1|<\varepsilon$ for every $x\in M$ and so the mapping $id:(\operatorname{span}M,\mu_B)\to(\operatorname{span}M,\nu')$ is $(1+\phi_1(5\varepsilon))$-isomorphism as well. Finally, since $T_BG(T_A)^{-1}$ is $(1+\varepsilon)^2$-isomorphism between $(\operatorname{span}N,\mu)$ and $(\operatorname{span}g(N),\mu_B)$ and

$$(1+\phi_1(5\varepsilon))^2(1+\varepsilon)^2\le(1+3\phi_1(5\varepsilon))(1+3\varepsilon)\le 1+9\max\{\varepsilon,\phi_1(5\varepsilon)\}\le 1+\tfrac{1}{n'},$$

we have that $T_g:=id\circ T_B\circ G\circ(T_A)^{-1}\circ id:(\operatorname{span}N,\mu')\to(\operatorname{span}M,\nu')$ is $(1+\frac{1}{n'})$-isomorphism. □

Since $\mu\in G(n,n',P,P',g)$, there is a $\mathbb{Q}$-linear mapping $\Phi: V\cap(\operatorname{span}M,\nu)\to V$ such that $\mu(\Phi(gx)-x)<\frac{2}{n'}\mu(x)$ for every $x\in N$ and $|\nu(x)-\mu(\Phi(x))|<\frac{1}{n'}\nu(x)$ for every $x\in M$. It is easy to see that Φ extends to a bounded linear operator $\Phi':(\operatorname{span}M,\nu)\to X_\mu$. Finally, consider $\Psi:=\Phi'\circ T_B: B\to X_\mu$.

For every $x\in M$ we have

$$|\mu(\Phi(x))-1|\le|\mu(\Phi(x))-\nu(x)|+|\nu(x)-1|\overset{(9)}{\le}\tfrac{1}{n'}\nu(x)+\varepsilon\overset{(9)}{\le}\tfrac{1}{n'}(1+\varepsilon)+\varepsilon<\delta;$$

thus, $|\|\Psi(x)\|_{X_\mu}-1|<\delta$ for every $x\in(T_B)^{-1}(M)$ and so Ψ is $(1+\eta)$-isomorphism.

Further, we have

$$\begin{aligned}\|\Psi G(a_i)-a_i\|_{X_\mu}&\le\|\Phi(g(T_Aa_i))-[T_Aa_i]\|_{X_\mu}+\|[T_Aa_i]-a_i\|_{X_\mu}\\&<\tfrac{2}{n'}(1+\varepsilon)+\varepsilon\le\tfrac{4}{n'}+\varepsilon<\delta;\end{aligned}$$

hence, by Lemma 2.4(i), we have $\|\Phi' T_B G - I_A\| \leq \phi_2^{\mathfrak{b}_A}(\frac{2}{n'}(1+\varepsilon)+\varepsilon) < \eta$. □

Theorem 2.7. *Let $\mathbb{G}$ be the Gurariĭ space. The set $\langle\mathbb{G}\rangle_{\equiv}^{\mathcal{P}_\infty}$ is G_δ in $\mathcal{P}_\infty$.*

Proof. By Proposition 2.6, we have for the countable set T defined before Proposition 2.6 that

$$\langle\mathbb{G}\rangle_{\equiv}^{\mathcal{P}_\infty} = \bigcap_{(n,n',P,P',g)\in T} G(n,n',P,P',g),$$

where $G(n,n',P,P',g)$ is a union of a closed and an open set in $\mathcal{P}_\infty$ (here we use the observation that the set $\{\mu \in \mathcal{P}_\infty \colon \mu$ restricted to $\operatorname{span}(\operatorname{dom} P) \subseteq c_{00}$ is a norm$\}$ is open due to Lemma 1.4); thus it is a countable intersection of G_δ sets. □

Proof of Theorem 2.1. Let us recall that $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ in $\mathcal{P}$, see Corollary 1.5. Thus, since we have $\langle\mathbb{G}\rangle_{\equiv}^{\mathcal{B}} = \langle\mathbb{G}\rangle_{\equiv}^{\mathcal{P}_\infty} \cap \mathcal{B}$, it follows from Proposition 2.6 that $\langle\mathbb{G}\rangle_{\equiv}^{\mathcal{I}}$ is G_δ in any $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.

By Proposition 1.9 we also have that $\langle\mathbb{G}\rangle_{\equiv}^{\mathcal{I}}$ is dense in $\mathcal{I}$ for every $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$. □

2.2. **Generic objects in $SB(X)$.** In this subsection, we address Problem 5.5 from [23] which suggests to investigate generic properties of admissible topologies. We have both positive and negative results. The positive result is Theorem 2.10 which shows that the Gurariĭ space is dense G_δ in the Wijsman topology. The negative results are Propositions 2.11 and 2.13, and Theorem 2.12 .

Definition 2.8. Given a closed set H in X we denote by $E^-(H)$ the set $SB(X) \setminus E^+(X \setminus H)$, that is, $E^-(H) = \{F \in SB(X) :\ F \subseteq H\}$. Obviously, this is a closed set in any admissible topology on $SB(X)$.

Definition 2.9. Let X be isometrically universal separable Banach space. By τ_W we denote *the Wijsman topology* on $SB(X)$, that is, the minimal topology such that the mappings $SB(X) \ni F \mapsto \operatorname{dist}_X(x,F)$ are continuous for every $x \in X$. Note that τ_W is admissible, see [23, Section 2].

Theorem 2.10. *$\langle\mathbb{G}\rangle_{\equiv}$ is dense G_δ in $(SB(\mathbb{G}), \tau_W)$.*

Proof. The class is G_δ since it is G_δ in $\mathcal{P}$ by Theorem 2.1 and there is a continuous reduction from $(SB(\mathbb{G}), \tau_W)$ to $\mathcal{P}$ by Theorem 1.17. So we must show that it is dense.

Choose a basic open set N in τ_W which is given by some closed subspace $X \subseteq \mathbb{G}$, finitely many points $x_1, \ldots, x_n \in \mathbb{G}$ and $\varepsilon > 0$ so that

$$N = \{Z \in SB(\mathbb{G}) \colon \forall i \leq n\ (|\operatorname{dist}_{\mathbb{G}}(x_i, X) - \operatorname{dist}_{\mathbb{G}}(x_i, Z)| < \varepsilon)\}.$$

Let us find a space G isometric to $\mathbb{G}$ such that $G \in N$. Let Y be $\operatorname{span}\{X \cup \{x_i \colon i \leq n\}\}$. Since X embeds into both Y and $\mathbb{G}$ we can consider the push-out of that diagram, i.e. the amalgamated sum of Y and $\mathbb{G}$ along the common subspace X. Recall this is nothing but the quotient $(\mathbb{G} \oplus_1 Y)/Z$, where $Z = \{(z, -z) \colon z \in X\}$. Denote this space by G' and notice that $\mathbb{G}$ is naturally embedded into G'. It is straightforward to verify that for each $i \leq n$, $\operatorname{dist}_{G'}(x_i, \mathbb{G}) = \operatorname{dist}_{\mathbb{G}}(x_i, X)$. Since $\mathbb{G}$ is universal, there is a linear isometric embedding $\iota : G' \hookrightarrow \mathbb{G}$. As there is a linear isometry $\phi : \iota[\operatorname{span}\{x_i \colon i \leq n\}] \to \operatorname{span}\{x_i \colon i \leq n\}$, by [31, Theorem 1.1] there is a bijective linear isometry $\Phi : \mathbb{G} \to \mathbb{G}$ such that $\|\Phi \circ \iota(x_i) - x_i\| < \varepsilon$, for each $i \leq n$. By triangle inequality, it follows that $G := \Phi \circ \iota[\mathbb{G} \subseteq G']$ satisfies for each $i \leq n$, $|\operatorname{dist}_{\mathbb{G}}(x_i, G) - \operatorname{dist}_{\mathbb{G}}(x_i, X)| < \varepsilon$, so it is the desired space isometric to $\mathbb{G}$ lying in the open set N. □

The rest of the section is devoted to negative results. They show that the definition of admissible topology allows a lot of flexibility by which one can alter which properties should be meager or not.

Proposition 2.11. *Let X be an isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle \mathbb{G} \rangle_{\simeq}$ is nowhere dense in $(SB_\infty(X), \tau')$.*

Proof. By the definition of an admissible topology, we may pick $(U_n)_{n\in\mathbb{N}}$, a basis of the topology τ, such that for every $n \in \mathbb{N}$ there are nonempty open sets V_k^n, $k = 1, \dots, N_n$ and W_n in X such that the set U'_n defined by

$$U'_n = \bigcap_{k=1}^{N_n} E^+(V_k^n) \setminus E^+(W_n)$$

is a nonempty subset of U_n.

We *claim* that for every $n \in \mathbb{N}$ there is $F_n \in U'_n$ such that $\mathbb{G} \not\hookrightarrow F_n$. Indeed, pick an arbitrary $Z \in U'_n$. We may without loss of generality assume there is $H_0 \subseteq Z$ with $H_0 \simeq \mathbb{G}$ and since $\mathbb{G}$ is isometrically universal, there is $H_1 \subseteq H_0$ with $H_1 \simeq \ell_2$. Now, pick points $v_k \in Z \cap V_k^n$, $k = 1, \dots, N_n$. Then we put $F_n := \overline{\operatorname{span}}\{v_1, \dots, v_{N_n}, u : \ u \in H_1\}$. Since F_n is a subset of Z, we have $F_n \notin E^+(W_n)$ and since it contains the points $v_1, \dots, v_{N_n}$, we have $F_n \in U'_n$. Moreover, it is a space isomorphic to ℓ_2 and so $\mathbb{G} \not\hookrightarrow F_n$.

Thus, for every $n \in \mathbb{N}$ there is a closed subspace F_n of X such that $U_n \cap E^-(F_n)$ is a nonempty set disjoint from $\langle \mathbb{G} \rangle_{\simeq}$.

It is a classical fact, see e.g., [29, Lemma 13.2 and Lemma 13.3], that the topology τ' generated by $\tau \cup \{E^-(F_n) : \ n \in \mathbb{N}\}$ is Polish. It is easy to check it is admissible. Moreover, for every $n \in \mathbb{N}$ we have that $U_n \cap E^-(F_n)$ is a nonempty τ'-open set in U_n disjoint from $\langle \mathbb{G} \rangle_{\simeq}$. It follows that nonempty sets of the form $U_n \cap \bigcap_{m\in I} E^-(F_m)$, for finite $I \subseteq \mathbb{N}$, give us a π-basis of τ'. Since obviously each element of the form $U_n \cap \bigcap_{m\in I} E^-(F_m)$ is disjoint from $\langle \mathbb{G} \rangle_{\simeq}$, the set $\langle \mathbb{G} \rangle_{\simeq}$ is τ'-nowhere dense. $\square$

Actually, one may observe that the same proof gives the following more general result, where the pair $(\mathbb{G}, \ell_2)$ is replaced by a more general pair of Banach spaces.

Theorem 2.12. *Let X be an isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Let Y and Z be infinite-dimensional Banach spaces such that $Y \hookrightarrow Z$ and $Z \not\hookrightarrow Y \oplus F$ for every finite-dimensional space F.*

Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle Z \rangle_{\simeq}$ is nowhere dense in $(SB_\infty(X), \tau')$.

It is even possible to find an admissible topology τ such that $\langle \ell_2 \rangle_{\equiv}$ is not a meager set in $(SB_\infty(X), \tau)$ which is an immediate consequence of the following more general observation (the property (P) bellow would be "X is isometric to ℓ_2").

Proposition 2.13. *Let X be isometrically universal separable Banach space and τ be an admissible topology on $SB_\infty(X)$. Let (P) be a non-void property (i,e. there are spaces with such a property) of infinite-dimensional Banach spaces closed under taking subspaces. Then there is an admissible topology $\tau' \supseteq \tau$ such that the set $\{Y \in SB_\infty(X) \colon Y \text{ has } (P)\}$ has non-empty interior in $(SB_\infty(X), \tau')$.*

Proof. Pick $F \in SB_\infty(X)$ with (P). Using again the classical fact, see e.g. [29, Lemma 13.2], that the topology τ' generated by $\tau \cup \{E^-(F)\}$ is Polish, it is easy to check it is admissible. Then the τ'-open set $E^-(F)$ is a subset of $\{Y \in SB_\infty(X) \colon Y \text{ has } (P)\}$. $\square$

3. Spaces with closed isometry classes

From this section on, we start our investigation of descriptive complexity of isometry classes. Let us first observe that no isometry class can be open as every isometry class actually has an empty interior. Indeed, it follows from Proposition 1.9 that the isometry class of every isometrically universal separable Banach space is dense. Since there are obviously many pairwise non-isometric universal Banach spaces we get that every open set (in all $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$) contains norms, resp. pseudonorms, defining different Banach spaces. The same argument can be also used to show that every isomorphism class has an empty interior.

Lemma 3.1. *$\langle \ell_2 \rangle_\equiv$ is closed in $\mathcal{B}$ and $\mathcal{P}_\infty$.*

Proof. Hilbert spaces are characterized among Banach spaces as those Banach spaces whose norm satisfies the parallelogram law, i.e. $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for any pair of elements x, y. It is clear that a norm satisfies the parallelogram law if and only if it satisfies it on a dense set of vectors, therefore every norm, resp. pseudonorm, from $\mathcal{B}$, resp. $\mathcal{P}_\infty$, satisfying the parallelogram law on V defines a Hilbert space. Since norms, resp. pseudonorms, from $\mathcal{B}$, resp. $\mathcal{P}_\infty$, define only infinite-dimensional spaces, they define spaces isometric to $\ell_2(\mathbb{N})$. Since the parallelogram law is clearly a closed condition, we are done. $\square$

Remark 3.2. We note that here we need to work with the spaces $\mathcal{B}$ or $\mathcal{P}_\infty$, since in $\mathcal{P}$ the only space with closed isometry class is the trivial space. To show it, first notice that the trivial space is indeed closed. Next we show that any open neighborhood of a pseudonorm defining trivial space contains a pseudonorm defining arbitrary Banach space, which will finish our claim. Let such an open neighborhood be fixed. We may assume that it is of the form $\{\mu \in \mathcal{P} \colon \mu(v_i) < \varepsilon, i \leq n\}$, where $v_1, \ldots, v_n \in V$ and $\varepsilon > 0$. Let m be such that all v_i, $i \leq n$, are in $\operatorname{span}_{\mathbb{Q}}\{e_j \colon j \leq m\}$. Let X be an arbitrary separable Banach space and let $(f_i)_{i\in\mathbb{N}} \subseteq X$ be a sequence whose span is dense in X. We define $\mu \in \mathcal{P}$ by $\mu(e_j) = 0$, for $j \leq m$, and $\mu(\sum_{i\in I} \alpha_i e_{m+i}) = \|\sum_{i\in I} \alpha_i f_i\|_X$, where $I \subseteq \mathbb{N}$ is finite and $(\alpha_i)_{i\in I} \subseteq \mathbb{Q}$. This defines μ separately on $\operatorname{span}_{\mathbb{Q}}\{e_i \colon i \leq m\}$ and $\operatorname{span}_{\mathbb{Q}}\{e_i \colon i > m\}$, however the extension to the whole V is unique. It is clear that μ is in the fixed open neighborhood and that $X_\mu \equiv X$.

One may be interested whether there are other Banach spaces whose isometry class is closed. The answer is negative. First, let us state another corollary of Proposition 1.9.

Lemma 3.3. *Let X be a separable infinite-dimensional Banach space. Then $\langle \ell_2 \rangle_\equiv^{\mathcal{B}} \subseteq \overline{\langle X \rangle_\equiv^{\mathcal{B}}} \cap \mathcal{B}$. The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$.*

Proof. By the Dvoretzky's theorem, ℓ_2 is finitely representable in every separable infinite-dimensional Banach space (see e.g. [1, Theorem 13.3.7]). So we are done by applying Proposition 1.9. $\square$

The following theorem is now an immediate consequence of Lemmas 3.1 and 3.3.

Theorem 3.4. *ℓ_2 is the only separable infinite-dimensional Banach space whose isometry class is closed in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ by $\mathcal{P}_\infty$.*

On the other hand, no isomorphism class can be closed. We show something stronger. Let us first start with the following simple lemma from descriptive set theory. Although it should be well known, we could not find a proper reference, so we provide a sketch of the proof.

Lemma 3.5. *Suppose that X is a Polish space and $B \subseteq X$ is a Borel set which is not G_δ. Then B is F_σ-hard. The same with the roles of G_δ and F_σ interchanged.*

Proof. By Hurewicz theorem (see e.g. [29, Theorem 21.18]), there is a set $C \subseteq X$ homeomorphic to the Cantor space such that $C \cap B$ is countable dense in C. Then $C \cap B$ is F_σ but not G_δ in the zero-dimensional Polish space C, and so it is F_σ-complete in C by Wadge's theorem (see e.g. [29, Theorem 22.10]). So for any zero-dimensional Polish space Y and any F_σ-subset A of Y, there is a Wadge reduction of $A \subseteq Y$ to $C \cap B \subseteq C$. But any such reduction is also a reduction of $A \subseteq Y$ to $B \subseteq X$, and so B is F_σ-hard.

The argument with the roles of F_σ and G_δ interchanged is similar. □

Proposition 3.6. *No isomorphism class can be closed in $\mathcal{P}_\infty$, $\mathcal{B}$ and $\mathcal{P}$, and with the possible exception of spaces isomorphic to $\mathbb{G}$ for which we do not know the answer, no isomorphism class can even be G_δ.*

Moreover, $\langle \ell_2 \rangle_\simeq$ is F_σ-complete in both $\mathcal{P}_\infty$ and $\mathcal{B}$.

