Finite volume method for first order hyperbolic problems

V. Dolejší

Charles University Prague, Faculty of Mathematics and Physics

Lecture 2

Overview of PDEs: physics, analysis, numerics

Elliptic/parabolic PDEs

- $-\nabla \cdot (\mathbb{K}(u)\nabla u) = f$
- describe diffusive effects
- "everything" is spread out in all directions
- effect is decreasing w.r.t. distance

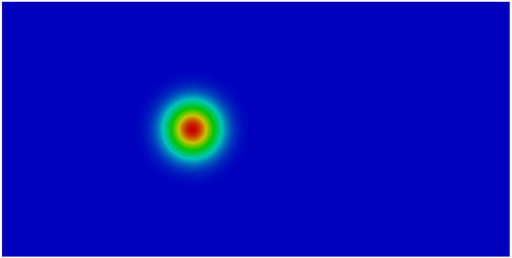
Hyperbolic PDEs (1st order)

- $\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) = 0$
- describe advection/convection
- "everything" is spread out in the dominant direction
- effect is decreasing w.r.t. distance only due to diffusion

Diffusion vs. convection

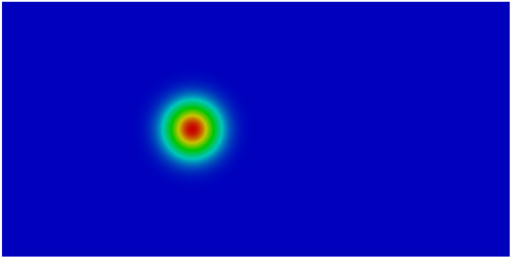
diffusion

$\frac{\partial u}{\partial t} - \Delta u = 0$



convection

$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} = 0$



Diffusion vs. convection

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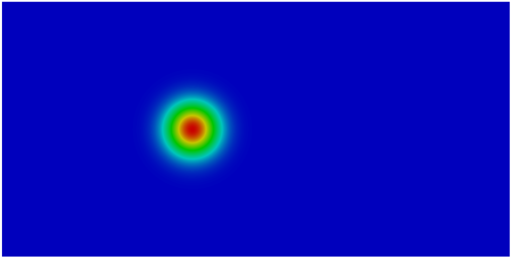

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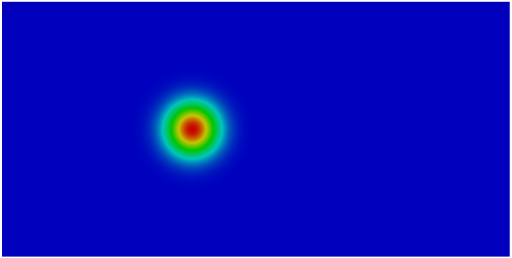


First order hyperbolic problems – linear case

Scalar linear problem

Find a function $u : \Omega \times (0, t) \to \mathbb{R}$ ($\Omega \in \mathbb{R}^d$) such that

$$\frac{\partial u}{\partial t} + \boldsymbol{a} \cdot \nabla u = g(= 0), \qquad (1)$$

where g is given functions, $0 \neq \boldsymbol{a} = (a_1, \ldots, a_d)^{\mathrm{T}} \in \mathbb{R}^d$ is a given vector, $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d})^{\mathrm{T}}$, $\boldsymbol{a} \cdot \nabla u = \sum_{s=1}^{d} a_s \frac{\partial u}{\partial x_s}$

Vector (system) linear problem

Find a function $\boldsymbol{u} : \Omega \times (0, t) \to \mathbb{R}^n$ ($\Omega \in \mathbb{R}^d$) such that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \mathbb{A}_s \frac{\partial \boldsymbol{u}}{\partial x_s} = 0, \qquad (2)$$

where $\mathbb{A}_s \in \mathbb{R}^{n \times n}$, $s = 1, \ldots, d$ are given matrices.

First order hyperbolic problems – nonlinear case

Scalar non-linear problem

Find a function $u : \Omega \times (0, t) \to \mathbb{R}$ ($\Omega \in \mathbb{R}^d$) such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(\boldsymbol{u})}{\partial x_s} = 0, \tag{3}$$

where $f_s : \mathbb{R} \to \mathbb{R}$, $s = 1, \dots, d$ is a given function.

