BDF-DGM for convection-diffusion equation

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Lecture 5

Model problem

- model scalar convection-diffusion equation,
- Let $\Omega \subset \mathbb{R}^d$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \mathbb{R}$ such that

Scalar convection-diffusion problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \varepsilon\, \Delta u &= g \qquad \text{in } Q_T, \qquad (1)\\
u &= u_D \qquad \text{on } \partial\Omega_D,\ t \in (0, T),\\
\nabla(u) \cdot \boldsymbol{n} &= g_N \qquad \text{on } \partial\Omega_N,\ t \in (0, T),\\
u(x, 0) &= u^0(x) \qquad x \in \Omega,
\end{aligned}
$$

where: $\vec{f} = (f_1, \ldots, f_d)$, $f_s \in C^1(\mathbb{R}), s = 1, \ldots, d$, $\varepsilon > 0$.
Suitable assumptions on $\vec{f}$, g, u_D, g_N and u^0 guarantee the existence and uniqueness of the weak solution.

Semi-discrete solution (see pres4.pdf)

Definition

We say that u_h is a DGFE solution iff

$$\begin{aligned}
&\text{a)} \quad u_h \in C^1(0,T;S_{hp}), \\
&\text{b)} \quad \left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + b_h(u_h(t), v_h) + \varepsilon\, a_h(u_h(t), v_h) \\
&\qquad + \varepsilon\, J_h^\sigma(u_h(t), v_h) = \ell_h(v_h)(t) \qquad \forall\, v_h \in S_{hp},\ t \in (0,T) \\
&\text{c)} \quad u_h(0) = \Pi_{hp} u^0,
\end{aligned} \tag{2}$$

where $\Pi_{hp} u^0$ is a projection of IC in S_{hp}

- system of ODEs,
- number of equations $= \dim S_{hp}$
- (semi)-implicit ODE solver advantageous,

Time discretization

BDF – backward differential formula

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = F(t, Y), \quad Y : (0, T) \to \mathbb{R}^m$$

- let $t_k,\ k = 0, \dots, r$ be a partition of $(0, T)$, let $Y^k \approx Y(t_k)$
- n-step BDF scheme (for constant time step τ)

$$\frac{1}{\tau}\sum_{l=0}^{n} \alpha_k Y^{k+1-l} = F(t_{k+1}, Y^{k+1}), \quad \alpha_k \in \mathbb{R},\ k = 0, \dots, r$$

Examples

- $n = 1$: $\frac{1}{\tau}(Y^{k+1} - Y^k) = F(t_{k+1}, Y^{k+1})$, implicit Euler,
  (1st order, error $= O(\tau)$)
- $n = 2$: $\frac{1}{\tau}(\frac{3}{2}Y^{k+1} - 2Y^k + \frac{1}{2}Y^{k-1}) = F(t_{k+1}, Y^{k+1})$,
  (2nd order, error $= O(\tau^2)$)

Semi-implicit scheme

- $A_h(u,v) := \varepsilon\, a_h(u,v) + \varepsilon J_h^\sigma(u,v) - \ell_h(v), \ u,v \in H^2(\Omega,\mathcal{T}_h)$

Space semi-discretization

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + b_h(u_h(t), v_h) + A_h(u_h(t), v_h) = 0 \qquad (3)$$

- direct application of 1-BDF to (3): $(u_h^k \approx u_h(t_k),\ k = 0,\dots)$

$$\frac{1}{\tau}\left(u_h^{k+1} - u_h^k, v_h\right) + A_h(u_h^{k+1}, v_h) + b_h(u_h^{k+1}, v_h) = 0$$

- $\Longrightarrow$ system of non-linear equations

Idea: Semi-implicit scheme

- an implicit treatment of linear terms,
- an explicit extrapolation for nonlinear terms
- $\Longrightarrow$ system of linear equations

Space-time discretization

First order implicit scheme

$$\frac{1}{\tau}\left(u_h^{k+1} - u_h^k, v_h\right) + A_h(u_h^{k+1}, v_h) + b_h(u_h^{k+1}, v_h) = 0$$

- $u_k^k \approx u_h(t_k), \quad u_h^k \in S_{hp}, \quad t_k \in [0, T]$

First order semiimplicit scheme

The approximate solution are functions u_h^k, $t_k \in [0, T]$:

a) $u_h^{k+1} \in S_{hp}$,

b) $\frac{1}{\tau}\left(u_h^{k+1} - u_h^k, v_h\right) + A_h(u_h^{k+1}, v_h) + b_h(u_h^k, v_h) = 0 \ \forall\, v_h \in S_{hp}$,

c) $u_h^0 = \Pi^{L^2} u^0, \quad \Pi^{L^2}$ is the L^2-projection.