Proof. Let X be a separable infinite-dimensional Banach space. We show that $\langle X \rangle_\simeq$ is dense (we show the argument only for $\mathcal{P}_\infty$, the other cases are analogous). Let F be a finite-dimensional Banach space. It is well known that every finite-dimensional space is complemented in any infinite-dimensional Banach space, so we have $X \simeq F \oplus_1 Y$ for some Banach space Y. Since F was arbitrary, it follows that every separable Banach space is finitely representable in $\langle X \rangle_\simeq$, so by Proposition 1.9, $\overline{\langle X \rangle_\simeq} = \mathcal{P}_\infty$, hence $\langle X \rangle_\simeq$ is dense.

It follows that $\langle X \rangle_\simeq$ cannot be closed for any X because it is dense and there are obviously two non-isomorphic spaces. Moreover, if X is not isomorphic to the Gurariĭ space then $\langle X \rangle_\simeq$ cannot be G_δ since by Theorem 2.1, the isometry class of the Gurariĭ space is dense G_δ, so it would have non-empty intersection with $\langle X \rangle_\simeq$ otherwise.

Finally suppose that $X = \ell_2$. The isomorphism class of ℓ_2 is proved to be F_σ in an admissible topology on SB_∞ in [23, Theorem 4.3]. The same proof, which we briefly sketch, works also for $\mathcal{P}_\infty$ and $\mathcal{B}$. By Kwapień's theorem (see e.g. [1, Theorem 7.4.1]) a separable infinite-dimensional Banach space is isomorphic to ℓ_2 if and only if it is of type 2 and of cotype 2. It is clear from the definition of type and cotype (see e.g. [1, Definition 6.2.10]) that these properties are F_σ. So to show that $\langle \ell_2 \rangle_\simeq$ is F_σ-complete, by Lemma 3.5 it suffices to show that $\langle \ell_2 \rangle_\simeq$ is not G_δ, which we have already proved. □

Later, in Theorem 6.1, we prove that ℓ_2 is actually the unique, up to isomorphism, separable infinite-dimensional Banach space whose isomorphism class is F_σ.

Remark 3.7. An alternative proof showing that the isomorphism class $\langle \ell_2 \rangle_\simeq$ is F_σ follows from [34, Theorem 2'] (see also Remark 4 therein) which provides a formula defining spaces isomorphic to ℓ_2 and which obviously defines an F_σ set (in $\mathcal{P}_\infty$ and $\mathcal{B}$).

3.1. QSL_p**-spaces.** We finish the section by considering some natural closed subspaces of $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$.

In [34], Kwapień denotes by S_p, resp. SQ_p, for $1 \leq p < \infty$, the class of all Banach spaces isometric to a subspace of $L_p(\mu)$, resp. to a subspace of some quotient of $L_p(\mu)$, for some measure μ.

Let us address the class S_p first. We have the following simple lemma.

Lemma 3.8. *Let $1 \leq p < \infty$. Put*

$$M := \{\mu \in \mathcal{B} : \ X_\mu \textit{ is isometric to a subspace of } L_p[0,1]\}.$$

Then M is a closed set in $\mathcal{B}$ and we have

$$M = \overline{\langle \ell_p \rangle^{\mathcal{B}}_{\equiv}} \cap \mathcal{B} = \overline{\{\mu \in \mathcal{B} : \ X_\mu \textit{ is } \mathcal{L}_{p,1+} \textit{ space}\}} \cap \mathcal{B}.$$

The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$.

Proof. We recall the fact that a separable infinite-dimensional Banach space is isometric to a subspace of $L_p[0,1]$ if and only if it is finitely representable in ℓ_p, see e.g. [1, Theorem 12.1.9]. The rest follows from Proposition 1.9. We refer the reader to Section 4 for a definition of the class $\mathcal{L}_{p,1+}$. □

In the rest, we focus on the class SQ_p. Notice that for $p = 1$, this class coincides with the class of all Banach spaces, and for $p = 2$, this class consists of Hilbert spaces.

These Banach spaces are also called QSL_p-spaces in literature and since it seems this is the more recent terminology, this is what we will use further. It seems to be well known, see e.g. [46], that this class of spaces is characterized by Proposition 3.9 below. This result was probably essentially proved by Kwapien [34] (however, in his paper he considered the isomorphic variant only), for a more detailed explanation of the proof (and even for a generalization) one may consult e.g. the proof in [37, Theorem 3.2] which uses the ideas from [45] and [26]. Let us note that, by Proposition 3.9 and [27, Proposition 0], the class of QSL_p spaces coincides with the class of p-spaces considered already in 1971 by Herz [27].

Proposition 3.9. *A Banach space X is a QSL_p-space, if and only if for every real valued (m,n)-matrix M satisfying*

$$\sum_{i=1}^{n}\left|\sum_{j=1}^{m} M(i,j) r_j\right|^p \le \sum_{k=1}^{m} |r_k|^p,$$

for all m-tuples $r_1,\dots,r_m \in \mathbb{R}$, we have

$$\sum_{i=1}^{n}\left\|\sum_{j=1}^{m} M(i,j) x_j\right\|_X^p \le \sum_{k=1}^{m} \|x_k\|_X^p,$$

for all m-tuples $x_1,\dots,x_m \in X$.

Since it is clear that it suffices to verify the condition from Proposition 3.9 only on dense tuples of vectors, and that this condition is closed, we immediately obtain the following.

Proposition 3.10. *For every $1 < p < \infty$, the set*

$$\{\mu \in \mathcal{P}_\infty \colon X_\mu \text{ is } QSL_p\}$$

is closed in $\mathcal{P}_\infty$.

The same is true if $\mathcal{P}_\infty$ is replaced by $\mathcal{B}$.

Denote now the set $\{\mu \in \mathcal{P}_\infty \colon X_\mu \text{ is } QSL_p\}$ by QSL_p. By Lemma 3.8, for $1 \le p < \infty$, the set $M_p := \{\mu \in \mathcal{P}_\infty \colon X_\mu \text{ is isometric to a subspace of } L_p[0,1]\}$ is closed. Clearly, $M_p \subseteq QSL_p$ (and for $p = 2$ there is an equality).

If $p \neq 2$ then $M_p \neq QSL_p$ because there exists an infinite-dimensional separable Banach space which is isomorphic to a quotient of $L_p[0,1]$ but not to its subspace. Indeed, if $p = 1$ this is easy since every separable Banach space is isomorphic to a quotient of ℓ_1, see e.g. [1, Theorem 2.3.1]. If $2 < q < p < \infty$ then ℓ_q is isometric to a quotient of $L_p[0,1]$ (because its dual $\ell_{q'}$ embeds isometrically into $L_{p'}[0,1]$) but is not isomorphic to a subspace of $L_p[0,1]$, see e.g. [1, Theorem 6.4.18]. Finally, if $1 < p < 2$ then by [15, Corollary 2] there exists a subspace X of $\ell_{p'} \subseteq L_{p'}$ which is not isomorphic to a quotient of $L_{p'}$[1] and so X^* is isometric to a quotient of L_p which is not isomorphic to a subspace of L_p. We would like to thank Bill Johnson for providing us those examples.

Moreover, we have the following.

Proposition 3.11. *For $p \in [1,2) \cup (2,\infty)$, the set M_p has an empty interior in QSL_p.*

Proof. Fix $p \in [1,2) \cup (2,\infty)$. Pick $\mu \in QSL_p$ such that X_μ does not isometrically embed as a subspace into $L_p[0,1]$ (such a space exists, see the examples above). Let U be now a basic open neighborhood of some $\nu \in QSL_p$. Since the class of QSL_p-spaces is clearly closed under taking ℓ_p-sums (see e.g. [46]), $X_\nu \oplus_p X_\mu$ is still a QSL_p-space. It is easy to define $\nu' \in U$ so that $X_{\nu'}$ is isometric to $X_\nu \oplus_p X_\mu$. Now since X_μ does not isometrically embed as a subspace into $L_p[0,1]$, neither $X_{\nu'}$ does. By [1, Theorem 12.1.9], $X_{\nu'}$ is not finitely representable in ℓ_p, so also not in $L_p[0,1]$ (by [1, Proposition 12.1.8]). It follows from Proposition 1.9 that there exists a basic open neighborhood U' of ν' avoiding M_p. Now $U \cap U'$ is a non-empty open subsets of U avoiding M_p and we are done. □

Corollary 3.12. *For $p \in [1,2) \cup (2,\infty)$, $L_p[0,1]$ is not a generic QSL_p-space.*

4. Spaces with G_δ isometry classes

In this section, we investigate Banach spaces whose isometry classes are G_δ, or even G_δ-complete. Besides ℓ_2, whose isometry class is actually closed, we have already proved in Theorem 2.1 that the isometry class of the Gurariĭ space is G_δ in $\mathcal{P}_\infty$ and $\mathcal{B}$. We start the section with some basic corollaries of that result; in particular, that the isometry class of $\mathbb{G}$ is even G_δ-complete. The main results of the section however concern the Lebesgue spaces $L_p([0,1],\lambda)$, for $1 \le p < \infty$.

Since for any separable infinite-dimensional Banach space X we obviously have $\langle X\rangle_\equiv^{\mathcal{B}} = \langle X\rangle_\equiv^{\mathcal{P}_\infty} \cap \mathcal{B}$, it is sufficient to formulate our positive result in the coding of $\mathcal{P}_\infty$ and negative results in the coding of $\mathcal{B}$.

Lemma 4.1. *Let X, Y be separable infinite-dimensional Banach spaces such that X is finitely representable in Y and Y is finitely representable in X. If $\langle X\rangle_\equiv$ is G_δ in $\mathcal{B}$ and $X \not\equiv Y$, then*

(i) $\langle Y\rangle_\equiv$ is not G_δ in $\mathcal{B}$.

(ii) $\langle X\rangle_\equiv$ is G_δ-complete in $\mathcal{B}$.

Proof. Recall that by Proposition 1.9 we have that both $\langle X\rangle_\equiv$ and $\langle Y\rangle_\equiv$ are dense in

$$N := \{\nu \in \mathcal{B} :\ X_\nu \text{ is finitely representable in } X\}.$$

(i): If both $\langle X\rangle_\equiv$ and $\langle Y\rangle_\equiv$ are G_δ, by the Baire theorem we have that $\langle X\rangle_\equiv \cap \langle Y\rangle_\equiv$ is comeager in N. Thus, the intersection cannot be an empty set and we obtain $X \equiv Y$.

(ii): Since $X \not\equiv Y$, we have that $\langle X\rangle_\equiv$ has empty interior in N. But it is also

[1] More precisely, by [15, Theorem 1] (see also e.g. [14, Corollary 3.2]) for every $n \in \mathbb{N}$ there exists a subspace E_n of ℓ_∞^{2n} such that $gl(E_n) \ge K\sqrt{n}$ where $K > 0$ is a constant independent of n and $gl(E_n)$ is a quantity related to the notion of a "GL-space" (or space with the "Gordon-Lewis property"). This implies that if we denote by $E_n^{p'}$ the space E_n endowed with the $\ell_{p'}$-norm, we obtain $gl(E_n^{p'}) \ge K\sqrt{n}d_{BM}(\ell_\infty^{2n}, \ell_{p'}^{2n})^{-1} = K2^{-1/p'}n^{1/2-1/p'} \to \infty$; hence, $X := (\bigoplus E_n^{p'})_{p'}$, the $\ell_{p'}$-sum of spaces $E_n^{p'}$, is isometric to a subspace of $\ell_{p'}$ but it is not a GL-space. If X was isomorphic to a quotient of $L_{p'}$ then X^* would be isomorphic to a subspace of L_p which would imply that X^* and X are GL-spaces (see e.g. [11, Proposition 17.9 and Proposition 17.10]), a contradiction.

comeager in N, and so it cannot be F_σ. Therefore it is G_δ-complete by Lemma 3.5. □

Corollary 4.2. *$\mathbb{G}$ is the only isometrically universal separable Banach space whose isometry class is G_δ in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ by $\mathcal{P}_\infty$.*

Moreover, $\langle\mathbb{G}\rangle_\equiv$ is G_δ-complete in both $\mathcal{P}_\infty$ and $\mathcal{B}$.

Proof. By Theorem 2.1, the isometry class of $\mathbb{G}$ is G_δ. Let X be an isometrically universal separable Banach space. By Lemma 4.1, if $X \not\equiv \mathbb{G}$ then $\langle X\rangle_\equiv^{\mathcal{B}}$ is not G_δ in $\mathcal{B}$ (and so not in $\mathcal{P}_\infty$ either).

For the "moreover" part we use Lemma 4.1 and any Banach space X not isometric to $\mathbb{G}$ that is finitely representable in $\mathbb{G}$ and vice versa (e.g. any other universal separable Banach space or c_0). □

The same proof gives us actually the following strengthening. Let us recall that by Maurey–Pisier theorem, see [40] or [1, Theorem 12.3.14], a Banach space X has no nontrivial cotype if and only if ℓ_∞ is finitely-representable in X (and yet equivalently, c_0 is finitely-representable in X).

Theorem 4.3. *$\mathbb{G}$ is the only separable Banach space with no nontrivial cotype whose isometry class is G_δ in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ by $\mathcal{P}_\infty$.*

Proof. Any separable Banach space is finitely representable in c_0, so by Lemma 4.1 there is at most one Banach space X such that c_0 is finitely representable in X and $\langle X\rangle_\equiv$ is G_δ. By Theorem 2.1, $\langle\mathbb{G}\rangle_\equiv$ is G_δ. □

4.1. **L_p-spaces.** Let us recall that a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space (with $1 \le p \le \infty$ and $\lambda \ge 1$) if every finite-dimensional subspace of X is contained in another finite-dimensional subspace of X whose Banach-Mazur distance d_{BM} to the corresponding ℓ_p^n is at most λ. A space X is said to be an $\mathcal{L}_p$-space, resp. $\mathcal{L}_{p,\lambda+}$-space, if it is an $\mathcal{L}_{p,\lambda'}$-space for some for some $\lambda' \ge 1$, resp. for every $\lambda' > \lambda$.

The main result of this subsection is the following.

Theorem 4.4. *For every $1 \le p < \infty$, $p \ne 2$, the isometry class of $L_p[0,1]$ is G_δ-complete in $\mathcal{B}$ and $\mathcal{P}_\infty$.*

Moreover, $L_p[0,1]$ is the only separable $\mathcal{L}_{p,1+}$ space whose isometry class is G_δ in $\mathcal{B}$, and the same holds if we replace $\mathcal{B}$ by $\mathcal{P}_\infty$.

The next theorem is a crucial step in proving Theorem 4.4. However, it is also of independent interest and its corollary improves the related result from [23].

Remark 4.5. It is easy to see (e.g. using [35, Section 17, Theorem 6]) that for every $p \in [1,\infty]$ and $\lambda \ge 1$ we have that a separable infinite-dimensional Banach space Y is a $\mathcal{L}_{p,\lambda+}$ space if and only if for every $\varepsilon > 0$ there is an increasing sequence $\{F_k\}_{k=1}^\infty$ of finite-dimensional subspaces whose union is dense in Y such that $d_{BM}(\ell_p^{\dim F_k}, F_k) \le \lambda + \varepsilon$ for every $k \in \mathbb{N}$.

Let us note that the following result admits a generalization (see Proposition 6.3). This is the reason why we use in the proof the characterization of $\mathcal{L}_{p,\lambda+}$ spaces mentioned in Remark 4.5.

Theorem 4.6. *Let $1 \le p \le \infty$ and $\lambda \ge 1$. The class of separable $\mathcal{L}_{p,\lambda+}$ spaces is G_δ in $\mathcal{P}$. In particular, the class of separable infinite-dimensional $\mathcal{L}_{p,\lambda+}$ spaces is G_δ in $\mathcal{P}_\infty$.*

Proof. For each finite tuple $\vec{v}$ of elements from V we set $S_{\vec{v}}$ to be the set of all finite tuples $\vec{w}$, $\mathbb{Q}$-linearly independent in V, such that each element of $\vec{v}$ (considered as an element of c_{00}) lies in span $\vec{w}$. For $m \in \mathbb{N}$ and a finite tuple $\vec{v}$ of elements from V we set

$$\begin{aligned} P(\vec{v}, m) := \big\{\mu \in \mathcal{P} \colon & \text{ if } \mu \text{ restricted to } \operatorname{span}\{v_1, \ldots, v_{|\vec{v}|}\} \subseteq c_{00} \text{ is a norm, then} \\ & \text{there exist } \vec{w} \in S_{\vec{v}} \text{ and } n \in \mathbb{N} \text{ such that} \\ & \mu \text{ restricted to } \operatorname{span}\{w_1, \ldots, w_{|\vec{w}|}\} \subseteq c_{00} \text{ is a norm and} \\ & (((\operatorname{span}\{w_1, \ldots, w_{|\vec{w}|}\}, \mu), \vec{w}) \overset{\sqrt{\lambda + \frac{1}{m}}}{\sim} \ell_p^{|\vec{w}|})\big\}. \end{aligned}$$

Then, using the observation that $\{\mu \in \mathcal{P} \colon \mu$ restricted to $\operatorname{span}\{v_1, \ldots, v_{|\vec{v}|}\} \subseteq c_{00}$ is a norm$\}$ is open due to Lemma 1.4, $P(\vec{v}, m)$ is a union of a closed and an open set, so it is G_δ.