Vector (system) non-linear problem

Find a function $\boldsymbol{u} : \Omega \times (0, t) \to \mathbb{R}^n$ ($\Omega \in \mathbb{R}^d$) such that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = 0, \tag{4}$$

where $\boldsymbol{f}_s : \mathbb{R}^n \to \mathbb{R}^n$, $s = 1, \dots, d$ are given functions.

Initial and boundary conditions

Initial conditions

- scalar case: $u(x,0) = u_0(x)$ for $x \in \Omega$
- vector case: $\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)$ for $x \in \Omega$

Boundary conditions

- for linear problem can be set
- non-linear problem, a difficult task, some heuristics

Properties

for nonlinear problems:

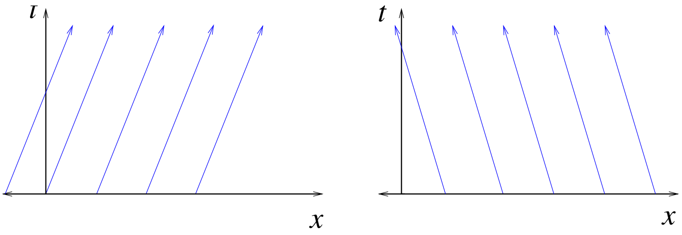
- solution may be not unique
- solution can contain discontinuities even for smooth data

Cauchy problem

- let $\Omega = \mathbb{R}$, $u_0 : \mathbb{R} \to \mathbb{R}$ be the initial condition, $a \neq 0$

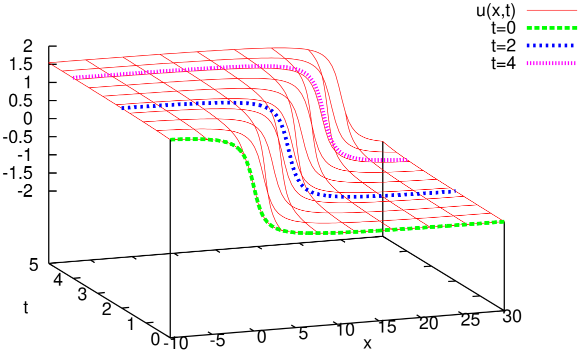
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad \text{for } (x, t) \in (-\infty, \infty) \times (0, \infty), \tag{5}$$

- exact solution $u(x, t) = u_0(x - at)$,
- solution is constant along lines $x - at = \text{const}$ (characteristics)



$a > 0$ $\qquad$ $a < 0$

Example of the propagation



$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = 0, \qquad u_0(x) = \arctan(-x)$$

$$\Longrightarrow u(x,t) = \arctan(-(x-5t))$$

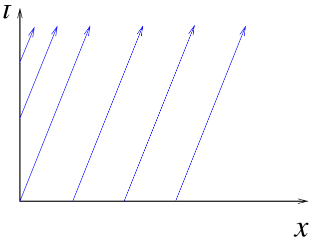
1D scalar linear equation + boundary conditions

Problem in half domain

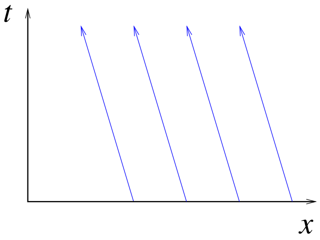
- let $\Omega = (0, \infty)$, $u_0 : \Omega \to \mathbb{R}$ be the initial condition, $a \neq 0$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad \text{for } (x,t) \in (0,\infty) \times (0,\infty), \qquad (6)$$

- boundary condition in $x = 0$?

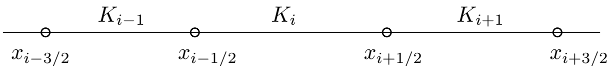


$a > 0$



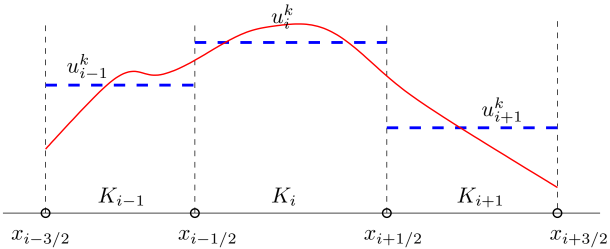
$a < 0$

Finite volume approximation

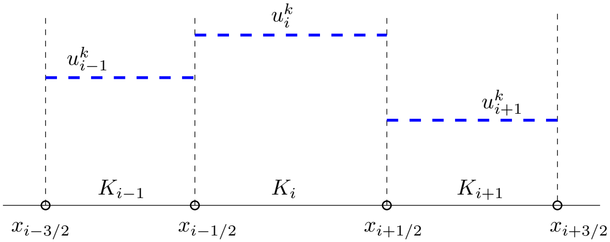


- space partition: $\{x_{i+1/2}\}_{i\in\mathbb{Z}}$, $K_i = (x_{i-1/2}, x_{i+1/2})$,
- time partition: $0 = t_0 < t_1 < \cdots < t_r = T$, $\tau_k := t_{k-1} - t_k$,
- piecewise constant approximation:

$$u_i^k := \frac{1}{|K_i|}\int_{K_i} u(x,t_k)\,\mathrm{d}x \approx u(\cdot,t_k)|_{K_i} \qquad (7)$$



Finite volume approximation (2)



FVM discretization

- integrating

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{8}$$

over $K_i \times (t_k, t_{k+1})$, Green's

$$|K_i|(u_i^{k+1} - u_i^k) = \int_{t_k}^{t_{k+1}} [au(\cdot, t)]_{x_{i-1/2}}^{x_{i+1/2}} \,\mathrm{d}t \approx \tau_k [au(\cdot, t_k)]_{x_{i-1/2}}^{x_{i+1/2}}$$

Finite volume approximation (2)

central differences

$$au(\cdot, t_k)|_{x_{i+1/2}} \approx a(u_i^k + u_{i+1}^k)/2$$

unconditionally unstable scheme

upwinding

information from the opposite direction of characteristics

$$au(\cdot, t_k)|_{x_{i+1/2}} \approx \begin{cases} au_i^k & \text{if } a > 0, \\ au_{i+1}^k & \text{if } a < 0. \end{cases}$$

conditionally stable scheme if $\tau_k < a \max_i |K_i|$

Numerical flux

$$au(\cdot, t_k)|_{x_{i+1/2}} \approx H(u_i^k, u_{i+1}^k, \boldsymbol{n}_i) := \max(a, 0)u_i^k + \min(a, 0)u_{i+1}^k$$

Extension to the system of equations

System of equations

we seek $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^n$ such that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathbb{A}\frac{\partial \boldsymbol{u}}{\partial x} = 0, \qquad \mathbb{A} \in \mathbb{R}^{n\times n} \tag{9}$$

Properties

let $\mathbb{A} = \mathbb{T}^{-1}\mathbb{L}\mathbb{T}$, we put $\mathbb{A}^{\pm} := \mathbb{T}^{-1}\mathbb{L}^{\pm}\mathbb{T}$,

Numerical flux

$$\mathbb{A}\boldsymbol{u}(\cdot, t_k)|_{x_{i+1/2}} \approx \mathrm{H}(\boldsymbol{u}_i^k, \boldsymbol{u}_{i+1}^k, \boldsymbol{n}_i) = \mathbb{A}^{+}\boldsymbol{u}_i^k + \mathbb{A}^{-}\boldsymbol{u}_{i+1}^k$$

1D nonlinear hyperbolic problem

Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad f : \mathbb{R} \to \mathbb{R}, \tag{10}$$
$$u(x, 0) = u_0(x)$$

Exact solution

$$u(x,t) = u_0(x - f'(u(x,t))t) \quad (x,t) \in \mathbb{R} \times (0,T) \tag{11}$$

implicit relation

Characteristics

$$x - f'(u(x,t))t = \text{const}, \quad (x,t) \in \mathbb{R} \times (0,T) \tag{12}$$

FVM for nonlinear equation

Idea: linearization

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial t} + f'(\bar{u})\frac{\partial u}{\partial x} = 0,$$

Explicit FVM discretization

$$u_i^{k+1} - u_i^k = \tau_k [f(u(\cdot, t_k))]_{x_{i-1/2}}^{x_{i+1/2}}$$

numerical flux

$$\begin{aligned} f(u(\cdot, t_k))|_{x_{i+1/2}} &\approx H(u_i^k, u_{i+1}^k, \boldsymbol{n}_i) \\ &:= \max(f'(\langle u \rangle), 0) u_i^k + \min(f'(\langle u \rangle), 0) u_{i+1}^k \\ &= (f'(\langle u \rangle))^+ u_i^k + (f'(\langle u \rangle))^- u_{i+1}^k \end{aligned}$$

$$\langle u \rangle = (u_i^k + u_{i+1}^k)/2$$

Multidimensional non-linear hyperbolic problem

- Let $\Omega \subset \mathbb{R}^d$, $T > 0$, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^n$, we seek $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^n$