Higher order schemes

2-BDF (second order)

$$\left(\frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau_k}, v_h\right) + A_h(u_h^{k+1}, v_h) + b_h(\bar{u}_h^{k+1}, v_h) = 0$$

$$\bar{u}_h^{k+1} = 2u_h^k - u_h^{k-1}.$$

3-BDF (third order)

$$\left(\frac{11u_h^{k+1} - 18u_h^k + 9u_h^{k-1} - 2u_h^{k-2}}{6\tau}, v_h\right) + A_h(u_h^{k+1}, v_h)$$

$$+ b_h(\bar{u}_h^{k+1}, v_h) = 0,$$

$$\bar{u}_h^{k+1} = 3u_h^k - 3u_h^{k-1} + u_h^{k-2}.$$

Higher order semi-implicit methods

General higher order scheme

$$\frac{1}{\tau_k}\left(\sum_{l=0}^{n}\alpha_l u_h^{k+1-l}, v_h\right) + A_h(u_h^{k+1}, v_h) + b_h(\bar{u}_h^{k+1}, v_h) = 0, \quad (4)$$

$$\forall\, v_h \in S_{hp}, \qquad \bar{u}_h^{k+1} = \sum_{l=1}^{n}\beta_l u_h^{k+1-l}, \quad \alpha_l, \beta_l \in \mathbb{R}.$$

Formal order of accuracy

$\|u - u_h\|_{L^2} = O(h^{p+1} + \tau^n)$, $p =$ space approx., $n =$ time approx.

- for $n > 6$, n-step BDF methods are unstable
- stability is decreasing for increasing n
- remark: matlab code *ode15*

Assumptions

Geometry of meshes

- let $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$ be a system of meshes,
- $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$ is shape regular,
- $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$ is locally quasi-uniform

Numerical flux

- consistent, conservative, Lipschitz continuous

Regularity of the weak solution

$$u \in W^{1,\infty}(0,T;H^{p+1}(\Omega)) \cap W^{n,\infty}(0,T;H^1(\Omega)) \\ \cap W^{n+1,\infty}(0,T;L^2(\Omega)),$$

$p \in \mathbb{N}$ is the space "degree" and $n \in \mathbb{N}$ is the time "degree"

Main result

Theorem: (SIPG variant of the BDF-DGM)

Let $p \geq 1$ and $n = 1, 2, 3$ (n-step BDF-DGM(p)) and

- u be the exact regular weak solution,
- u_h^k be the approximate solutions at t_k, $k = 0, 1, \ldots, r$,
- assumptions on meshes and numerical fluxes,

$$e = \{e_h^k\}_{k=0}^r = \{u_h^k - u^k\}_{k=0}^r, \quad u^k := u(t_k)$$

$$\|e\|^2_{h,\tau,L^\infty(L^2)} = \max_{k=0,\ldots,r} \|e_h^k\|^2_{L^2(\Omega)},$$

$$\|e\|^2_{h,\tau,L^2(H^1)} = \varepsilon\tau \sum\nolimits_{k=0}^{r} \left(|e_h^k|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(e_h^k, e_h^k) \right).$$

Then

$$\|e\|^2_{h,\tau,L^\infty(L^2)} = \exp(T(1 + C/\varepsilon))\ O(h^{p+1} + \tau^n), \quad \text{(optimal)} \quad (5)$$

$$|e\|^2_{h,\tau,L^2(H^1)} = \exp(T(1 + C/\varepsilon))\ O(h^p + \tau^n), \quad \text{(optimal)}$$

Numerical results

- 2D viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} = \varepsilon \Delta u + g \quad \text{in} \;\; \Omega \times (0, T),$$

  where $\Omega = (0,1)^2$.
- Dirichlet boundary condition over $\partial\Omega$,
- initial boundary condition $u(0,x) = u_0(x), \;\; x \in \Omega$.

- We verify:
  - order of convergence with respect τ,
  - order of convergence with respect h.