Denote by LI the set of all finite tuples $\vec{v} = (v_1, \ldots, v_{|\vec{v}|})$ of elements from V which are linearly independent in c_{00}. We now set

$$\mathcal{G} := \bigcap_{m \in \mathbb{N}} \bigcap_{\vec{v} \in LI} P(\vec{v}, m),$$

which is clearly G_δ. We shall prove that it defines the class of separable $\mathcal{L}_{p,\lambda+}$ spaces.

If $\mu \in \mathcal{G}$, it is clear that for every m, we can recursively build an increasing sequence $\{F_k\}_{k=1}^\infty$ of finite-dimensional subspaces whose union is dense in X_μ such that we have $d_{BM}(F_k, \ell_p^{\dim F_k}) \leq \lambda + \frac{1}{m}$ for every $k \in \mathbb{N}$. It follows that X_μ is $\mathcal{L}_{p,\lambda+}$ space.

On the other hand, let $\mu \in \mathcal{P}$ be such that X_μ is $\mathcal{L}_{p,\lambda+}$ space. In what follows for $x \in c_{00}$ we denote by $[x] \in X_\mu$ the equivalence class corresponding to x. Pick some $m \in \mathbb{N}$ and an n-tuple $\vec{v} \in LI$ such that μ restricted to $\operatorname{span}\{v_1, \ldots, v_n\} \subseteq c_{00}$ is a norm, so $\{v_1, \ldots, v_n\}$ is a basis of $\operatorname{span}\{v_1, \ldots, v_n\}$. Pick $\lambda' \in (\lambda, \lambda + \frac{1}{m})$ and $\delta > 1$ with $\delta\lambda' < \lambda + \frac{1}{m}$. Since X_μ is $\mathcal{L}_{p,\lambda+}$ space, there is an increasing sequence $\{F_k\}_{k=1}^\infty$ of finite-dimensional subspaces whose union is dense in X_μ such that $\sup_{k \in \mathbb{N}} d_{BM}(F_k, \ell_p^{\dim F_k}) \leq \lambda'$. By [35, Section 17, Theorem 6], we can find a finite dimensional subspace $\operatorname{span}\{[v_1], \ldots, [v_n]\} \subseteq Y \subseteq X_\mu$ and $k \in \mathbb{N}$ such that $d_{BM}(Y, F_k) \leq \delta$ so $d_{BM}(Y, \ell_p^{\dim Y}) \leq \delta\lambda'$. Select $y_{n+1}, \ldots, y_{\dim Y} \in Y$ such that $\mathfrak{b} = \{[v_1], \ldots, [v_n], y_{n+1}, \ldots, y_{\dim Y}\}$ is a basis of Y. Let $\phi_2^{\mathfrak{b}}$ be the function from Lemma 2.4(i) and let $\eta > 0$ be such that $\delta\lambda'(1 + \phi_2^{\mathfrak{b}}(\eta))^2 < \lambda + \frac{1}{m}$. Further, for every $n + 1 \leq i \leq \dim Y$ pick $v_i \in V$ with $\|[v_i] - y_i\|_{X_\mu} < \eta$. Then $(\mu, [v_1], \ldots, [v_{\dim Y}]) \overset{1+\phi_2^{\mathfrak{b}}(\eta)}{\sim} Y$ so μ restricted to $\operatorname{span}\{v_1, \ldots, v_{\dim Y}\} \subseteq c_{00}$ is a norm and $\operatorname{span}\{[v_1], \ldots, [v_{\dim Y}]\} \subseteq X_\mu$ is isometric to $(\operatorname{span}\{v_1, \ldots, v_{\dim Y}\}, \mu)$. Since $d_{BM}((\operatorname{span}\{v_1, \ldots, v_{\dim Y}\}, \mu), \ell_p^{\dim Y}) < \delta\lambda' \cdot (1 + \phi_2^{\mathfrak{b}}(\eta))^2 < \lambda + \frac{1}{m}$, there exists a surjective isomorphism $T : \ell_p^{\dim Y} \to (\operatorname{span}\{v_1, \ldots, v_{\dim Y}\}, \mu)$ with $\max\{\|T\|, \|T^{-1}\|\} < \sqrt{\lambda + \frac{1}{m}}$. By Lemma 2.4(i), we may without loss of generality assume that $w_i := T(e_i) \in V$ for every $i \leq \dim Y$. Then μ restricted to $\operatorname{span}\{w_1, \ldots, w_{\dim Y}\} \subseteq c_{00}$ is a norm, $\vec{w} \in S_{\vec{v}}$ and $((\mu, \vec{w}) \overset{\sqrt{\lambda + \frac{1}{m}}}{\sim} \ell_p^{\dim Y})$. ∎

Note that for $1 \leq p \leq \infty$ the class of $\mathcal{L}_p$ spaces is obtained as the union $\bigcup_{\lambda \geq 1} \mathcal{L}_{p,\lambda+}$. It is shown in [23, Proposition 4.5] that $\mathcal{L}_p$ is $\mathbf{\Sigma}_4^0$ in an admissible topology. It is immediate from Theorem 4.6 (and using Theorem 1.17) that we have a better estimate.

Corollary 4.7. *For every $1 \leq p \leq \infty$ the class of separable $\mathcal{L}_p$ spaces is $G_{\delta\sigma}$ in $\mathcal{P}$ and any admissible topology.*

Let us recall the following classical result.

Theorem 4.8 (Lindenstrauss, Pełczyński)**.** *For every $1 \leq p < \infty$ and a separable infinite-dimensional Banach space X the following assertions are equivalent.*

- *X is $\mathcal{L}_{p,1+}$ space.*
- *X is isometric to a separable $L_p(\mu)$ space for some σ-additive measure μ.*
- *X is isometric to one of the following spaces*

$$L_p[0,1], \quad L_p[0,1] \oplus_p \ell_p, \quad \ell_p, \quad L_p[0,1] \oplus_p \ell_p^n \ \text{ (for some } n \in \mathbb{N}\text{)}.$$

Proof. By [38, Section 7, Corollaries 4 and 5], a separable Banach space is $\mathcal{L}_{p,1+}$ space if and only if it is isometric to an $L_p(\mu)$ space for some measure μ. Finally, note that every separable infinite-dimensional $L_p(\mu)$ space is isometric to one of the spaces mentioned above, see e.g. [1, p. 137-138]. □

Recall that given a finite sequence $(z_n)_{n\in N}$ in a Banach space Z, the symbol $(z_n) \overset{K}{\sim} \ell_p^N$ means that $K^{-1}\left(\sum_{i\in N}|a_i|^p\right)^{1/p} < \|\sum_{i\in N} a_i z_i\| < K\left(\sum_{i\in N}|a_i|^p\right)^{1/p}$ for every $a \in c_{00}^N$. If (z_n) is isometrically equivalent to the ℓ_p^N basis (that is, $(z_n) \overset{1+\varepsilon}{\sim} \ell_p^N$ for every $\varepsilon > 0$), we write $(z_n) \equiv \ell_p^N$.

Theorem 4.9. *Let $1 \leq p < \infty$, $p \neq 2$, and let X be a separable infinite-dimensional $\mathcal{L}_{p,1+}$ space. Then the following assertions are equivalent.*

(i) X is isometric to $L_p[0,1]$.

(ii) For every $x \in S_X$ the following condition is satisfied

$$\forall N \in \mathbb{N}\ \exists x_1,\dots,x_N \in X: \quad (x_i)_{i=1}^N \equiv \ell_p^N \ \text{ and } N^{1/p}\cdot x = \sum_{i=1}^N x_i.$$

(iii) For every $x \in S_X$ the following condition is satisfied

$$\forall \varepsilon > 0\ \exists x_1, x_2 \in X: \quad (x_1,x_2) \overset{1+\varepsilon}{\sim} \ell_p^2 \ \text{ and } 2^{1/p}\cdot x = x_1 + x_2.$$

(iv) For every $x \in S_X$ the following condition is satisfied

$$\forall \varepsilon > 0\ \forall \delta > 0\ \exists x_1, x_2 \in X: \quad (x_1,x_2) \overset{1+\varepsilon}{\sim} \ell_p^2 \ \text{ and } \|2^{1/p}\cdot x - x_1 - x_2\| < \delta.$$

Proof. $(i) \implies (ii)$: Pick $f \in S_{L_p[0,1]}$ and $N \in \mathbb{N}$. Then, using the continuity of the mapping $[0,1] \ni x \mapsto \int_0^x |f|$, we find $0 = x_0 < x_1 < \ldots < x_N = 1$ such that $\int_{x_{i-1}}^{x_i} |f|^p = \frac{1}{N}\int_0^1 |f|^p$ for every $i = 1,\dots,N$. We put $f_i := N^{1/p}\cdot f \cdot \chi_{[x_{i-1},x_i]}$, $i = 1,\dots,N$. Then, since the supports of f_i are disjoint and since f_i are normalized, we have $(f_i)_{i=1}^N \equiv \ell_p^N$. Further, we obviously have $N^{1/p}\cdot f = \sum_{i=1}^N f_i$.

Obviously, we have $(ii) \implies (iii)$ and $(iii) \implies (iv)$.

$(iii) \implies (i)$: In order to get a contradiction, let us assume that X is not isometric to $L_p[0,1]$ which, by Theorem 4.8, implies that X is isometric to $L_p(\mu)$, where (Ω, S, μ) is a measure space for which there is $\omega \in \Omega$ with $\mu(\{\omega\}) = 1$. Fix $\varepsilon > 0$ small enough (to be specified later). Suppose to the contrary that there are $f, g \in L_p(\mu)$ such that $(f,g) \overset{1+\varepsilon}{\sim} \ell_p^2$ and $2^{\frac{1}{p}}\cdot\delta_\omega = f + g$, where δ_ω is the Dirac function supported by the point ω. For μ-a.e. $x \in \Omega\setminus\{\omega\}$, we have $f(x)+g(x) = 0$, so we assume this holds for all $x \in \Omega\setminus\{\omega\}$. We without loss of generality assume that $f(\omega) \geq g(\omega)$.

We claim that both $f(\omega)$ and $g(\omega)$ are positive and $|f(\omega) - g(\omega)|^p < \frac{1}{2}$ if $\varepsilon > 0$ is chosen sufficiently small. Indeed, we have

$$(1+\varepsilon)^p - \frac{1}{(1+\varepsilon)^p} \geq \Big|\|f\|_p^p - \|g\|_p^p\Big| = \Big||f(\omega)|^p - |g(\omega)|^p\Big|,$$

which implies $||f(\omega)| - |g(\omega)|| < 2^{-1/p}$ for sufficiently small $\varepsilon > 0$. The claim follows since if both $f(\omega)$ and $g(\omega)$ were not positive we would have $2^{1/p} > 2^{-1/p} > ||f(\omega)| - |g(\omega)|| = |(f+g)(\omega)| = 2^{1/p}$, a contradiction.

First, let us handle the case when $1 \leq p < 2$. We have

$$\begin{aligned}\|2f\|_p^p &= \int_{\Omega\setminus\{\omega\}} |2f|^p \,\mathrm{d}\mu + (2f(\omega))^p = \int_{\Omega\setminus\{\omega\}} |f-g|^p \,\mathrm{d}\mu + (2f(\omega))^p \\ &= \|f-g\|_p^p + (2f(\omega))^p - ((f-g)(\omega))^p \\ &\geq \|f-g\|_p^p + ((f+g)(\omega))^p = \|f-g\|_p^p + \|f+g\|_p^p,\end{aligned}$$

where in the inequality we used superadditivity of the function $[0,\infty) \ni t \mapsto t^p$. Thus, $(f,g) \overset{1+\varepsilon}{\sim} \ell_2^p$ implies

$$(1+\varepsilon)^p \geq \|f\|_p^p \geq \frac{\|f-g\|_p^p + \|f+g\|_p^p}{2^p} \geq \frac{4}{2^p(1+\varepsilon)^p};$$

hence, if $1 \leq p < 2$ we get a contradiction for sufficiently small $\varepsilon > 0$.

Finally, let us handle the case when $p > 2$. Note that since $f(\omega) \geq g(\omega) \geq 0$ and $f(\omega) + g(\omega) = 2^{1/p}$, we have $g(\omega) \leq 2^{1/p-1}$. Further, we have

$$\|2g\|_p^p = \int_{\Omega\setminus\{\omega\}} |f-g|^p \,\mathrm{d}\mu + (2g(\omega))^p \leq \|f-g\|_p^p + 2.$$

Thus, $(f,g) \overset{1+\varepsilon}{\sim} \ell_2^p$ implies

$$\frac{1}{(1+\varepsilon)^p} \leq \|g\|_p^p \leq \frac{\|f-g\|_p^p + 2}{2^p} \leq \frac{2(1+\varepsilon)^p + 2}{2^p};$$

hence, if $p > 2$ we get a contradiction for sufficiently small $\varepsilon > 0$.

$(iv) \implies (iii)$: Fix $x \in S_X$ and $\varepsilon > 0$. Pick $\delta > 0$ small enough (to be specified later). Applying the condition (iv) we obtain $x_1', x_2' \in X$ such that $(x_1', x_2') \overset{1+\frac{\varepsilon}{2}}{\sim} \ell_p^2$ and $\|2^{1/p}\cdot x - x_1' - x_2'\| < \delta$. Now set $x_1 = x_1' + (2^{1/p}\cdot x - (x_1' + x_2'))/2$ and $x_2 = x_2' + (2^{1/p}\cdot x - (x_1' + x_2'))/2$. If δ was chosen sufficiently small, we have $(x_1, x_2) \overset{1+\varepsilon}{\sim} \ell_p^2$ and clearly $2^{1/p}\cdot x = x_1 + x_2$. □

Let us note the following easy observation. The proof is easy and so omitted.

Fact 4.10. *Let $v, w \in V$, $v \neq 0$ and $a, b \in \mathbb{R}$. Then the set*

$$\Big\{\mu \in \mathcal{P} : \ \mu(v) \neq 0 \text{ and } \mu(a\cdot\frac{v}{\mu(v)} - w) < b\Big\}$$

is open in $\mathcal{P}$.

Proof of Theorem 4.4. Let $\mathcal{F}$ be the set of those $\nu \in \mathcal{P}_\infty$ for which X_ν is a $\mathcal{L}_{p,1+}$ space. By Theorem 4.6, $\mathcal{F} \subseteq \mathcal{P}_\infty$ is a G_δ set. By Theorem 4.9, using the obvious observation that condition (iv) may be verified on a dense subset, we have

$$\langle L_p[0,1]\rangle_{\equiv}^{\mathcal{P}_\infty} = \mathcal{F} \cap \bigcap_{v\in V}\bigcap_{n,k\in\mathbb{N}} U_{v,n,k},$$

where $U_{v,n,k}$ are open sets (using Fact 4.10 and Lemma 1.4) defined as

$$U_{v,n,k} := \Big\{\mu \in \mathcal{P}_\infty : \ \exists v_1, v_2 \in V : \ (v_1, v_2) \overset{1+\frac{1}{n}}{\sim} \ell_p^2 \text{ and } \mu\big(2^{1/p}\cdot\frac{v}{\mu(v)} - v_1 - v_2\big) < \frac{1}{k}\Big\}.$$

Thus, $\langle L_p[0,1]\rangle_{\equiv}^{\mathcal{P}_\infty}$ is a G_δ set.

On the other hand, since any $L_p(\mu)$ is finitely representable in ℓ_p and vice versa (see e.g. [1, Proposition 12.1.8]), from Lemma 4.1 and Theorem 4.8 we obtain that there is at most one (up to isometry) $\mathcal{L}_{p,1+}$ space X such that $\langle X\rangle_\equiv$ is G_δ in $\mathcal{B}$ and that $\langle L_p[0,1]\rangle_\equiv$ is G_δ-complete. □

5. Spaces with $F_{\sigma\delta}$ isometry classes

In this section we focus on another classical Banach spaces, namely ℓ_p spaces, for $p \in [1,2) \cup (2,\infty)$, and c_0. The main result of this section is the following.

Theorem 5.1. *The sets $\langle c_0\rangle_\equiv$ and $\langle \ell_p\rangle_\equiv$ (for $p \in [1,2) \cup (2,\infty)$) are $F_{\sigma\delta}$-complete in both $\mathcal{P}_\infty$ and $\mathcal{B}$.*

Note that in order to obtain that result we prove Proposition 5.6 and Theorem 5.13, which are of independent interest and where the "easiest possible" isometric characterizations of the Banach spaces ℓ_p, resp. c_0, among $\mathcal{L}_{p,1+}$ spaces, resp. $\mathcal{L}_{\infty,1+}$ spaces are given. The proof of Theorem 5.1 follows immediately from Proposition 5.3, Proposition 5.4 and Proposition 5.11.

Let us emphasize that in subsection 5.2 we compute the Borel complexity of the operation assigning to a given Banach space a Szlenk derivative of its dual unit ball, which could be of an independent interest as well. See e.g. subsection 7.2 for some consequences. The reason why we need to do it here is obviously that our isometric characterization of the space c_0 involves Szlenk derivatives.

We start with the part which is common for both cases – that is, for $\langle c_0\rangle_\equiv$ and $\langle \ell_p\rangle_\equiv$.

Lemma 5.2. *Let $p \in [1,\infty)$ and let $X = (\bigoplus_{n\in\mathbb{N}} X_n)_p$ be the ℓ_p-sum of the family $(X_n)_{n\in\mathbb{N}}$ of separable infinite-dimensional Banach spaces. Then $X \equiv \ell_p$ if and only if $X_n \equiv \ell_p$ for every $n \in \mathbb{N}$.*

Similarly, let $X = (\bigoplus_{n\in\mathbb{N}} X_n)_0$ be the c_0-sum of the family $(X_n)_{n\in\mathbb{N}}$ of separable infinite-dimensional Banach spaces. Then $X \equiv c_0$ if and only if $X_n \equiv c_0$ for every $n \in \mathbb{N}$.