Hyperbolic problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = 0, \qquad (\boldsymbol{f}_1, \dots, \boldsymbol{f}_d) : \mathbb{R}^n \to \mathbb{R}^n, \tag{13}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x)$$

boundary conditions

complicated non-linear problem, n different characteristics

FVM for nonlinear system in $\Omega \subset \mathbb{R}^d$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = 0, \qquad (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_d) : \mathbb{R}^n \to \mathbb{R}^n,$$

- finite volume mesh $\mathcal{T}_h = \{K_i\}$, time partition
- $\boldsymbol{u}_i^k = \frac{1}{|K_i|} \int_{K_i} \boldsymbol{u}(x, t_k) \, \mathrm{d}x, \;\; \partial K = \sum_{j \in s(i)} \Gamma_{ij}$

Explicit FVM discretization

$$\boldsymbol{u}_i^{k+1} - \boldsymbol{u}_i^k = \frac{\tau_k}{|K_i|} \int_{\partial K_i} \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot, t_k)) n_s \, \mathrm{d}S \tag{14}$$

numerical flux

$$\sum_{s=1}^{d} \boldsymbol{f}_i(\boldsymbol{u}(\cdot, t_k)) n_s|_{\Gamma_{ij}} \approx \mathrm{H}(\boldsymbol{u}_i^k, \boldsymbol{u}_j^k, \boldsymbol{n}_{ij})$$

K_j is neighbour of K_i through face Γ_{ij} with normal $\boldsymbol{n}_{ij}$

Numerical flux $\mathrm{H}(\cdot,\cdot,\cdot)$

Properties of H

- consistent: $\mathrm{H}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{n}) = \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}) n_s$,
- conservative: $\mathrm{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = -\mathrm{H}(\boldsymbol{u}_2,\boldsymbol{u}_1 - \boldsymbol{n})$.
- (local) Lipschitz continuity, monotonicity, etc.

Examples of H

- Lax-Friedrichs

$$\mathrm{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = \frac{1}{2}\sum_{s=1}^{d} \left(\boldsymbol{f}_s(\boldsymbol{u}_1) + \boldsymbol{f}_s(\boldsymbol{u}_2)\right) n_s - \frac{1}{\lambda}(\boldsymbol{u}_1 - \boldsymbol{u}_2),$$

- Vijayasundaram

$$\mathrm{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = \mathbb{P}^{+}(\langle \boldsymbol{u}\rangle,\boldsymbol{n})\boldsymbol{u}_1 + \mathbb{P}^{-}(\langle \boldsymbol{u}\rangle,\boldsymbol{n})\boldsymbol{u}_2,$$

where $\mathbb{P}^{\pm}$ are the pos/neg parts of $\mathbb{P} := \frac{D}{D\boldsymbol{u}}(\sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}) n_s)$

FEM vs. FVM

Cauchy problem

$$u(x,t) : \mathbb{R} \times (0,T) \to \mathbb{R} : \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 10^{-5}\frac{\partial^2 u}{\partial x^2}$$

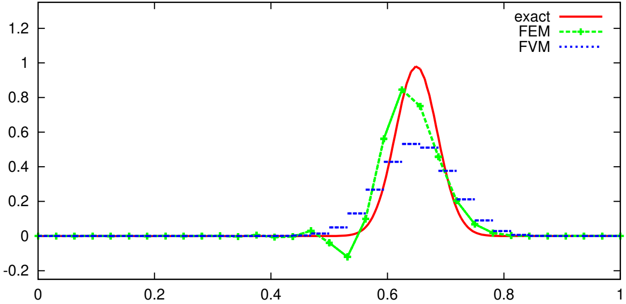
$$u(x,0) = \exp[(x - 1/4)^2]$$

Numerical methods

- FEM – P_1 continuous approximation
- FVM – P_0 approximation

numerical solutions

FEM vs. FVM (2)



Numerical methods – observation

- FEM – numerical solution oscillates
- FVM – very diffusive method

FEM vs. FVM (3)

FEM

- high order of accuracy
- many theoretical results
- efficient for elliptic and parabolic problems

FVM

- low order of accuracy
- lack of theory
- works for hyperbolic problems with discontinuities

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discontinuous Galerkin method

- piecewise polynomial discontinuous approximation
- theoretical justification
- higher freedom (adaptation, parallelization, etc.)