Convergence with respect to τ

- we choose the initial condition u_0 and the source term g in such a way that

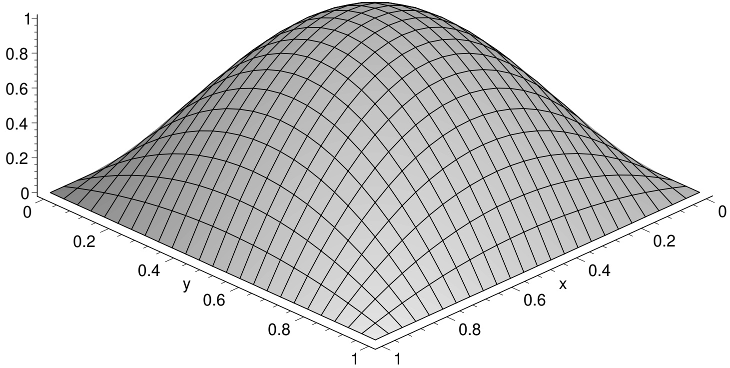
$$u(x_1, x_2, t) = \frac{e^{10t} - 1}{e^{10} - 1} \hat{u}(x_1, x_2),$$

where

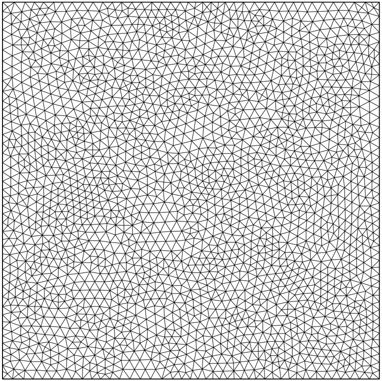
$$\hat{u}(x_1, x_2) = 16\, x_1(1 - x_1)x_2(1 - x_2).$$

  - we investigate solution at $T = 1$, $u(\cdot, \cdot, 1) \equiv \hat{u}(\cdot, \cdot)$,
  - we put $\varepsilon = 0.01$.

Exact solution at $T = 1$



Convergence with respect to τ



Computational data

- 6 different time steps (1/20, 1/40, 1/80, 1/160, 1/320, 1/640),
- P_3 approximation in space,
- mesh with 4219 elements

		L^2–norm		H^1–seminorm	
l	τ_l	error	EoC	error	EoC
1	$1/20$	1.452 E-01	–	6.712 E-01	–
2	$1/40$	6.700 E-02	1.054	3.218 E-01	1.061
3	$1/80$	3.431 E-02	1.028	1.573 E-01	1.032
4	$1/160$	1.698 E-02	1.014	7.778 E-02	1.016
5	$1/320$	8.449 E-03	1.007	3.687 E-02	1.008
6	$1/640$	4.231 E-03	1.004	1.928 E-02	1.004

triangular mesh with 4219 elements and P_3 approximation in space

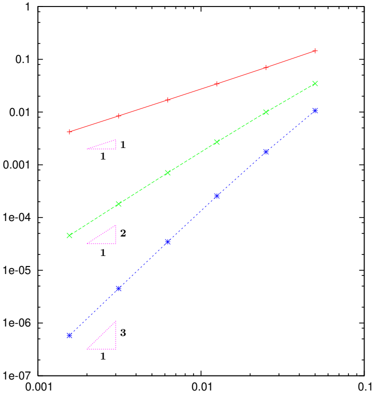
l	τ_l	L^2–norm error	L^2–norm EoC	H^1–seminorm error	H^1–seminorm EoC
1	$1/20$	3.474 E-02	–	1.680 E-01	–
2	$1/40$	9.964 E-03	1.802	4.819 E-02	1.801
3	$1/80$	2.701 E-03	1.883	1.309 E-02	1.880
4	$1/160$	7.062 E-04	1.936	3.429 E-03	1.933
5	$1/320$	1.808 E-04	1.966	8.789 E-04	1.964
6	$1/640$	4.575 E-05	1.982	2.227 E-04	1.981

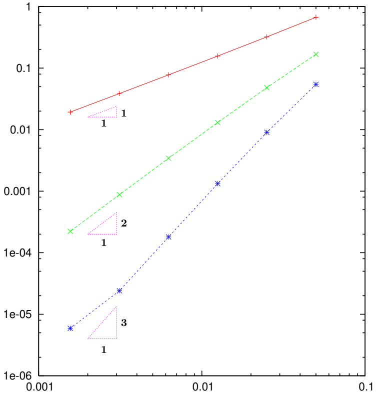
triangular mesh with 4219 elements and P_3 approximation in space