Proof. It is easy and well-known that the ℓ_p-sum of countably many ℓ_p spaces is isometric to ℓ_p, and that the c_0-sum of countably many c_0 spaces is isometric to c_0. The opposite implications follow from the facts that every 1-complemented infinite-dimensional subspace of ℓ_p is isometric to ℓ_p, and that every 1-complemented infinite-dimensional subspace of c_0 is isometric to c_0, see [39, page 54]. □

Proposition 5.3. *Let X be one of the spaces ℓ_p, $p \in [1,2) \cup (2,\infty)$, or c_0. Then the set $\langle X\rangle_\equiv$ is $F_{\sigma\delta}$-hard in $\mathcal{B}$.*

Proof. Our plan is to find a Wadge reduction of a known $F_{\sigma\delta}$-hard set to $\langle X\rangle_\equiv^{\mathcal{B}}$. For this purpose we will use the set

$$P_3 = \{x \in 2^{\mathbb{N}\times\mathbb{N}} \colon \forall m \text{ there are only finitely many } n\text{'s with } x(m,n) = 1\}$$

(see e.g. [29, Section 23.A] for the fact that P_3 is $F_{\sigma\delta}$-hard in $2^{\mathbb{N}\times\mathbb{N}}$). But before we start to construct the reduction of P_3 to $\langle X\rangle_\equiv^{\mathcal{B}}$ we need to do some preparation.

By Theorem 4.4 (in case $X = \ell_p$) and Theorem 4.3 (in case $X = c_0$) we know that $\langle X\rangle_\equiv^{\mathcal{B}}$ is not G_δ in $\mathcal{B}$. Therefore it is F_σ-hard in $\mathcal{B}$ by Lemma 3.5. Now as the set

$$N_2 = \{x \in 2^{\mathbb{N}} \colon \text{there are only finitely many } n\text{'s with } x(n) = 1\}$$

is F_σ in $2^{\mathbb{N}}$, it is Wadge reducible to $\langle X\rangle_\equiv^{\mathcal{B}}$, so there is a continuous function $\varrho\colon 2^{\mathbb{N}} \to \mathcal{B}$ such that

$$x \in N_2 \Leftrightarrow \varrho(x) \in \langle X\rangle_\equiv^{\mathcal{B}}.$$

We fix a bijection $b\colon \mathbb{N}^2 \to \mathbb{N}$. For every $x \in 2^{\mathbb{N}}$ and every $m \in \mathbb{N}$ we define $\varrho_m(x) \in \mathcal{P}_\infty$ as follows. Suppose that $v = \sum_{n\in\mathbb{N}} \alpha_n e_n$ is an element of V (i.e., α_n is a rational number for every n, and $\alpha_n \neq 0$ only for finitely many n's), then we put

$$\varrho_m(x)(v) = \varrho(x)\left(\sum_{n\in\mathbb{N}} \alpha_{b(m,n)} e_n\right).$$

Note that the set $\{e_{b(m,n)}\colon n\in\mathbb{N}\}$ is both linearly independent and linearly dense in $X_{\varrho_m(x)}$, and that $\varrho_m(x)(e_k)=0$ if $k\notin\{b(m,n)\colon n\in\mathbb{N}\}$. Also, $X_{\varrho_m(x)}$ is isometric to $X_{\varrho(x)}$, where the isometry is induced by the operator

$$e_k\mapsto\begin{cases}e_n & k=b(m,n),\\ 0 & k\notin\{b(m,n)\colon n\in\mathbb{N}\}.\end{cases}$$

Now we are ready to construct the required reduction $f\colon 2^{\mathbb{N}\times\mathbb{N}}\to\mathcal{B}$. For every $x\in 2^{\mathbb{N}\times\mathbb{N}}$ and every $m\in\mathbb{N}$ we write $x^{(m)}$ for the sequence $(x(m,n))_{n\in\mathbb{N}}$. If $X=\ell_p$, we define

$$f(x)(v)=\sqrt[p]{\sum_{m\in\mathbb{N}}\big(\varrho_m(x^{(m)})(v)\big)^p},\qquad v\in V,$$

and if $X=c_0$ we put

$$f(x)(v)=\sup\{(\varrho_m(x^{(m)}))(v)\colon m\in\mathbb{N}\},\qquad v\in V.$$

This formula, together with the preceding considerations, easily imply that $f(x)\in\mathcal{B}$ and that $X_{f(x)}$ is isometric to the ℓ_p-sum, or to the c_0-sum (depending on whether $X=\ell_p$ or $X=c_0$), of the spaces $X_{\varrho(x^{(m)})}$, $m\in\mathbb{N}$. Continuity of the functions ϱ_m and $x\mapsto x^{(m)}$, $m\in\mathbb{N}$, immediately implies continuity of f. By Lemma 5.2, $f(x)\in\langle X\rangle^{\mathcal{B}}_{\equiv}$ if and only if $\varrho(x^{(m)})\in\langle X\rangle^{\mathcal{B}}_{\equiv}$ for every $m\in\mathbb{N}$. Hence,

$$x\in P_3\Leftrightarrow\forall m\in\mathbb{N}:\ x^{(m)}\in N_2\Leftrightarrow f(x)\in\langle X\rangle^{\mathcal{B}}_{\equiv}.\qquad\square$$

5.1. **The spaces ℓ_p.** The purpose of this subsection is to prove the following result.

Proposition 5.4. *For every $p\in[1,2)\cup(2,\infty)$ we have that $\langle\ell_p\rangle_{\equiv}$ is $F_{\sigma\delta}$ in $\mathcal{P}_\infty$.*

We start with the following classical result, which is sometimes named the Clarkson's inequality. The proof may be found on various places, the original one is in the paper by Clarkson, see [9]. In fact, we use only a very special case of the Clarkson's inequality where z,w are required to be elements of the real line instead of an L_p space (and this case is rather straightforward to prove).

Lemma 5.5 (Clarkson's inequality)**.** *Let $1\le p<\infty$, $p\neq 2$. If $p>2$, then for every $z,w\in\mathbb{R}$ we have*

$$|z+w|^p+|z-w|^p-2|z|^p-2|w|^p\ge 0.$$

If $p<2$ then reverse inequality holds. Moreover, the equality holds if and only if $zw=0$.

Proposition 5.6. *Let $1\le p<\infty$, $p\neq 2$, and let X be a separable infinite-dimensional $\mathcal{L}_{p,1+}$ space. Let D be a dense subset of X. Then the following assertions are equivalent.*

(i) X is isometric to ℓ_p.

(ii) For every $x\in S_X$ and every $\delta\in(0,1)$ the following condition is satisfied:

$$\exists N\in\mathbb{N}\ \exists\varepsilon>0\ \forall x_1,\dots,x_N\in X:\quad (N^{1/p}\cdot x_i)_{i=1}^N\overset{1+\varepsilon}{\sim}\ell_p^N\Rightarrow\|x-\sum_{i=1}^N x_i\|>\delta.$$

(iii) For every $x\in S_X$ the following condition is satisfied:

$$\exists N\in\mathbb{N}\ \forall x_1,\dots,x_N\in X:\quad (N^{1/p}\cdot x_i)_{i=1}^N\equiv\ell_p^N\Rightarrow x\neq\sum_{i=1}^N x_i.$$

(iv) For every $x\in D\setminus\{0\}$ and every $\delta\in(0,1)$ the following condition is satisfied:

$$\exists N\in\mathbb{N}\ \exists\varepsilon>0\ \forall x_1,\dots,x_N\in D:\quad (N^{1/p}\cdot x_i)_{i=1}^N\overset{1+\varepsilon}{\sim}\ell_p^N\Rightarrow\Big\|\frac{x}{\|x\|}-\sum_{i=1}^N x_i\Big\|\ge\delta.$$

Proof. $(i) \implies (ii)$: Fix $x \in S_{\ell_p}$ and $\delta \in (0,1)$. Pick $l \in \mathbb{N}$ with $\sum_{k=1}^{l} |x(k)|^p > \delta^p$ and $N \in \mathbb{N}$ such that $\sum_{k=1}^{l}(|x(k)| - \frac{3}{\sqrt[p]{N}})^p > \delta^p$. Fix a sequence $(\varepsilon_m)_{m\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ with $\varepsilon_m \to 0$. In order to get a contradiction, for every $m \in \mathbb{N}$, pick $x_1^{\varepsilon_m}, \ldots, x_N^{\varepsilon_m} \in \ell_p$ such that $(N^{1/p} \cdot x_i^{\varepsilon_m})_{i=1}^N \overset{1+\varepsilon_m}{\sim} \ell_p^N$ and $\|x - \sum_{i=1}^N x_i^{\varepsilon_m}\| \le \delta$ for every $m \in \mathbb{N}$.

We *claim* that there is $m \in \mathbb{N}$ such that $|x_i^{\varepsilon_m}(k) x_j^{\varepsilon_m}(k)| < \eta := N^{-(2+2/p)}$ for every $i, j \in \{1, \ldots, N\}$, $i \ne j$, and $k \in \{1, \ldots, l\}$. Indeed, otherwise there are i, j, k such that $|x_i^{\varepsilon_m}(k) x_j^{\varepsilon_m}(k)| \ge \eta$ for infinitely many m's. By passing to a subsequence, we may assume that this holds for every $m \in \mathbb{N}$. Since the sequences $(|x_i^{\varepsilon_m}(k)|)_m$ and $(|x_j^{\varepsilon_m}(k)|)_m$ are bounded, by passing to a subsequence we may assume there are numbers $a, b \in \mathbb{R}$ with $x_i^{\varepsilon_m}(k) \to a$, $x_j^{\varepsilon_m}(k) \to b$ and $|ab| \ge \eta > 0$. Since $(N^{1/p} \cdot x_i^{\varepsilon_m}, N^{1/p} \cdot x_j^{\varepsilon_m}) \overset{1+\varepsilon_m}{\sim} \ell_p^2$, using Lemma 5.5, for $p > 2$ we obtain

$$\begin{aligned}
0 &\le |a+b|^p + |a-b|^p - 2|a|^p - 2|b|^p \\
&= \lim_m \left(|x_i^{\varepsilon_m}(k) + x_j^{\varepsilon_m}(k)|^p + |x_i^{\varepsilon_m}(k) - x_j^{\varepsilon_m}(k)|^p - 2|x_i^{\varepsilon_m}(k)|^p - 2|x_j^{\varepsilon_m}(k)|^p\right) \\
&\le \lim_m \left(\|x_i^{\varepsilon_m} + x_j^{\varepsilon_m}\|^p + \|x_i^{\varepsilon_m} - x_j^{\varepsilon_m}\|^p - 2\|x_i^{\varepsilon_m}\|^p - 2\|x_j^{\varepsilon_m}\|^p\right) = 0;
\end{aligned}$$

hence, $|a+b|^p + |a-b|^p = 2|a|^p + 2|b|^p = 0$ which, by Lemma 5.5, is in contradiction with $|ab| > 0$. The case when $p < 2$ is similar.

From now on, we write x_i instead of $x_i^{\varepsilon_m}$, where $m \in \mathbb{N}$ is chosen to satisfy the claim above. Fix $k \le l$. By the claim above, there is at most one $i_0 \in \{1, \ldots, N\}$ with $|x_{i_0}(k)| \ge \sqrt{\eta}$ and for this i_0 we have $|x_{i_0}(k)| \le \|x_{i_0}\| \le 2N^{-1/p}$. Consequently, we have

$$\sum_{i=1}^N |x_i(k)| \le \frac{2}{N^{1/p}} + \sum_{i\in\{1,\ldots,N\} \& |x_i(k)| < \sqrt{\eta}} |x_i(k)| \le \frac{2}{N^{1/p}} + N \cdot \sqrt{\eta} = \frac{3}{N^{1/p}}.$$

Thus, we have

$$\|x - \sum_{i=1}^N x_i\|^p \ge \sum_{k=1}^l \Big(|x(k)| - \sum_{i=1}^N |x_i(k)|\Big)^p \ge \sum_{k=1}^l \Big(|x(k)| - \frac{3}{N^{1/p}}\Big)^p > \delta^p,$$

which is in contradiction with $\|x - \sum_{i=1}^N x_i\| = \|x - \sum_{i=1}^N x_i^{\varepsilon_m}\| \le \delta$.

$(ii) \implies (iii)$ is obvious.

$(iii) \implies (i)$: suppose that X is not isometric to ℓ_p. By Theorem 4.8, X is isometric to $L_p[0,1] \oplus_p Y$ for some (possibly trivial) Banach space Y. By abusing the notation, we may assume that $X = L_p[0,1] \oplus_p Y$. Let $\mathbf{1} \in L_p[0,1]$ be the constant 1 function, and define $x \in X = L_p[0,1] \oplus_p Y$ by $x = (\mathbf{1}, 0)$. Now fix $N \in \mathbb{N}$ arbitrarily. Define $x_1, \ldots, x_N \in X$ by $x_i = (\chi_{[\frac{i-1}{n}, \frac{i}{n}]}, 0)$. Clearly $(N^{1/p} \cdot x_i)_{i=1}^N \equiv \ell_p^n$ and we have $x = \sum_{i=1}^N x_i$.

$(ii) \implies (iv)$ is obvious, so it only remains to show that $(iv) \implies (ii)$. For every $x \in X \setminus \{0\}$, $\delta \in (0,1)$, $N \in \mathbb{N}$, $\varepsilon > 0$ and $x_1, \ldots, x_N \in X$ we denote by $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^N)$ the assertion that if $(N^{1/p} \cdot x_i)_{i=1}^N \overset{1+\varepsilon}{\sim} \ell_p^N$ then $\|\frac{x}{\|x\|} - \sum_{i=1}^N x_i\| \ge \delta$. The desired implication straightforwardly follows by the following two easy observations. First, if $x \in D \setminus \{0\}$, δ, N and ε are given such that $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^N)$ holds for every $x_1, \ldots, x_N \in D$ then $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^N)$ holds for every $x_1, \ldots, x_N \in X$. Second, if for every $x \in D \setminus \{0\}$ and δ there are N and ε such that $V(x, \frac{1+\delta}{2}, N, \varepsilon, (x_i)_{i=1}^N)$ holds for every $x_1, \ldots, x_N \in X$, then for every $x \in X \setminus \{0\}$ and δ there are N and ε such that $V(x, \delta, N, \varepsilon, (x_i)_{i=1}^N)$ holds for every $x_1, \ldots, x_N \in X$. $\square$

Proof of Proposition 5.4. Let $\mathcal{F}$ be the set of those $\nu \in \mathcal{P}_\infty$ for which X_ν is an $\mathcal{L}_{p,1+}$ space. By Theorem 4.6, $\mathcal{F} \subseteq \mathcal{P}_\infty$ is a G_δ set. By Proposition 5.6 $(i) \Leftrightarrow (iv)$, we have

$$\langle \ell_p \rangle_{\equiv}^{\mathcal{P}_\infty} = \mathcal{F} \cap \bigcap_{v \in V \setminus \{0\}} \bigcap_{m \in \mathbb{N}} \bigcup_{n,k \in \mathbb{N}} V_{v,m,n,k},$$

where the closed (see Fact 4.10 and Lemma 1.4) sets $V_{v,m,n,k}$ are given by

$$V_{v,m,n,k} := \Big\{\mu \in \mathcal{P}_\infty : \mu(v) = 0 \text{ or for every } (v_i)_{i=1}^n \in V^n \text{ we have}$$
$$\neg\Big((\sqrt[p]{n} v_i)_{i=1}^n \overset{1+\frac{1}{k}}{\sim} \ell_p^n \Big) \text{ or } \mu\Big(\frac{v}{\mu(v)} - \sum_{i=1}^n v_i\Big) \geq \frac{1}{m}\Big\}.$$

Thus, $\langle \ell_p \rangle_{\equiv}^{\mathcal{P}_\infty}$ is an $F_{\sigma\delta}$ set. $\square$

5.2. **Dual unit balls and the Szlenk derivative.** The purpose here is to show that mappings which assign a dual unit ball and its Szlenk derivative to a separable Banach space may be realized as a Borel map, see Lemma 5.10 and Lemma 5.9. This will be later used in order to estimate the Borel complexity of the isometry class of the space c_0 because the isometric characterization of the space c_0 we use involves Szlenk derivatives, see Theorem 5.13. Note that the issue of handling Szlenk derivations as Borel maps was previously considered also by Bossard in [4, page 141], but our approach is slightly different as we prefer to work with coding $\mathcal{P}$ and we also need to obtain an estimate on the Borel class of the mapping.

Let us recall that given a real Banach space X, w^*-compact set $F \subseteq X^*$ and $\varepsilon > 0$, the Szlenk derivative is given as

$$F'_\varepsilon = \big\{x^* \in F : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap F) \geq \varepsilon\big\}.$$

We start by coding dual unit balls as closed subsets of B_{ℓ_∞} equipped with the weak* topology, i.e., the topology generated by elements of the unique predual ℓ_1.

