

Discontinuous Galerkin method for first order hyperbolic problems

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Lecture 3

Nonlinear hyperbolic system of 1st order

$\Omega \subset \mathbb{R}^d$, $(\mathbf{f}_1, \dots, \mathbf{f}_d) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^n$

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{s=1}^d \frac{\partial \mathbf{f}_s(\mathbf{u})}{\partial x_s} = 0 \quad \text{in } \Omega \times (0, T),$$

- mesh of Ω : $\mathcal{T}_h = \{K_i\}$,
- $\mathbf{u}_i(t) := \frac{1}{|K_i|} \int_{K_i} \mathbf{u}(x, t) dx$, $t \in [0, T]$
- integration over K_i and the Green theorem

Space semi-discretization

$$\frac{\partial}{\partial t} \mathbf{u}_i(t) = \frac{1}{|K_i|} \int_{\partial K_i} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s dS, \quad t \in (0, T) \quad (1)$$

FVM for nonlinear system (2)

Space semi-discretization

$$\frac{\partial}{\partial t} \mathbf{u}_i(t) = \frac{1}{|K_i|} \sum_{\Gamma \subset \partial K_i} \int_{\Gamma} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s dS, \quad t \in (0, T) \quad (2)$$

Numerical flux

$$\sum_{s=1}^d \mathbf{f}_i(\mathbf{u}(\cdot, t_k)) n_s|_{\Gamma} \approx \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma})$$

$\mathbf{u}_{\Gamma}^{(L)}$ and $\mathbf{u}_{\Gamma}^{(R)}$ are values on $K_{\Gamma}^{(L)}$ and $K_{\Gamma}^{(R)}$ sharing Γ , normal \mathbf{n}_{Γ}

FV scheme

$$\frac{\partial}{\partial t} \mathbf{u}_i(t) = \frac{1}{|K_i|} \sum_{\Gamma \subset \partial K_i} \int_{\Gamma} \sum_{s=1}^d \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) dS, \quad t \in (0, T)$$

Numerical flux $\mathbf{H}(\cdot, \cdot, \cdot)$

Properties of \mathbf{H}

- consistent: $\mathbf{H}(\mathbf{u}, \mathbf{u}, \mathbf{n}) = \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}) n_s$,
- conservative: $\mathbf{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}) = -\mathbf{H}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{n})$.
- (local) Lipschitz continuity, monotonicity, etc.

Examples of \mathbf{H}

- Lax-Friedrichs

$$\mathbf{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}) = \frac{1}{2} \sum_{s=1}^d (\mathbf{f}_s(\mathbf{u}_1) + \mathbf{f}_s(\mathbf{u}_2)) n_s - \frac{1}{\lambda} (\mathbf{u}_1 - \mathbf{u}_2),$$

- Vijayasundaram

$$\mathbf{H}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}) = \mathbb{P}^+(\langle \mathbf{u}, \mathbf{n} \rangle) \mathbf{u}_1 + \mathbb{P}^-(\langle \mathbf{u}, \mathbf{n} \rangle) \mathbf{u}_2,$$

where \mathbb{P}^\pm are the pos/neg parts of $\mathbb{P} := \frac{D}{D\mathbf{u}} (\sum_{s=1}^d \mathbf{f}_s(\mathbf{u}) n_s)$

Nonlinear hyperbolic system of 1st order

$$\text{find } \mathbf{u}(x, t) : \frac{\partial \mathbf{u}}{\partial t} + \sum_{s=1}^d \frac{\partial \mathbf{f}_s(\mathbf{u})}{\partial x_s} = 0 \quad \text{in } \Omega \times (0, T), \quad (3)$$

- mesh of Ω : $\mathcal{T}_h = \{K_i\}$,
- multiply (3) by $\varphi \in \mathbf{H}^1(\Omega, \mathcal{T}_h)$, \int_K , Greens theorem, $\sum_{K \in \mathcal{T}_h}$