Third order scheme

l	τ_l	L^2–norm error	L^2–norm EoC	H^1–seminorm error	H^1–seminorm EoC
1	1/20	1.066 E-02	–	5.432 E-02	–
2	1/40	1.759 E-03	2.600	9.044 E-03	2.586
3	1/80	2.558 E-04	2.781	1.326 E-03	2.770
4	1/160	3.461 E-05	2.885	1.803 E-04	2.878
5	1/320	4.510 E-06	2.940	2.407 E-05	2.904
6	1/640	5.800 E-07	2.959	5.903 E-06	2.028

triangular mesh with 4219 elements and P_3 approximation in space





L^2-norm H^1-seminorm,

first order scheme, second order scheme, third order scheme

- we choose the initial condition u_0 and the source term g in such a way that

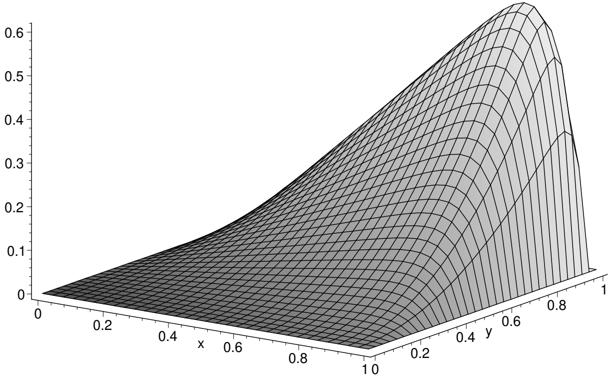
$$u(x_1, x_2, t) = (1 - e^{-10t})\hat{u}(x_1, x_2),$$

where

$$\hat{u}(x_1, x_2) = x_1 x_2^2 - x_2^2 \exp\left(\frac{2(x_1 - 1)}{\varepsilon}\right) - x_1 \exp\left(\frac{3(x_2 - 1)}{\varepsilon}\right) + \exp\left(\frac{2x_1 + 3x_2 - 5}{\varepsilon}\right),$$

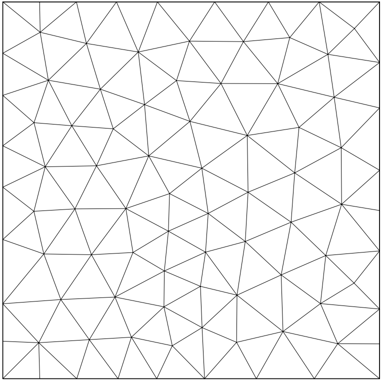
we put $\varepsilon = 0.1$.

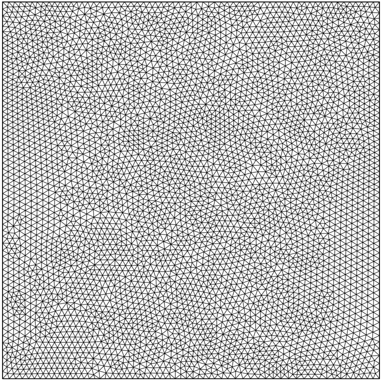
Exact solution



Computational grids

- 7 grids (148, 289, 591, 1056, 2360, 4219, 9372 elements),
- P_1, P_2 and P_3 polynomial approximation,
- third order scheme in time





Convergence with respect to h, P_1 approximation

		L^2–norm		H^1–seminorm	
l	h_l	e_h^2	EoC	e_h^1	EoC
1	1.898E-01	3.466 E-01	–	3.139 E-00	–
2	1.358E-01	1.858 E-01	1.864	2.208 E-00	1.051
3	9.500E-02	8.277 E-02	2.260	1.231 E-00	1.633
4	7.110E-02	5.232 E-02	1.580	9.605 E-01	0.855
5	4.754E-02	2.334 E-02	2.008	5.398 E-01	1.433
6	3.555E-02	1.580 E-02	1.342	4.436 E-01	0.675
7	2.386E-02	6.264 E-03	2.319	2.524 E-01	1.413
global order $\bar{\alpha}_n$			1.902		1.210