Lemma 5.7. *Let X be a separable Banach space and let $\{x_n : n \in \mathbb{N}\}$ be a dense set in B_X. Then the mapping $B_{X^*} \ni x^* \mapsto (x^*(x_n))_{n=1}^\infty \in B_{\ell_\infty}$ is $\|\cdot\|$-$\|\cdot\|$ isometry and w^*-w^* homeomorphism onto the set*

$$\Omega(X) := \Big\{(a_n)_{n=1}^\infty \in B_{\ell_\infty} : M \subseteq \mathbb{N} \text{ finite} \Rightarrow \Big|\sum_{n \in M} a_n\Big| \leq \Big\|\sum_{n \in M} x_n\Big\|\Big\}.$$

Proof. That the mapping is w^*-w^* homeomorphism onto its image follows from the fact that B_{X^*} is w^*-compact and the mapping is one-to-one (because (x_n) separate the points of B_{X^*}) and w^*-w^* continuous (because on B_{ℓ_∞} the w^*-topology coincides with the topology of pointwise convergence). It is also straightforward to see that the mapping is isometry. Thus, it suffices to proof that

$$\{(x^*(x_n))_{n=1}^\infty : x^* \in B_{X^*}\} = \Omega(X).$$

The inclusion $\subseteq$ is easy, let us prove $\supseteq$. Given numbers $a_1, a_2, \dots$ satisfying $|\sum_{n \in M} a_n| \leq \|\sum_{n \in M} x_n\|$ for any finite $M \subseteq \mathbb{N}$, we need to find $x^* \in B_{X^*}$ such that $x^*(x_n) = a_n$ for each n.

Let us realize first that

- $|a_n - a_m| \leq \|x_n - x_m\|$ for every n, m,
- $|a_n + a_m - a_l| \leq \|x_n + x_m - x_l\|$ for every n, m, l.

We check the first inequality only, the second inequality can be checked in the same way. Given $\varepsilon > 0$, let n' different from n and m be such that $\|x_n + x_{n'}\| < \varepsilon$. We obtain $|a_n - a_m| = |(a_n + a_{n'}) - (a_m + a_{n'})| \leq |a_n + a_{n'}| + |a_m + a_{n'}| \leq \|x_n + x_{n'}\| + \|x_m + x_{n'}\| \leq 2\|x_n + x_{n'}\| + \|x_m - x_n\| < 2\varepsilon + \|x_m - x_n\|$. Since $\varepsilon > 0$ was chosen arbitrarily, we arrive at $|a_n - a_m| \leq \|x_m - x_n\|$.

It follows that there is a function $f : B_X \to \mathbb{R}$ with the Lipschitz constant 1 such that $f(x_n) = a_n$ for each n. We claim that $f(u + v) = f(u) + f(v)$ and $f(\alpha u) = \alpha f(u)$, whenever $u, \alpha u, v, u + v \in B_X$. Given $\varepsilon > 0$, we pick n, m, l such that $\|x_n-u\| < \varepsilon$, $\|x_m-v\| < \varepsilon$ and $\|x_l-(u+v)\| < \varepsilon$. Then $|f(u)+f(v)-f(u+v)| < |a_n+a_m-a_l|+3\varepsilon \leq \|x_n+x_m-x_l\|+3\varepsilon \leq \|u+v-(u+v)\|+3\varepsilon+3\varepsilon = 6\varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, we arrive at $|f(u)+f(v)-f(u+v)| = 0$. This also shows that $f(u/2) = f(u)/2$, therefore $f(\alpha u) = \alpha f(u)$, provided that α is a dyadic rational number. For general α, we use density of dyadic rationals and continuity of f.

Now, it is easy to see that f uniquely extends to a linear functional on X. □

By the above, every dual unit ball of a separable Banach space may be realized as a subset of the unit ball of ℓ_∞. Thus, in what follows we use the following convention.

Convention. Whenever we talk about open (closed, F_σ, etc.) subsets of B_{l_∞} we always mean open (closed, F_σ, etc.) subsets in the weak* topology. On the other hand, whenever we talk about the diameter of a subset of B_{l_∞}, or about the distance of two subsets of B_{l_∞}, we always mean the diameter, or the distance, with respect to the metric given by the norm of ℓ_∞. Also, we write only $\mathcal{K}(B_{l_\infty})$ instead of $\mathcal{K}(B_{l_\infty}, w^*)$.

Let us note the following easy observation for further references.

Lemma 5.8. *Let P be a Polish space, X a metrizable compact, $\alpha \in [1, \omega_1)$ and $f : P \to \mathcal{K}(X)$ a mapping such that $\{p \in P\colon f(p) \subseteq U\} \in \mathbf{\Sigma}^0_\alpha(P) \cup \mathbf{\Pi}^0_\alpha(P)$ for every open $U \subseteq X$. Then f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable.*

Proof. The sets of the form

$$\{F \in \mathcal{K}(X)\colon F \subseteq W\} \quad \text{and} \quad \{F \in \mathcal{K}(X)\colon F \cap W \neq \emptyset\},$$

where W ranges over all open subsets of X, form a subbasis of the topology of $\mathcal{K}(X)$. So we only need to check that $f^{-1}(U)$ is an $\mathbf{\Sigma}^0_{\alpha+1}$ set for every open set U of one of these forms. For the first case this follows immediately from the assumptions and for the second case, if $\{U_n\colon n \in \mathbb{N}\}$ is an open basis for the topology of X, we have

$$f^{-1}(\{F \in \mathcal{K}(X)\colon F \cap W \neq \emptyset\}) = \bigcup_{n\in\mathbb{N} \text{ such that } \overline{U_n} \subseteq W} P \setminus \{p \in P\colon f(p) \subseteq X \setminus \overline{U_n}\},$$

which, by the assumptions, is countable union of sets from $\mathbf{\Sigma}^0_\alpha(P) \cup \mathbf{\Pi}^0_\alpha(P)$. □

Lemma 5.9. *For every $\nu \in \mathcal{P}$ we can choose a countable dense subset $\{x^\nu_n\colon n \in \mathbb{N}\}$ of B_{X_ν} in such a way that the mapping $\Omega\colon \mathcal{P} \to \mathcal{K}(B_{l_\infty}, w^*)$ given by*

$$\Omega(\nu) = \left\{(a_n)_{n=1}^\infty \in B_{l_\infty} : M \subseteq \mathbb{N} \text{ finite} \Rightarrow \left|\sum_{n\in M} a_n\right| \leq \nu\left(\sum_{n\in M} x^\nu_n\right)\right\}$$

is continuous.

Proof. First of all, we describe the choice of the sets $\{x^\nu_n\colon n \in \mathbb{N}\}$, $\nu \in \mathcal{P}$. Let $g\colon [0,\infty) \to [1,\infty)$ be given by $g(t) = 1$ for $t \leq 1$ and $g(t) = t$ for $t > 1$. Let $\{v_n\colon n \in \mathbb{N}\}$ be an enumeration of all elements of the vector space V (which is naturally embedded into all Banach spaces X_ν, $\nu \in \mathcal{P}$). Now for every $\nu \in \mathcal{P}$ and every $n \in \mathbb{N}$ we define $x^\nu_n \in B_{X_\nu}$ by $x^\nu_n = \frac{v_n}{g(\nu(v_n))}$. Then for every $\nu \in \mathcal{P}$ we have that $\{x^\nu_n\colon n \in \mathbb{N}\}$ is a dense subset of B_{X_ν}. Note also that the set

$$\left\{(\nu, (a_n)_{n=1}^\infty) \in \mathcal{P} \times B_{l_\infty} : \left|\sum_{n\in M} a_n\right| > \nu\left(\sum_{n\in M} x^\nu_n\right)\right\}$$

is open in $\mathcal{P} \times (B_{l_\infty}, w^*)$ for every $M \subseteq \mathbb{N}$ finite (the proof is easy and, similarly as the proof of Fact 4.10, it is omitted).

Pick an open subset U of B_{ℓ_∞}. We have

$$\begin{aligned}&\Omega^{-1}\left(\{F\in\mathcal{K}(B_{l_\infty})\colon F\subseteq U\}\right)\\&=\left\{\nu\in\mathcal{P}\colon \forall_{(a_n)_{n=1}^\infty\in B_{l_\infty}}\left(\left(\exists_{\substack{M\subseteq\mathbb{N}\\ \text{finite}}}\left|\sum_{n\in M}a_n\right|>\nu\left(\sum_{n\in M}x_n^\nu\right)\right)\text{ or }\left((a_n)_{n=1}^\infty\in U\right)\right)\right\}.\end{aligned}$$

The complement of the last set is the projection of a closed subset of $\mathcal{P}\times(B_{l_\infty},w^*)$ onto the first coordinate. As the space (B_{l_∞},w^*) is compact, the complement is a closed subset of $\mathcal{P}$.

It remains to show that the set $\{\nu\in\mathcal{P}\colon\Omega(\nu)\cap U\neq\emptyset\}$ is open. Pick $\nu\in\mathcal{P}$ with $\Omega(\nu)\cap U\neq\emptyset$. By Lemma 5.7, there exists $x^*\in B_{X_\nu^*}$ such that the sequence $(a_n)_{n=1}^\infty$ given by $a_n=x^*(x_n^\nu)$, $n\in\mathbb{N}$, satisfies $(a_n)_{n=1}^\infty\in\Omega(\nu)\cap U$. Let $\varepsilon>0$ and $N\in\mathbb{N}$ be such that $(b_n)_{n=1}^\infty\in\ell_\infty$ is an element of U whenever $|b_n-a_n|<\varepsilon$ for every $1\leq n\leq N$. Let us consider subspaces of c_{00} given as $E=\operatorname{span}\{v_1,\dots,v_N\}$ and $F=\{x\in E\colon\nu(x)=0\}$. Let G be such that $F\oplus G=E$ and $\overline{G\cap V}=G$ (it is enough to pick a basis of E consisting of vectors from V, and using the Gauss elimination to determine which vectors from the basis generate the algebraic complement to F). Let $P_F:E\to F$ and $P_G:E\to G$ be linear projections onto F and G, respectively. Pick $\delta<\min\{1,\frac{\varepsilon}{3}\}$ such that $\delta\cdot|x^*(P_Gv_n)|<\varepsilon/3$ for every $1\leq n\leq N$. Finally, put

$$\begin{aligned}\mathcal{O}:=&\{\nu'\in\mathcal{P}\colon\frac{1}{1-\delta}\nu(x)>\nu'(x)>(1-\delta)\nu(x)\text{ for every }x\in G\setminus\{0\}\}\cap\\&\bigcap_{n=1}^N\{\nu'\in\mathcal{P}\colon\nu'(P_Fv_n)<\delta,|\nu'(v_n)-\nu(v_n)|<\delta\}.\end{aligned}$$

Then $\mathcal{O}$ is an open neighborhood of ν, which easily follows from Lemma 1.4 and the fact that $G\cap V$ is dense in G.

We will show that $\mathcal{O}\subseteq\{\nu'\in\mathcal{P}\colon\Omega(\nu')\cap U\neq\emptyset\}$. Pick $\nu'\in\mathcal{O}$. If we put $y^*(x):=(1-\delta)x^*(x)$ for $x\in G$, then $|y^*(x)|=(1-\delta)|x^*(x)|\leq(1-\delta)\nu(x)\leq\nu'(x)$ for every $x\in G$ and so by the Hahn-Banach theorem we may extend y^* to a functional (denoted again by y^*) from the dual unit ball of $X_{\nu'}$. By Lemma 5.7, the sequence $(b_n)_{n=1}^\infty$ given by $b_n:=y^*(x_n^{\nu'})$, $n\in\mathbb{N}$, is in $\Omega(\nu')$. Moreover, for every $1\leq n\leq N$ we have

$$\begin{aligned}|b_n-a_n|&=\left|\tfrac{1}{g(\nu'(v_n))}y^*(v_n)-\tfrac{1}{g(\nu(v_n))}x^*(v_n)\right|\\&=\left|\tfrac{1}{g(\nu'(v_n))}y^*(P_Fv_n)+\tfrac{1}{g(\nu'(v_n))}y^*(P_Gv_n)-\tfrac{1}{g(\nu(v_n))}x^*(P_Fv_n+P_Gv_n)\right|\\&=\left|\tfrac{1}{g(\nu'(v_n))}y^*(P_Fv_n)+\tfrac{1}{g(\nu'(v_n))}(1-\delta)x^*(P_Gv_n)-\tfrac{1}{g(\nu(v_n))}x^*(P_Gv_n)\right|\\&\leq\tfrac{1}{g(\nu'(v_n))}|y^*(P_Fv_n)|+\left|\tfrac{1}{g(\nu'(v_n))}(1-\delta)-\tfrac{1}{g(\nu(v_n))}\right||x^*(P_Gv_n)|\\&\leq\delta+\big(|g(\nu(v_n))-g(\nu'(v_n))|+\delta\big)|x^*(P_Gv_n)|\\&\leq\delta+2\delta|x^*(P_Gv_n)|<\varepsilon,\end{aligned}$$

and so $(b_n)_{n=1}^\infty\in\Omega(\nu')\cap U$. Hence, $\mathcal{O}\subseteq\{\nu'\in\mathcal{P}\colon\Omega(\nu')\cap U\neq\emptyset\}$, so $\{\nu\in\mathcal{P}\colon\Omega(\nu)\cap U\neq\emptyset\}$ is open set and Ω is a continuous mapping. $\square$

We close the first part of the subsection by realizing that the mapping which assigns to every compact subset of B_{ℓ_∞} its Szlenk derivative is Borel. Let us note that the result is almost optimal as the mapping from Lemma 5.10 is not F_σ-measurable, see Corollary 5.14.

Lemma 5.10. *For every $\varepsilon>0$, the function $s_\varepsilon\colon\mathcal{K}(B_{l_\infty},w^*)\to\mathcal{K}(B_{l_\infty},w^*)$ given by $s_\varepsilon(F)=F'_\varepsilon$ is $\mathbf{\Sigma}_3^0$-measurable.*

Proof. First, we claim that the set

$$\{F \in \mathcal{K}(B_{l_\infty})\colon \operatorname{diam}(U \cap F) < \varepsilon\}$$

is an F_σ set for every open subset U of B_{l_∞}. Indeed, the set above equals

$$\begin{aligned}&\bigcup_{k=1}^{\infty}\{F \in \mathcal{K}(B_{l_\infty})\colon \operatorname{diam}(U \cap F) \le \varepsilon - \tfrac{1}{k}\}\\ =&\bigcup_{k=1}^{\infty}\ \bigcap_{\substack{O_1,O_2 \text{ open subsets of } U\\ \operatorname{dist}(O_1,O_2)\ge\varepsilon-\frac{1}{k}}} \{F \in \mathcal{K}(B_{l_\infty})\colon F \cap O_1 = \emptyset \text{ or } F \cap O_2 = \emptyset\},\end{aligned}$$

and our claim immediately follows.

Now let W be an open subset of B_{l_∞}. Let $\{U_n\colon n \in \mathbb{N}\}$ be an open basis for the weak* topology of B_{l_∞}. Then we have (using a compactness argument in the last equality) that

$$\begin{aligned}&s_\varepsilon^{-1}(\{F \in \mathcal{K}(B_{l_\infty})\colon F \subseteq W\})\\ =&\{F \in \mathcal{K}(B_{l_\infty})\colon \underset{M\subseteq\mathbb{N}}{\exists}\big(\big(\underset{n\in M}{\forall} \operatorname{diam}(U_n \cap F) < \varepsilon\big) \text{ and } \big(F \subseteq W \cup \bigcup_{n\in M} U_n\big)\big)\}\\ =&\{F \in \mathcal{K}(B_{l_\infty})\colon \underset{\substack{M\subseteq\mathbb{N}\\ \text{finite}}}{\exists}\big(\big(\underset{n\in M}{\forall} \operatorname{diam}(U_n \cap F) < \varepsilon\big) \text{ and } \big(F \subseteq W \cup \bigcup_{n\in M} U_n\big)\big)\},\end{aligned}$$

and our previous claim implies that the last set is F_σ.

Thus, by Lemma 5.8, the mapping s_ε is $\mathbf{\Sigma}^0_3$-measurable. $\square$

5.3. **The space c_0.** The main goal of this subsection is to prove the following.

Proposition 5.11. *$\langle c_0\rangle_\equiv$ is an $F_{\sigma\delta}$ set in $\mathcal{P}_\infty$.*

Our estimate on the Borel complexity of the isometry class of c_0 is based on an isometric characterization of c_0 among $\mathcal{L}_{\infty,1+}$ spaces. Let us recall that $\mathcal{L}_{\infty,1+}$ spaces are often called the *Lindenstrauss spaces* or *L_1 predual spaces.* There are many different characterizations of this class of spaces. Let us recall one which we will use further, see e.g. [35, p. 232] (the "in particular" part follows from the easy part of Theorem 4.8 applied to X^* and the fact that $L_1[0,1]$ is not isomorphic to a subspace of a separable dual Banach space, see e.g. [1, Theorem 6.3.7]).

Theorem 5.12. *Let X be a Banach space. Then the following conditions are equivalent.*

- *(i) X is $\mathcal{L}_{\infty,1+}$ space.*
- *(ii) X^* is isometric to $L_1(\mu)$ for some measure μ.*

In particular, if X is a $\mathcal{L}_{\infty,1+}$ space with X^* separable then X^* is isometric to ℓ_1.

The isometric characterization of c_0 which we use for our upper estimate follows.

Theorem 5.13. *Let X be a separable $\mathcal{L}_{\infty,1+}$-space and let $0 < \varepsilon < 1$. Then X is isometric to c_0 if and only if*

$$(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}.$$

Proof. First, we show that $(B_{c_0^*})'_{2\varepsilon} = (1-\varepsilon)B_{c_0^*}$ (this must be known but we were unable to find any reference). By a standard argument, $(1-\varepsilon)B_{X^*} \subseteq (B_{X^*})'_{2\varepsilon}$ for any infinite-dimensional X. (Let $x^* \in (1-\varepsilon)B_{X^*}$. Any w^*-open set U containing 0 contains also both y^* and $-y^*$ for some $y^* \in S_{X^*}$, and so $\operatorname{diam}(U \cap B_{X^*}) = 2$. For this reason, any w^*-open set V containing x^* fulfills $\operatorname{diam}(V \cap (x^* + \varepsilon B_{X^*})) = 2\varepsilon$, in particular, $\operatorname{diam}(V \cap B_{X^*}) \ge 2\varepsilon$. This proves that $x^* \in (B_{X^*})'_{2\varepsilon}$.)