Space semi-discretization

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx \\ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s \varphi \, dS = 0 \quad \forall t \end{aligned} \quad (4)$$

Employing the identity: $\sum_{K \in \mathcal{T}_h} \int_{\partial K} = \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma}$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx & \quad (5) \\ + \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s \varphi \, dS = 0 \end{aligned}$$

If \mathbf{u} is continuous over all Γ then a manipulation gives

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} & \quad (6) \\ + \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s [\varphi] \, dS = 0, \quad \forall t, \end{aligned}$$

Physical flux \approx numerical flux

$$\sum_{s=1}^d \mathbf{f}_i(\mathbf{u}(\cdot, t_k)) n_s|_{\Gamma} \approx \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma})$$

DGM for nonlinear system – an alternative

back to relation (5)

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx \\ + \sum_{K \in \mathcal{T}_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) n_s \varphi \, dS = 0 \end{aligned}$$

$$\sum_{s=1}^d \mathbf{f}_i(\mathbf{u}(\cdot, t_k)) n_s|_{\Gamma} \approx \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma})$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx \quad (7) \\ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \sum_{s=1}^d \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) \varphi \, dS = 0 \end{aligned}$$

relation (7):

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx \\ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \sum_{s=1}^d \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) \varphi \, dS = 0 \end{aligned}$$

Using the conservativity: $\mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) = -\mathbf{H}(\mathbf{u}_{\Gamma}^{(R)}, \mathbf{u}_{\Gamma}^{(L)}, -\mathbf{n}_{\Gamma})$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \mathbf{u}(\cdot, t) \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}(\cdot, t)) \frac{\partial \varphi}{\partial x_s} \, dx \\ + \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \mathbf{H}(\mathbf{u}_{\Gamma}^{(L)}, \mathbf{u}_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) [\varphi] \, dS = 0, \quad \forall t, \end{aligned} \quad (8)$$

Approximate solution by DGM

- $\mathbf{S}_{hp} \subset \mathbf{H}^1(\Omega, \mathcal{T}_h)$... discontinuous pw polynomial functions
- $\mathbf{u}_h \in C^1([0, T]; \mathbf{S}_{hp})$
- such that

$$\frac{\partial}{\partial t}(\mathbf{u}_h(t), \boldsymbol{\varphi}_h) + \mathbf{b}_h(\mathbf{u}_h(t), \boldsymbol{\varphi}_h) = 0 \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_{hp} \quad \forall t \in (0, T)$$

where

$$\begin{aligned} \mathbf{b}_h(\mathbf{u}_h, \boldsymbol{\varphi}_h) = & - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^d \mathbf{f}_s(\mathbf{u}_h) \frac{\partial \varphi_h}{\partial x_s} dx \\ & + \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \sum_{s=1}^d \mathbf{H}(\mathbf{u}_h|_{\Gamma}^{(L)}, \mathbf{u}_h|_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) [\varphi_h] dS. \end{aligned}$$

- $\mathbf{u}_h(0)$ is given by initial condition

System of ODEs, has to be solved

Numerical flux through $\partial\Omega$

$$\sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} \sum_{s=1}^d \mathbf{H}(\mathbf{u}_h|_{\Gamma}^{(L)}, \mathbf{u}_h|_{\Gamma}^{(R)}, \mathbf{n}_{\Gamma}) [\varphi_h] dS.$$

- if $\Gamma \subset \partial\Omega$, how to set $\mathbf{u}_h|_{\Gamma}^{(R)}$?
- number of BCs = the number of incoming characteristics
- physical flux

$$\mathbb{P}(\mathbf{u}, \mathbf{n}) := \sum_{s=1}^2 \frac{D}{D\mathbf{u}} \mathbf{f}_s(\mathbf{u}) \mathbf{n}_s$$

- if eigenvalue of \mathbb{P} on $\partial\Omega$ is negative then the corresponding characteristic is incoming