Let us show that the opposite inclusion takes place for $X = c_0$. Assuming $1 - \varepsilon < \|x^*\| \leq 1$, we need to check that $x^* \notin (B_{c_0^*})'_{2\varepsilon}$. Let $e_1, e_2, \dots$ be the canonical basis of c_0. Let n be large enough that $\sum_{i=1}^n |x^*(e_i)| > 1 - \varepsilon$ and let $\delta > 0$ satisfy $2\delta n < [\sum_{i=1}^n |x^*(e_i)|] - (1 - \varepsilon)$. Let

$$U = \{y^* \in c_0^* : 1 \leq i \leq n \Rightarrow |y^*(e_i) - x^*(e_i)| < \delta\}.$$

For $y^*, z^* \in U \cap B_{c_0^*}$, we have

$$\sum_{i=n+1}^{\infty} |y^*(e_i)| = \|y^*\| - \sum_{i=1}^{n} |y^*(e_i)| \leq 1 - \sum_{i=1}^{n} |x^*(e_i)| + \delta n,$$

and the same for z^*, thus

$$\|y^* - z^*\| \leq \sum_{i=1}^{n} |y^*(e_i) - z^*(e_i)| + \sum_{i=n+1}^{\infty} |y^*(e_i)| + \sum_{i=n+1}^{\infty} |z^*(e_i)|$$

$$\leq 2\delta n + 2\Big[1 - \sum_{i=1}^{n} |x^*(e_i)|\Big] + 2\delta n.$$

We get $\operatorname{diam}(U \cap B_{c_0^*}) \leq 4\delta n + 2[1 - \sum_{i=1}^n |x^*(e_i)|] < 2\varepsilon$.

Now, let us assume that X satisfies $(B_{X^*})'_{2\varepsilon} = (1-\varepsilon)B_{X^*}$. Clearly, X is infinite-dimensional, as $(B_{X^*})'_{2\varepsilon}$ is non-empty. Moreover, X^* is separable because the Szlenk index of X is ω, see e.g. [36, Proposition 3 and Theorem 1]. Thus, by Theorem 5.12, the dual X^* is isometric to ℓ_1. Let $e_1^*, e_2^*, \dots$ be a basis of X^* that is 1-equivalent to the canonical basis of ℓ_1, and let $e_1^{**}, e_2^{**}, \dots$ be the dual basic sequence in X^{**}. We claim that the functionals e_n^{**} are w^*-continuous.

Suppose that e_n^{**} is not w^*-continuous for some n. It means that $\{x^* \in X^* : e_n^{**}(x^*) = 0\}$ is not w^*-closed. By the Banach-Dieudonné theorem, the set $\{x^* \in B_{X^*} : e_n^{**}(x^*) = 0\}$ is not w^*-closed, too. The space (B_{X^*}, w^*) is metrizable, so there is a sequence x_k^* in B_{X^*} with $e_n^{**}(x_k^*) = 0$ which w^*-converges to some x^* with $e_n^{**}(x^*) \neq 0$. Without loss of generality, let us assume that $e_n^{**}(x^*) > 0$ and that $e_i^{**}(x_k^*)$ converges to some a_i for every i. Then clearly $a_n = 0$. Note that $\sum_{i=1}^{\infty} |a_i| \leq 1$, which follows from the fact that $\sum_{i=1}^{\infty} |e_i^{**}(x_k^*)| = \|x_k^*\| \leq 1$ for every k. Let us put $a^* = \sum_{i=1}^{\infty} a_i e_i^*$, $y_k^* = x_k^* - a^*$ and $y^* = x^* - a^*$. Then $e_n^{**}(y_k^*) = 0, e_n^{**}(y^*) > 0$, the sequence y_k^* is w^*-convergent to y^* and, moreover, $e_i^{**}(y_k^*)$ converges to 0 for every i. Choosing a subsequence and making a small perturbation, we can find a sequence z_l^* which is a block sequence with respect to the basis e_i^* and which still w^*-converges to y^*. Without loss of generality, let us assume that $\|z_l^*\|$ converges to some λ, clearly with $\lambda \geq \|y^*\| > 0$, and let us consider $u_l^* = \frac{1}{\|z_l^*\|} z_l^*$ and $u^* = \frac{1}{\lambda} y^*$.

So, we have seen that there is a normalized block sequence u_l^* in X^* which w^*-converges to some u^* with $e_n^{**}(u^*) > 0$. We put

$$v_l^* = (1-\varepsilon)e_n^* + \varepsilon u_l^*, \quad v^* = (1-\varepsilon)e_n^* + \varepsilon u^*.$$

Then v_l^* is a sequence in B_{X^*} that w^*-converges to v^*. Since $\|v_l^* - v_{l'}^*\| = 2\varepsilon$ for $l \neq l'$, any w^*-open set U containing v^* fulfills $\operatorname{diam}(U \cap B_{X^*}) \geq 2\varepsilon$. It follows that $v^* \in (B_{X^*})'_{2\varepsilon}$ and, by our assumption, $v^* \in (1-\varepsilon)B_{X^*}$. At the same time,

$$\|v^*\| \geq e_n^{**}(v^*) = (1-\varepsilon) + \varepsilon e_n^{**}(u^*) > 1 - \varepsilon,$$

which is not possible.

Hence, the functionals e_n^{**} are w^*-continuous indeed. Every e_n^{**} is therefore the evaluation of some $e_n \in X$. Finally, it is easy to check that $e_1, e_2, \dots$ is a basis of X that is 1-equivalent to the canonical basis of c_0. $\square$

Proof of Proposition 5.11. Let $\mathcal{F}$ be the set of those $\mu \in \mathcal{P}_\infty$ for which X_μ is $\mathcal{L}_{\infty,1+}$-space. By Theorem 4.6, $\mathcal{F}$ is G_δ in $\mathcal{P}_\infty$. Let Ω be the mapping from Lemma 5.9 and let us denote by $\triangle$ the closed set $\{(x,x)\colon x \in \mathcal{K}(\ell_\infty)\}$ in $\mathcal{K}(\ell_\infty) \times \mathcal{K}(\ell_\infty)$. By Lemma 5.7 and Theorem 5.13, we have that

$$\langle c_0 \rangle_\equiv = \mathcal{F} \cap \{\nu \in \mathcal{P}\colon (\tfrac{1}{2}\Omega(\nu), \Omega_1'(\nu)) \in \triangle\}.$$

By Lemma 5.9 and Lemma 5.10, the mapping $\mathcal{P} \ni \nu \mapsto (\frac{1}{2}\Omega(\nu), \Omega_1'(\nu)) \in \mathcal{K}(\ell_\infty) \times \mathcal{K}(\ell_\infty)$ is $\mathbf{\Sigma}^0_3$-measurable, so we obtain that $\langle c_0 \rangle_\equiv$ is $F_{\sigma\delta}$ in $\mathcal{P}_\infty$. □

Corollary 5.14. *Let $\varepsilon > 0$. Then the mapping s_ε from Lemma 5.10 is not $\mathbf{\Sigma}^0_2$-measurable.*

Proof. Otherwise, similarly as in the proof of Proposition 5.11 we would prove that $\langle c_0 \rangle_\equiv$ is G_δ in $\mathcal{P}_\infty$, which is not possible due to Theorem 4.3. □

6. Spaces with descriptively simple isomorphism classes

While there are several Banach spaces whose isometry classes have low complexity, there are reasons to suspect that isomorphism classes are rather complicated in general. The main result of this section is the following.

Theorem 6.1. *The Hilbert space ℓ_2 is characterized as the unique, up to isomorphism, infinite-dimensional separable Banach space X such that $\langle X \rangle_\simeq$ is F_σ in $\mathcal{B}$. The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$.*

Recall that the isomorphism class of ℓ_2 is F_σ, see Proposition 3.6. Besides this space, it is proved in [23, Theorem 4.12] that separable Banach spaces determined by their pavings have $\mathbf{\Sigma}^0_4$ isomorphism classes in any admissible topology. We refer the interested reader to the text below for a definition of spaces determined by their pavings. Here we just briefly note that this class of spaces was introduced by Johnson, Lindenstrauss and Schechtman in [28] and that there are known examples of separable Banach spaces determined by their pavings not isomorphic to ℓ_2 (e.g. certain ℓ_2-sums of finite-dimensional spaces are such). The second main result of this section is the following improvement of the estimate mentioned above.

Theorem 6.2. *Let X be a separable infinite-dimensional Banach space that is determined by its pavings. Then $\langle X \rangle_\simeq$ is $G_{\delta\sigma}$ in $\mathcal{P}_\infty$. In particular, it is $G_{\delta\sigma}$ in $\mathcal{P}$ and in any admissible topology.*

Let us start with the proof of Theorem 6.1.

Proof of Theorem 6.1. We only need to show that if a separable infinite-dimensional Banach space X is not isomorphic to ℓ_2, then $\langle X \rangle_\simeq$ is not F_σ in $\mathcal{B}$. In what follows, we denote by $\mathcal{T}$ the set of finite tuples (including empty) of natural numbers without repetition. The length of $\gamma \in \mathcal{T}$ is denoted by $|\gamma|$ and its range by $\operatorname{rng}(\gamma)$. Moreover, for every $\gamma \in \mathcal{T}$ and every $\mu \in \mathcal{B}$ we put

$$\mathbb{M}^\gamma_\mu := \Big\{\nu \in \mathcal{B}\colon \text{for every } (a_i)_{i=1}^{|\gamma|} \in \mathbb{Q}^{|\gamma|} \text{ we have } \nu\Big(\sum_{i=1}^{|\gamma|} a_i e_i\Big) = \mu\Big(\sum_{i=1}^{|\gamma|} a_i e_{\gamma(i)}\Big)\Big\}.$$

In order to get a contradiction assume that $(F_n)_{n=1}^\infty$ are closed sets in $\mathcal{B}$ such that $\langle X \rangle_\simeq = \bigcup_{n=1}^\infty F_n$.

Claim. For every $\mu \in \mathcal{B}$ with $\langle X_\mu \rangle_\equiv \subseteq \bigcup_{n=1}^\infty F_n$ there exist $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ such that we have $\mathbb{M}^{\gamma'}_\mu \cap F_m \neq \emptyset$ for every $\gamma' \in \mathcal{T}$ with $\gamma' \supseteq \gamma$.

Proof of the claim. Suppose the statement is not true. In particular, it does not hold for $\gamma = \emptyset$ and $m = 1$. That is, there is some $\gamma_1' \in \mathcal{T}$ so that $\mathbb{M}_\mu^{\gamma_1'} \cap F_1 = \emptyset$. If $1 \in \operatorname{rng}(\gamma_1')$, we set $\gamma_1 = \gamma_1'$. Otherwise, we set $\gamma_1 = \gamma_1'^\frown(1)$.

In the next step, we use that the statement is not true for γ_1 and $m = 2$ to obtain $\gamma_2' \in \mathcal{T}$, $\gamma_2' \supseteq \gamma_1$ so that $\mathbb{M}_\mu^{\gamma_2'} \cap F_2 = \emptyset$. If $2 \in \operatorname{rng}(\gamma_2')$, we set $\gamma_2 = \gamma_2'$. Otherwise, we set $\gamma_2 = \gamma_2'^\frown(2)$.

We continue analogously. At the end of the recursion, we a obtain a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi \supseteq \gamma_n$ for every $n \in \mathbb{N}$ and $\mathbb{M}_\mu^{\gamma_n} \cap F_n = \emptyset$ for every $n \in \mathbb{N}$. Consider $\mu_0 \in \mathcal{B}$ given as

$$\mu_0\Big(\sum_{i=1}^k a_i e_i\Big) := \mu\Big(\sum_{i=1}^k a_i e_{\pi(i)}\Big), \quad k \in \mathbb{N},\ (a_i)_{i=1}^k \in \mathbb{Q}^k.$$

Then the linear mapping given by $e_i \mapsto e_{\pi(i)}$, $i \in \mathbb{N}$, witnesses that $X_{\mu_0} \equiv X_\mu$ and $\mu_0 \in \mathbb{M}_\mu^{\gamma_n}$ for every $n \in \mathbb{N}$. Thus, $\mu_0 \notin \bigcup_{n=1}^\infty F_n$ which is in contradiction with $\mu_0 \in \langle X_\mu \rangle_\equiv \subseteq \bigcup_{n=1}^\infty F_n$. $\square$

Since $X \not\simeq \ell_2$, by the celebrated solution to the homogeneous subspace problem following from the results of Komorowski and Tomczak-Jaegermann ([30]) and of Gowers ([24]), it must contain an infinite-dimensional closed subspace $Y \subseteq X$ that is not isomorphic to X. Let $I \subseteq \mathbb{N}$ be an infinite subset and $\{x_n\}_{n\in\mathbb{N}}$ a sequence of linearly independent vectors in X so that $\overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{N}} = X$ and $\overline{\operatorname{span}}\{x_n \colon n \in I\} = Y$. We define $\mu \in \mathcal{B}$ as

$$\mu\Big(\sum_{i=1}^k a_i e_i\Big) := \Big\|\sum_{i=1}^k a_i x_i\Big\|_X, \quad k \in \mathbb{N},\ (a_i)_{i=1}^k \in \mathbb{Q}^k.$$

Then $\langle X_\mu \rangle_\equiv = \langle X \rangle_\equiv \subseteq \bigcup_{n=1}^\infty F_n$ and so, by the claim above, there exist $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ with $\mathbb{M}_\mu^{\gamma'} \cap F_m \neq \emptyset$ for every $\gamma' \in \mathcal{T}$ with $\gamma' \supseteq \gamma$. Consider now the space $Z := (\overline{\operatorname{span}}\{e_i \colon i \in I \cup \operatorname{rng}(\gamma)\}, \mu) \subseteq X_\mu$ which is isomorphic to $Y \oplus F$ for some finite-dimensional Banach space F. Fix some $\widetilde{I} \subseteq I$ such that $|I \setminus \widetilde{I}| = \dim F$ and such that $(I \setminus \widetilde{I}) \cap \operatorname{rng}(\gamma) = \emptyset$. Denote by Y' the subspace $\overline{\operatorname{span}}\{e_i \colon i \in \widetilde{I}\}$ and by E the finite dimensional subspace $\operatorname{span}\{e_i \colon i \in I \setminus \widetilde{I}\}$ isomorphic to F. It is easy to check that we have $Y \simeq Y' \oplus E$. Define $\widetilde{Z} := (\overline{\operatorname{span}}\{e_i \colon i \in \widetilde{I} \cup \operatorname{rng}(\gamma)\}, \mu)$. Then we have

$$\widetilde{Z} \simeq Y' \oplus F \simeq Y' \oplus E \simeq Y,$$

and so $\widetilde{Z} \not\simeq X$. Let $\varphi : \mathbb{N} \to \operatorname{rng}(\gamma) \cup \widetilde{I}$ be a bijection with $\varphi \supseteq \gamma$. We define $\nu \in \mathcal{B}$ by

$$\nu\Big(\sum_{i=1}^k a_i e_i\Big) := \mu\Big(\sum_{i=1}^k a_i e_{\varphi(i)}\Big), \quad k \in \mathbb{N},\ (a_i)_{i=1}^k \in \mathbb{Q}^k.$$

Clearly, $X_\nu \equiv \widetilde{Z} \not\simeq X$.

We *claim* that $\nu \in F_m$. This will be in contradiction with the fact that $F_m \subseteq \langle X \rangle_\simeq$. Since F_m is closed, it suffices to check that each basic open neighborhood of ν intersects F_m. Pick $v_1, \ldots, v_l \in V$ and $\varepsilon > 0$. We need to find $\mu' \in F_m$ so that $|\mu'(v_j) - \nu(v_j)| < \varepsilon$ for every $j \leq l$.

Let $L \in \mathbb{N}$, $L \geq |\gamma|$, be such that $v_1, \ldots, v_l \in \operatorname{span}\{e_i \colon i \leq L\}$. Since $\varphi|_{\{1,\ldots,L\}} \supseteq \gamma$, we may pick $\mu' \in \mathbb{M}_\mu^{\varphi|_{\{1,\ldots,L\}}} \cap F_m$. Then

$$\mu'\Big(\sum_{i=1}^L a_i e_i\Big) = \mu\Big(\sum_{i=1}^L a_i e_{\varphi(i)}\Big) = \nu\Big(\sum_{i=1}^L a_i e_i\Big), \quad (a_i)_{i=1}^L \in \mathbb{Q}^L.$$

In particular, $\mu'(v_j) = \nu(v_j)$, $j \leq l$, as desired. $\square$

In the remainder of this section we head towards the proof of Theorem 6.2. Following [28], we say that an increasing sequence $E_1 \subseteq E_2 \subseteq \dots$ of finite-dimensional subspaces of a separable Banach space X whose union is dense is a *paving of* X. A separable Banach space X is *determined by its pavings* if whenever Y is a Banach space for which there are pavings $\{E_n\}_{n=1}^\infty$ of X and $\{F_n\}_{n=1}^\infty$ of Y with $\sup_{n\in\mathbb{N}} d_{BM}(E_n, F_n) < \infty$, then Y is isomorphic to X. We refer the reader to [28] for details and examples.

We start with a straightforward generalization of Theorem 4.6.

Proposition 6.3. *Let X be a separable infinite-dimensional Banach space, $\{E_n\}_{n=1}^\infty$ paving of X and $\lambda \geq 1$. Then the set*

$$\big\{\mu \in \mathcal{P}_\infty : \text{for every } \varepsilon > 0 \text{ there is a paving } \{F_k\}_{k=1}^\infty \text{ of } X_\mu \text{ and an increasing sequence } (n_k)_{k=1}^\infty \in \mathbb{N}^\mathbb{N} \text{ with } \sup_{k\in\mathbb{N}} d_{BM}(F_k, E_{n_k}) \leq \lambda + \varepsilon\big\}$$

is G_δ in $\mathcal{P}_\infty$.

Proof. The proof is verbatim the same as the proof of Theorem 4.6 with the only exception that instead of ℓ_p^n we write E_n and we suppose we have some fixed basis of each E_n. □

Proof of Theorem 6.2. Pick a paving $\{E_n\}_{n=1}^\infty$ of X. It is easy to see that for every $\mu \in \mathcal{P}_\infty$ the Banach space X_μ is isomorphic to X if and only if μ belongs to the set from Proposition 6.3 for some $\lambda \geq 1$. The "In particular" part follows since $\mathcal{P}_\infty$ is G_δ in $\mathcal{P}$, see Corollary 1.5. For admissible topologies, the result follows by applying Theorem 1.17. □

7. Miscellaneous

7.1. **Superreflexive spaces.** Recall that a map $f : M \to N$ between metric spaces is called a C-bilipschitz embedding if

$$\forall x \neq y \in M : \; C^{-1} d_M(x, y) < d_N(f(x), f(y)) < C d_M(x, y).$$

Lemma 7.1. *Let M be a finite metric space and $C > 0$. The set $E(M, C)$ consisting of those $\mu \in \mathcal{P}$ such that M admits a C-bilipschitz embedding into X_μ is open in $\mathcal{P}$.*

Proof. Let $\mu \in E(M, C)$. Thus, there is a C-bilipschitz embedding $f : M \to X_\mu$. By perturbing the image of f if necessary, we may without loss of generality assume that $f(M) \subseteq V$.

Consider $\varepsilon > 0$ and the open neighborhood U_ε consisting of those $\mu' \in \mathcal{P}$ for which $|\mu(f(x) - f(y)) - \mu'(f(x) - f(y))| < \varepsilon$ for every $x, y \in M$. Then $U_\varepsilon \subseteq E(M, C)$ for $\varepsilon > 0$ small enough. Indeed, it suffices to choose ε smaller than

$$\min\{\min_{x\neq y\in M} C d_M(x, y) - \mu(f(x) - f(y)), \min_{x\neq y\in M} \mu(f(x) - f(y)) - C^{-1} d_M(x, y)\}.$$

The easy verification is left to the reader. □

Proposition 7.2. *Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of finite metric spaces and let $\mathcal{X}$ be the class of those Banach spaces X for which there exists a constant C such that for every $n \in \mathbb{N}$, M_n admits a C-bilipschitz embedding into X.*

Then $\mathcal{F} := \{\mu \in \mathcal{P}_\infty : X_\mu$ is in $\mathcal{X}\}$ is $G_{\delta\sigma}$ in $\mathcal{P}_\infty$.

Proof. Follows immediately from Lemma 7.1, because we have

$$\mathcal{F} = \mathcal{P}_\infty \cap \bigcup_{C>0} \bigcap_{n\in\mathbb{N}} E(M_n, C).$$

□

Bourgain in his seminal paper [7] found a sequence of finite metric spaces $(M_n)_{n\in\mathbb{N}}$ such that a separable Banach space is not superreflexive if and only if there exists a constant C such that for every $n \in \mathbb{N}$, M_n admits C-bilipschitz embedding into X. We refer the interested reader to [44, Section 9] for some more related facts and results. Thus, combining this result with Proposition 7.2 we obtain immediately the following.

Theorem 7.3. *The class of all superreflexive spaces is $F_{\sigma\delta}$ in $\mathcal{P}_\infty$.*

A metric space M is called *locally finite* if it is uniformly discrete and all balls in M are finite sets (in particular, every such M is at most countable). Let us mention a result by Ostrovskii by which a locally finite metric space bilipschitz embeds into a Banach space X if and only if all of its finite subsets admit uniformly bilipschitz embeddings into X, see [43] or [44, Theorem 2.6]. Thus, from Proposition 7.2 we obtain also the following.

Corollary 7.4. *Let M be a locally finite metric space. Then the set of those $\mu \in \mathcal{P}_\infty$ for which M admits a bilipschitz embeddings into X_μ is $G_{\delta\sigma}$ in $\mathcal{P}_\infty$.*

It is well-known that many important classes of separable Banach spaces are not Borel. This concerns e.g. reflexive spaces, spaces with separable dual, spaces containing ℓ_1, spaces with the Radon-Nikodým property, spaces isomorphic to $L_p[0,1]$ for $p \in (1,2) \cup (2,\infty)$, or spaces isomorphic to c_0. We refer to [4, page 130 and Corollary 3.3] and [33, Theorem 1.1] for papers which contain the corresponding results and to the monograph [12] and the survey [21] for some more information. Thus, e.g. in combination with Corollary 7.4, we see that none of those classes might be characterized as a class into which a given locally finite metric space bilipchitz embeds. Let us give an example of such a result which is related to [42, Problem 12.5(b)]. This is an elementary, but interesting application of the whole theory.

Corollary 7.5. *There does not exist a locally finite metric space M such that any separable Banach space X is not reflexive if and only if M admits a bilipschitz embeddings into X.*

Remark 7.6. Let us draw attention of the reader once more to the remarkable paper [42], where the authors found a metric characterization of reflexivity even though such a condition is necessarily non Borel (as mentioned above).

7.2. **Szlenk indices.** In this subsection we give estimates on the Borel classes of spaces with Szlenk index less than or equal to a given ordinal number. Note that it is a result by Bossard, see [4, Section 4], that those sets are Borel and their Borel classes are unbounded. So our contribution here is that we provide certain quantitative estimates from above. Similarly, we give an estimate on the Borel class of spaces with summable Szlenk index, which is a quantitative improvement of the result mentioned in [20, page 367]. Let us start with the corresponding definitions. Let X be a real Banach space and $K \subseteq X^*$ a w^*-compact set. Following [36], for $\varepsilon > 0$ we define $s_\varepsilon(K)$ as the Szlenk derivative of the set K (see Subsection 5.2) and then we inductively define $s_\varepsilon^\alpha(K)$ for an ordinal α by $s_\varepsilon^{\alpha+1}(K) := s_\varepsilon(s_\varepsilon^\alpha(K))$ and $s_\varepsilon^\alpha(K) := \bigcap_{\beta<\alpha} s_\varepsilon^\beta(K)$ if α is a limit ordinal. Given a real Banach space X, $Sz(X,\varepsilon)$ is the least ordinal α such that $s_\varepsilon^\alpha(B_{X^*}) = \emptyset$, if such an ordinal exists (otherwise we write $Sz(X,\varepsilon) = \infty$). The Szlenk index is defined by $Sz(X) = \sup_{\varepsilon>0} Sz(X,\varepsilon)$.

Recall that for a separable infinite-dimensional Banach space X the Szlenk index is either ∞ or ω^α for some $\alpha \in [1,\omega_1)$, see [36, Section 3].

Theorem 7.7. *Let $\alpha \in [1,\infty)$ be an ordinal. Then*

$$\{\mu \in \mathcal{P}_\infty \colon Sz(X_\mu) \leq \omega^\alpha\}$$

is a $\mathbf{\Pi}^0_{\omega^\alpha+1}$ *set in* $\mathcal{P}_\infty$.

Proof. Using Lemma 5.10, it is easy to prove by induction on n that the mapping $\mathcal{K}(B_{\ell_\infty}) \ni F \mapsto s^n_\varepsilon(F) \in \mathcal{K}(B_{\ell_\infty})$ is $\mathbf{\Sigma}^0_{2n+1}$-measurable for every $n \in \mathbb{N}$. Further, the mapping $\mathcal{K}(B_{\ell_\infty}) \ni F \mapsto s^\omega_\varepsilon(F) \in \mathcal{K}(B_{\ell_\infty})$ is $\mathbf{\Sigma}^0_{\omega+1}$-measurable. Indeed, for every open $V \subseteq B_{\ell_\infty}$, by compactness argument, we have

$$\{F\colon s^\omega_\varepsilon(F) \subseteq V\} = \bigcup_{n=1}^{\infty} \{F\colon s^n_\varepsilon(F) \subseteq V\}$$

which is a $\mathbf{\Sigma}^0_\omega$ set, so by Lemma 5.8 the mapping s^ω_ε is $\mathbf{\Sigma}^0_{\omega+1}$-measurable. Similarly, we prove by transfinite induction that s^β_ε is $\mathbf{\Sigma}^0_{\beta+1}$-measurable whenever $\beta \in [\omega, \omega_1)$ is a limit ordinal.

Let Ω be the mapping from Lemma 5.9. Then by Lemma 5.7 we have

$$\begin{aligned}\{\mu \in \mathcal{P}_\infty\colon Sz(X_\mu) \leq \omega^\alpha\} &= \bigcap_{k\in\mathbb{N}} \{\mu \in \mathcal{P}_\infty\colon Sz(X_\mu, \tfrac{1}{k}) \leq \omega^\alpha\} \\ &= \bigcap_{k\in\mathbb{N}} \{\mu \in \mathcal{P}_\infty\colon s^{\omega^\alpha}_{1/k}(\Omega(\mu)) = \emptyset\},\end{aligned}$$

which, by the above and Lemma 5.9, is a countable intersection of preimages of closed sets under $\mathbf{\Sigma}^0_{\omega^\alpha+1}$-measurable mapping, so it is a $\mathbf{\Pi}^0_{\omega^\alpha+1}$ set in $\mathcal{P}_\infty$. □

Let us recall that a Banach space X has *summable Szlenk index* if there is a constant M such that for all positive $\varepsilon_1, \dots, \varepsilon_n$ with $s_{\varepsilon_1} \dots s_{\varepsilon_n} B_{X^*} \neq \emptyset$ we have $\sum_{i=1}^n \varepsilon_i \leq M$.

Proposition 7.8. *The set* $\{\mu \in \mathcal{P}_\infty\colon X_\mu \text{ has a summable Szlenk index}\}$ *is a* $\mathbf{\Sigma}^0_{\omega+2}$ *set in* $\mathcal{P}_\infty$.

Proof. Let Ω be the mapping from Lemma 5.9. It is easy to see that the set $\{\mu \in \mathcal{P}_\infty\colon X_\mu \text{ has a summable Szlenk index}\}$ is equal to

$$\bigcup_{M\in\mathbb{N}} \bigcap_{\substack{\varepsilon_1,\dots,\varepsilon_n \in \mathbb{Q}_+ \\ \sum_{i=1}^n \varepsilon_i > M}} \{\mu \in \mathcal{P}_\infty\colon s_{\varepsilon_1} \dots s_{\varepsilon_n} \Omega(\mu) = \emptyset\},$$

which by Lemma 5.9 and Lemma 5.10 is a $\mathbf{\Sigma}^0_{\omega+2}$ set in $\mathcal{P}_\infty$. □

Finally, let us note that similarly one can of course estimate Borel complexity of various other classes of spaces related to Szlenk derivations, e.g. spaces with Szlenk power type at most p etc.

7.3. **Spaces having Schauder basis-like structures.** It is an open problem whether the class of spaces with Schauder basis is a Borel set in $\mathcal{B}$ (see e.g. [12, Problem 8]) and note that by the results from Section 1 it does not matter whether we use the coding $SB(C([0,1]))$ or $\mathcal{B}$. However, it was proved by Ghawadrah that the class of spaces with π-property is Borel (actually, it is $\mathbf{\Sigma}^0_6$ in $\mathcal{P}_\infty$ which follows immediately from [16, Lemma 2.1], see also [19]) and that the class of spaces with the bounded approximation property (BAP) is Borel (actually, it is $\mathbf{\Sigma}^0_7$ in $\mathcal{P}_\infty$ which follows immediately from [18, Lemma 2.1] and this estimate has recently been improved to $\mathbf{\Sigma}^0_6$ in any admissible topology, see [19]).

One is therefore led to the question of finding examples of Banach spaces having BAP but not the Schauder basis. Such an example was constructed by Szarek [47]. Actually, Szarek considered classes of separable spaces with *local basis structure (LBS)* and *local* Π-*basis structure (L*Π*BS)* for which we have

$$\text{basis} \implies (\text{L}\Pi\text{BS}) \implies \Big((\text{LBS}) \text{ and } (\text{BAP})\Big) \implies (\text{BAP})$$

and he proved that the converse to the second and the third implication does not hold in general. The problem of whether the converse to the first implication holds seems to be open, see [47, Problem 1.8]. In this subsection we prove that both (LBS) and (LΠBS) give rise to a Borel class of separable Banach spaces (we even compute an upper bound on their Borel complexities, see Theorem 7.13). Note that this result somehow builds a bridge between both open problems mentioned above, that is, between the problem of whether ⟨spaces with Schauder basis⟩ is a Borel set in $\mathcal{B}$ and the problem of whether every separable Banach space with (LΠBS) has a basis.

Let us start with the definitions as they are given in [47].

Definition 7.9. By the *basis constant* of a basis $(x_i)_{i=1}^d$ of a Banach space X of dimension $d \in [0,\infty]$ we mean the least number $C \geq 1$ such that $\|\sum_{i=1}^n a_i x_i\| \leq C\|\sum_{i=1}^m a_i x_i\|$ whenever $n,m \in \mathbb{N}$, $n \leq m \leq d$ and $a_1,\dots,a_m \in \mathbb{R}$. The basis constant of $(x_i)_{i=1}^d$ is denoted by $\mathrm{bc}((x_i)_{i=1}^d)$. We further denote

$$\mathrm{bc}(X) = \inf\big\{\mathrm{bc}((x_i)_{i=1}^d) : (x_i)_{i=1}^d \text{ is a basis of } X\big\}.$$

Definition 7.10. A Banach space X is said to have the *local basis structure (LBS)* if $X = \overline{\bigcup_{n=1}^\infty E_n}$, where $E_1 \subseteq E_2 \subseteq \dots$ are finite-dimensional subspaces satisfying $\sup_{n\in\mathbb{N}} \mathrm{bc}(E_n) < \infty$.

Further, X is said to have the *local* Π-*basis structure (L*Π*BS)* if $X = \overline{\bigcup_{n=1}^\infty E_n}$, where $E_1 \subseteq E_2 \subseteq \dots$ are finite-dimensional subspaces satisfying $\sup_{n\in\mathbb{N}} \mathrm{bc}(E_n) < \infty$ for which there are projections $P_n : X \to E_n$ such that $P_n(X) = E_n$ and $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$.

Lemma 7.11. *Whenever E is a finite-dimensional subspace of a Banach space X, $\delta \in (0,1)$, $K > 0$, $T : E \to X$ is a $(1+\delta)$-isomorphism (not necessarily surjective) with $\|T - I\| < \delta$ and $P : X \to E$ is a projection with $P(X) = E$ and $\|P\| \leq K$, then for every subspace F of E we have $\|TP|_{T(F)} - I_{T(F)}\| \leq 4\delta K$.*

Moreover, whenever $\|TP|_{T(E)} - I_{T(E)}\| \leq q < 1$ then $T(E)$ is $\frac{(1+\delta)K}{1-q}$-complemented in X.

Proof. Let $f_1,\dots,f_n$ be a basis of F. Then for every $x = \sum_{i=1}^n a_i T(f_i) \in T(F)$ we have

$$\|TPx - x\| = \|TP(\sum_{i=1}^n a_i(T(f_i) - f_i))\| \leq (1+\delta)K\|(T-I)T^{-1}x\| \leq (1+\delta)^2\delta K\|x\|.$$

Moreover, if $\|TP|_{T(E)} - I_{T(E)}\| < 1$ then the mapping $TP|_{T(E)}$ is an isomorphism with $\|(TP|_{T(E)})^{-1}\| \leq \sum_{i=0}^\infty q^i = \frac{1}{1-q}$. It is now straightforward to prove that $P' := (TP|_{T(E)})^{-1}TP : X \to T(E)$ is a projection onto $T(E)$ with $\|P'\| \leq \frac{(1+\delta)K}{1-q}$. □

Lemma 7.12. *For every $\mu \in \mathcal{B}$, $K, l \in \mathbb{N}$ and $v_1,\dots,v_m \in V$, let us denote by $\Phi(\mu,K,v_1,\dots,v_m)$ and $\Psi(\mu,K,l,v_1,\dots,v_m)$ the formulae*

$$\Phi(\mu,K,v_1,\dots,v_m) = \forall a_1,\dots,a_m \in \mathbb{R} : \max_{1\leq k\leq m} \mu\Big(\sum_{i=1}^k a_i v_i\Big) \leq K\mu\Big(\sum_{i=1}^m a_i v_i\Big)$$

and

$$\Psi(\mu,K,l,v_1,\dots,v_m) = \exists u_1,\dots,u_l \in \mathbb{Q}\text{-span}\{v_1,\dots,v_m\}\,\forall a_1,\dots,a_m,b_1,\dots,b_l \in \mathbb{R} :$$
$$\mu\Big(\sum_{i=1}^m a_i v_i + \sum_{i=1}^l b_i u_i\Big) \leq K\mu\Big(\sum_{i=1}^m a_i v_i + \sum_{i=1}^l b_i e_i\Big).$$

Then for every $\nu \in \mathcal{B}$ the following holds.