Number of BCs

- 2D inviscid compressible flow, $\mathbf{u} = (\rho, v_1, v_2, e)^T$
- eigenvalues: $\mathbf{v} \cdot \mathbf{n}$, $\mathbf{v} \cdot \mathbf{n}$, $\mathbf{v} \cdot \mathbf{n} - a$, $\mathbf{v} \cdot \mathbf{n} + a$,
 a speed of sound
- we distinguish:
 - subsonic inlet $(\mathbf{v} \cdot \mathbf{n} < 0, |\mathbf{v} \cdot \mathbf{n}| < a)$: 3 BCs
 - supersonic inlet $(\mathbf{v} \cdot \mathbf{n} < 0, |\mathbf{v} \cdot \mathbf{n}| > a)$: 4 BCs
 - subsonic outlet $(\mathbf{v} \cdot \mathbf{n} > 0, |\mathbf{v} \cdot \mathbf{n}| < a)$: 1 BCs
 - supersonic outlet $(\mathbf{v} \cdot \mathbf{n} > 0, |\mathbf{v} \cdot \mathbf{n}| > a)$: 0 BCs

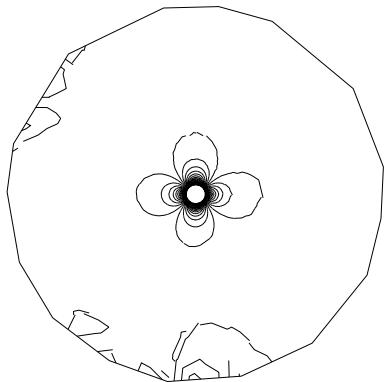
Setting of BC

Problem dependent

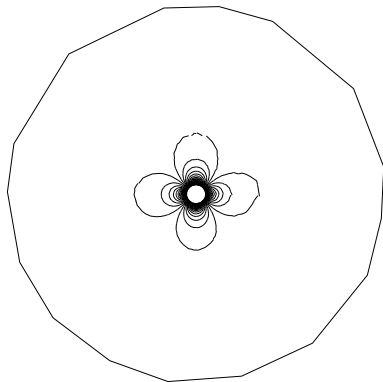
- flow around isolated profile: far-field BC (flow at ∞)
- flow in a channel (tube): ration between inlet/outlet pressure
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Example of far-filed BCs: flow around isolated circle

artificial boundary 20 times larger then diameter of the circle



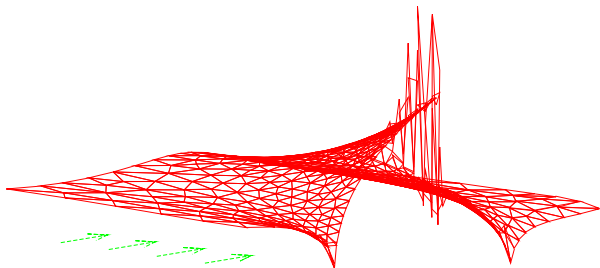
not correct BC



correct BC

Observations

- nonlinear hyperbolic problems: solution can contain discontinuities (shock waves)
- numerical approximation by FVM is fine (but smeared)
- numerical approximation by higher order methods can oscillate

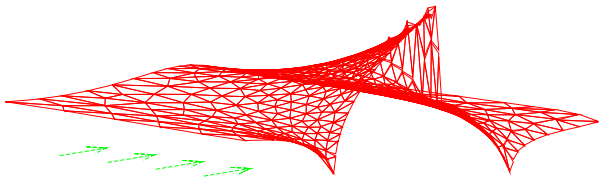


flow through a channel

Artificial diffusion approach

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{s=1}^d \frac{\partial \mathbf{f}_s(\mathbf{u})}{\partial x_s} = G(\mathbf{u}) \Delta \mathbf{u} \quad (9)$$

$G(\mathbf{u})$ vanish for smooth \mathbf{u}



flow through a channel