(a) The space X_ν has LBS if and only if

$$\exists K \in \mathbb{N} \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists v_1, \dots, v_m \in V, \{e_1, \dots, e_n\} \subseteq \operatorname{span}\{v_1, \dots, v_m\}$$

$$\Phi(\nu, K, v_1, \dots, v_m).$$

(b) The space X_ν has LΠBS if and only if

$$\exists K \in \mathbb{N} \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists v_1, \dots, v_m \in V, \{e_1, \dots, e_n\} \subseteq \operatorname{span}\{v_1, \dots, v_m\}$$

$$\Phi(\nu, K, v_1, \dots, v_m) \wedge \forall l \in \mathbb{N} \Psi(\nu, K, l, v_1, \dots, v_m).$$

Proof. We prove only the more difficult part (b). Since $\nu \in \mathcal{B}$, the space X_ν is just the completion of (c_{00}, ν) (it is not necessary to consider a quotient). So, the notions of linear span and of linear independence have the same meaning in c_{00} and in X_ν, if performed on subsets of c_{00}.

Let us suppose that $\nu \in \mathcal{B}$ satisfies the formula in (b) for some $K \in \mathbb{N}$. We put $E_0 = \{0\}$ and choose recursively subspaces $E_1 \subseteq E_2 \subseteq \dots$ of X_ν, each of which is generated by a finite number of elements of V, in the following way. Assuming that E_j has been already chosen, we pick first $n_{j+1} \geq j+1$ such that $E_j \subseteq \operatorname{span}\{e_1, \dots, e_{n_{j+1}}\}$. Then we can pick $m_{j+1} \in \mathbb{N}$ and $v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1} \in V$ with $\{e_1, \dots, e_{n_{j+1}}\} \subseteq \operatorname{span}\{v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1}\}$ such that $\Phi(\nu, K, v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1})$ and for every $l \in \mathbb{N}$, $\Psi(\nu, K, l, v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1})$ hold.

We put $E_{j+1} = \operatorname{span}\{v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1}\}$. In this way, we obtain $E_j \subseteq E_{j+1}$. Also, $X_\nu = \overline{\bigcup_{n=1}^\infty E_n}$ (we have $e_{j+1} \in E_{j+1}$, as $n_{j+1} \geq j+1$). If we take all non-zero vectors $v_i^{j+1}, 1 \leq i \leq m_{j+1}$, we obtain a basis of E_{j+1} with the basis constant at most K.

To show that the sequence $E_1 \subseteq E_2 \subseteq \dots$ witnesses that X_ν has LΠBS, it remains to find a projection P_{j+1} of X_ν onto E_{j+1} such that $\|P_{j+1}\| \leq K$. Let us pick some $l \in \mathbb{N}$ and put $E^{(l)} = \operatorname{span}\{v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1}, e_1, \dots, e_l\}$. By $\Psi(\nu, K, l, v_1^{j+1}, \dots, v_{m_{j+1}}^{j+1})$, there exists a projection $P^{(l)}$ of $E^{(l)}$ onto E_{j+1} with $\|P^{(l)}\| \leq K$. Since the norms of $P^{(l)}$, for $l \in \mathbb{N}$, are uniformly bounded and have a fixed finite-dimensional range, there exists their accumulation point in SOT which is a projection $P_{j+1} : X_\nu \to E_{j+1}$ of norm bounded by K as desired.

Conversely, suppose that X_ν has LΠBS as witnessed by some $C > 1$ and a sequence $(E_n)_{n \in \mathbb{N}}$ of finite-dimensional subspaces satisfying $X_\nu = \overline{\bigcup_n E_n}$ and $\sup_{n \in \mathbb{N}} \operatorname{bc}(E_n) < C$, for which there are projections $P_n : X_\nu \to E_n$ such that $P_n(X_\nu) = E_n$ and $\sup_{n \in \mathbb{N}} \|P_n\| < C$. Pick $D > 0$ such that $H_n := (\operatorname{span}\{e_1, \dots, e_n\}, \nu)$ is D-complemented in X_ν and let $\phi_1 := \phi^{e_1, \dots, e_n}$ be the function from Lemma 2.4(i). Fix $\varepsilon > 0$ such that $\phi_1(t)$ is small enough (to be specified later) whenever $t < \varepsilon$. Find $k \in \mathbb{N}$ such that there are $h_1, \dots, h_n \in E_k$ with $\nu(e_i - h_i) < \varepsilon$. If $\phi_1(\varepsilon)$ is small enough we have $\frac{(1+\phi_1(\varepsilon))D}{1-4\phi_1(\varepsilon)D} \leq 2D$ (this value refers to the "Moreover" part in Lemma 7.11). By Lemma 7.11, $\operatorname{span}\{h_i \colon i \leq n\}$ is $2D$-complemented in X_ν, so let $Q : X_\nu \to \operatorname{span}\{h_i \colon i \leq n\}$ be the corresponding projection. Pick a basis $h_{n+1}, \dots, h_{\dim E_k}$ of the space $E_k \cap Q^{-1}(0)$ which is $(2D+1)$-complemented in E_k. Let $\phi_2 := \phi^{h_{n+1}, \dots, h_{\dim E_k}}$ be the function from Lemma 2.4(i). Fix $\delta > 0$ such that $\phi_2(t)$ is small enough (to be specified later) whenever $t < \delta$. Finally, find $f_{n+1}, \dots, f_{\dim E_k} \in V$ with $\nu(f_j - h_j) < \delta$ for $j = n+1, \dots, \dim E_k$.

We *claim* that the space $F_n := (\operatorname{span}\{e_1, \dots, e_n, f_{n+1}, \dots, f_{\dim E_k}\}, \nu)$ is $2C$-complemented in X_ν and $d_{BM}(F_n, E_k) < 2$. If we denote by $T : E_k \to F_n$ the linear mapping given by $h_i \mapsto e_i$, $i \leq n$, and $h_j \mapsto f_j$, $n+1 \leq j \leq \dim E_k$, then for every $y \in \operatorname{span}\{h_i : i \le n\}$ and $z \in \operatorname{span}\{h_j : j = n+1, \dots, \dim E_k\}$ we have

$$\begin{aligned}\nu(T(y+z) - y - z) &\le \nu(Ty - y) + \nu(Tz - z) \le \phi_1(\varepsilon)\nu(y) + \phi_2(\delta)\nu(z)\\ &\le \Big(\phi_1(\varepsilon)2D + \phi_2(\delta)(2D+1)\Big)\nu(y+z);\end{aligned}$$

hence, if $\eta := \Big(\phi_1(\varepsilon)2D + \phi_2(\delta)(2D+1)\Big) < 1$, we obtain $\|T\| \le 1 + \|I - T\| \le 1 + \eta$ and $\|Tx\| \ge \|x\| - \|(I-T)x\| \ge (1-\eta)\|x\|$ for every $x \in E_k$ so T is an isomorphism with $\|T\|^{-1} \le (1-\eta)^{-1}$. Thus, by Lemma 7.11, if $\phi_1(\varepsilon)$ and $\phi_2(\delta)$ are small enough (and so η is small enough), we obtain $\|T\|\|T^{-1}\| < 2$ and F_n is $2C$-complemented in X_ν.

Thus, $\operatorname{bc}(F_n) \le \operatorname{bc}(E_k) d_{BM}(E_k, F_n) < 2C$ which is witnessed by some basis $v_1, \dots, v_m \in V$ of F_n. This shows that $\Phi(\nu, 2C, v_1, \dots, v_m)$ holds. Let $P : X_\nu \to F_n$ be a projection with $P[X_\nu] = F_n$ and $\|P\| \le 2C$. Given $l \in \mathbb{N}$, let $T \subseteq \{1, \dots, l\}$ be a set such that $(e_i)_{i\in T}$ together with $(v_i)_{i=1}^m$ form a basis of $\operatorname{span}(\{v_1, \dots, v_m\} \cup \{e_1, \dots, e_l\})$. Pick $A > 0$ such that $(v_i)_{i=1}^m \cup (e_i)_{i\in T} \overset{A}{\sim} \ell_1^{m+|T|}$. For $i \in T$ pick $u_i \in \operatorname{span}_{\mathbb{Q}}\{v_1, \dots, v_m\}$ such that $\nu(u_i - P(e_i)) < \frac{C}{A}$. Then for every $a_1, \dots, a_m \in \mathbb{R}$ and every $(b_i)_{i\in T} \in \mathbb{R}^T$ we have

$$\begin{aligned}\nu\Big(\sum_{i=1}^m a_i v_i + \sum_{i\in T} b_i u_i\Big) &\le 2C\nu\Big(\sum_{i=1}^m a_i v_i + \sum_{i\in T} b_i e_i\Big) + \nu\big(\sum_{i\in T} b_i(u_i - P(e_i))\big)\\ &\le 3C\nu\Big(\sum_{i=1}^m a_i v_i + \sum_{i\in T} b_i e_i\Big).\end{aligned}$$

Thus, the linear mapping $O : \operatorname{span}(\{v_1, \dots, v_m\} \cup \{e_1, \dots, e_l\}) \to \operatorname{span}\{v_1, \dots, v_m\}$ given by $v_i \mapsto v_i$, $i \le m$, and $e_i \mapsto u_i$, $i \in T$, is a linear projection, and if we put $u_i := O(e_i) \in V$ for every $i \in \{1, \dots, l\}$, we see that $\Psi(\nu, 3C, l, v_1, \dots, v_m)$ holds and the formula in (b) is satisfied with $K = 3C$. □

Theorem 7.13. *(a) The class of spaces which have LBS is* $\mathbf{\Sigma}_4^0$ *in* $\mathcal{B}$.
(b) The class of spaces which have LΠBS is $\mathbf{\Sigma}_6^0$ *in* $\mathcal{B}$.

Proof. This follows from Lemma 7.12 because the conditions given by formulas Φ and Ψ are obviously closed and F_σ, respectively. □

8. Open questions and remarks

In Section 1 we investigated three ways of formalizing the class of all separable infinite-dimensional Banach spaces as a Polish space. Those were denoted by $\mathcal{P}_\infty$, $\mathcal{B}$ and $SB_\infty(X)$. We obtained an optimal reduction from $\mathcal{B}$ to $SB_\infty(X)$ and from $SB_\infty(X)$ to $\mathcal{P}_\infty$. However, our reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$ seems not to be optimal, so one is tempted to ask the following.

Question 1. Does there exist a continuous mapping $\Phi : \mathcal{P}_\infty \to \mathcal{B}$ such that for every $\mu \in \mathcal{P}_\infty$ we have $X_\mu \equiv X_{\Phi(\mu)}$?

Note that a positive answer to Question 1 would imply a positive answer to Question 2 and that a sufficient condition for a positive solution of Question 2 is provided by Proposition 1.29.

Question 2. Let X be an isometrically universal separable Banach space and let τ be an admissible topology on $SB(X)$. Does there exist a $\mathbf{\Sigma}_2^0$-measurable mapping $\Phi : \mathcal{P}_\infty \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}_\infty$ we have $X_\mu \equiv \Phi(\mu)$?

In Theorem 6.1 we proved that ℓ_2 is the unique separable infinite-dimensional Banach space (up to isomorphism) whose isomorphism class is F_σ. Following [28], we say that a separable infinite-dimensional Banach space X is *determined by its* *finite dimensional subspaces* if it is isomorphic to every separable Banach space Y which is finitely crudely representable in X and for which X is finitely crudely representable in Y. Note that ℓ_2 is determined by its finite dimensional subspaces and that if a separable infinite-dimensional Banach space is determined by its finite dimensional subspaces then it is obviously determined by its pavings and so, by Theorem 6.2, its isomorphism class is $G_{\delta\sigma}$. Johnson, Lindenstrauss, and Schechtman conjectured (see [28, Conjecture 7.3]) that ℓ_2 is the unique, up to isomorphism, separable infinite-dimensional Banach space which is determined by its finite dimensional subspaces. We believe that Theorem 6.1 could be instrumental for proving this conjecture, since it follows from this theorem that the conjecture is equivalent to the positive answer to the following question. We thank Gilles Godefroy who suggested to us that there might be a relation between having F_σ isomorphism class and being determined by finite dimensional subspaces.

Question 3. Let X be a separable infinite-dimensional Banach space determined by its finite dimensional subspaces. Is $\langle X\rangle_{\simeq}$ F_σ in $\mathcal{B}$?

It would be interesting to know whether there is a separable infinite-dimensional Banach space X such that $\langle X\rangle_{\simeq}$ is G_δ in $\mathcal{B}$ or in $\mathcal{P}_\infty$. Note that the only one possible candidate is the Gurariĭ space, see Section 3 for more details. One of the possible strategies to answer Question 4 in negative for $\mathcal{P}_\infty$ would be to find an admissible topology τ on $SB(X)$ such that $\langle \mathbb{G}\rangle_{\simeq}$ is a dense and meager set in $(SB(X),\tau)$. However, we do not even know whether $\langle \mathbb{G}\rangle_{\simeq}$ is Borel.

Question 4. Is $\langle \mathbb{G}\rangle_{\simeq}$ a G_δ set in $\mathcal{P}_\infty$ or in $\mathcal{B}$? Is it at least Borel?

Solving the homogeneous Banach space problem, Komorowski and Tomczak-Jaegermann ([30]), and Gowers ([24]) proved that if a separable infinite-dimensional Banach space is isomorphic to all of its closed infinite-dimensional subspaces, then it is isomorphic to ℓ_2. It seems that the isometric variant of this result is open; that is, whether ℓ_2 is the only separable infinite-dimensional Banach space that is isometric to all of its infinite-dimensional closed subspaces. We note that any Banach space satisfying this criterion must be, by the Gowers' result, isomorphic to ℓ_2. Our initial interest in this problem was that we observed that a positive answer implies that whenever $\langle X\rangle_{\equiv}$ is closed in $\mathcal{P}_\infty$ then $X \equiv \ell_2$. Eventually we found another argument (see Section 3), but the question is clearly of independent interest.

Question 5. Let X be a separable infinite-dimensional Banach space which is isometric to all of its closed infinite-dimensional subspaces. Is then X isometric to ℓ_2?

In Sections 2, 3 and 4 we proved that $\langle \mathbb{G}\rangle_{\equiv}$, resp. $\langle L_p[0,1]\rangle_{\equiv}$, for $p \in [1,\infty)$, are G_δ; we even proved that they are dense G_δ in $\mathcal{P}_\infty$, resp. in $\mathcal{L}_{p,1+} \cap \mathcal{P}_\infty$. Coincidentally, all these spaces are Fraïssé limits (we refer to [13, Proposition 3.7] for this statement about $L_p[0,1]$. According to [13], no other examples of separable Banach spaces which are Fraïssé limits seem to be known. This motivates us to ask the following.

Question 6. Does there exist a separable infinite-dimensional Banach space X which is not isometric to $L_p[0,1]$, for $p \in [1,\infty)$, and to the Gurariĭ space, and $\langle X\rangle_{\equiv}$ is G_δ in $\mathcal{P}_\infty$ or in $\mathcal{B}$?

It also follows that for $1 \le p < \infty$, $L_p[0,1]$ is a generic $\mathcal{L}_{p,1+}$-space. On the other hand, by Corollary 3.12, for $p \in [1,2)\cup(2,\infty)$, $L_p[0,1]$ is not a generic QSL_p-space. For $p = 2$, ℓ_2 is obviously the generic QSL_2-space, and since QSL_1-spaces coincide with the class of all Banach spaces, for $p = 1$, $\mathbb{G}$ is the generic QSL_1-space. This leaves open the next question.

Question 7. For $p \in (1,2) \cup (2,\infty)$, does there exist a generic QSL_p-space in $\mathcal{B}$ or $\mathcal{P}_\infty$?

In Theorem 7.3, we have computed that the class of superreflexive spaces is $F_{\sigma\delta}$. It is easy to check that the class of superreflexive spaces is dense in $\mathcal{P}_\infty$ and $\mathcal{B}$, so it cannot be G_δ as then this class would have a non-empty intersection with the isometry class of $\mathbb{G}$ which is not superreflexive. However, the following is not known to us.

Question 8. Is the class of all superreflexive spaces $F_{\sigma\delta}$-complete in $\mathcal{P}_\infty$ or $\mathcal{B}$?

Taking into account that spaces with summable Szlenk index form a class of spaces which is $\mathbf{\Sigma}^0_{\omega+2}$, see Proposition 7.8, the following seems to be an interesting problem.

Question 9. Is the set $\{\mu \in \mathcal{P}_\infty \colon X_\mu \text{ has a summable Szlenk index}\}$ of a finite Borel class?

Even though we do not formulate it as a numbered question, a natural project to consider is to determine at least upper bounds for isometry classes of other (classical or less classical) separable infinite-dimensional Banach spaces, such as $C[0,1]$, $C([0,\alpha])$ with α countable ordinal, Orlicz sequence spaces, Orlicz function spaces, spaces of absolutely continuous functions, Tsirelson's space, etc.

Kechris in [29, page 189] mentions that there are not known any natural examples of Borel sets from topology or analysis that are $\mathbf{\Pi}^0_\xi$ or $\mathbf{\Sigma}^0_\xi$, for $\xi \geq 5$, and not of lower complexity. We think that the area of research investigated in this paper is a good one to find such examples.

Acknowledgements. We would like to thank William B. Johnson for his helpful advice concerning the structure of subspaces and quotients of L_p spaces (see Section 3), Gilles Godefroy for his suggestion to have a look at the paper [28], and Ghadeer Ghawadrah for sending us a preliminary version of her paper [19].

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