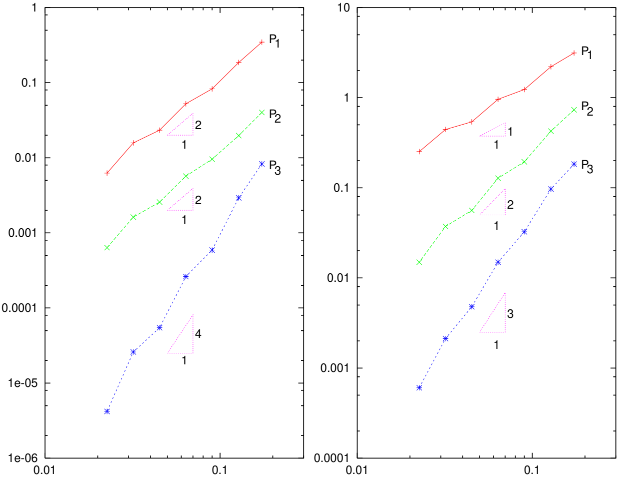
Convergence with respect to h, P_2 approximation

		L^2–norm		H^1–seminorm	
l	h_l	e_h^2	EoC	e_h^1	EoC
1	1.898E-01	4.012 E-02	–	7.348 E-01	–
2	1.358E-01	1.975 E-02	2.119	4.261 E-01	1.628
3	9.500E-02	9.573 E-03	2.024	1.942 E-01	2.197
4	7.110E-02	5.694 E-03	1.791	1.280 E-01	1.437
5	4.754E-02	2.571 E-03	1.977	5.618 E-02	2.048
6	3.555E-02	1.622 E-03	1.585	3.727 E-02	1.413
7	2.386E-02	6.355 E-04	2.348	1.493 E-02	2.292
global order $\bar{\alpha}_n$			1.955		1.858

Convergence with respect to h, P_3 approximation

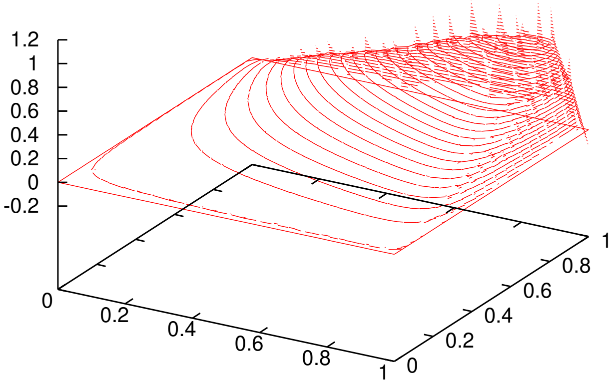
l	h_l	L^2–norm e_h^2	EoC	H^1–seminorm e_h^1	EoC
1	1.898E-01	8.290 E-03	–	1.835 E-01	–
2	1.358E-01	2.929 E-03	3.110	9.704 E-02	1.905
3	9.500E-02	5.899 E-04	4.480	3.250 E-02	3.058
4	7.110E-02	2.621 E-04	2.795	1.494 E-02	2.679
5	4.754E-02	5.463 E-05	3.900	4.783 E-03	2.832
6	3.555E-02	2.596 E-05	2.561	2.126 E-03	2.792
7	2.386E-02	4.191 E-06	4.570	6.040 E-04	3.153
global order $\bar{\alpha}_n$			3.611		2.785

Convergence with respect to h (NIPG method)

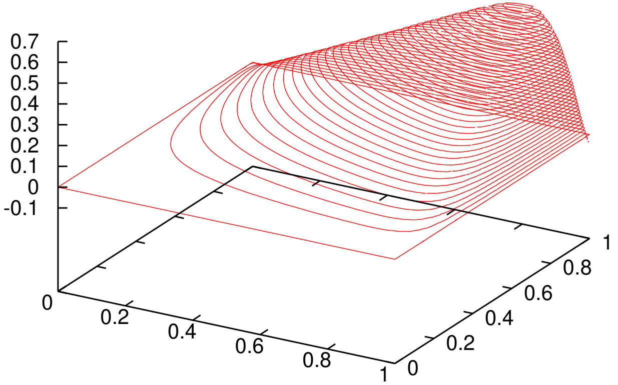


L^2-norm H^1-seminorm

Numerical result, P_1 approximation

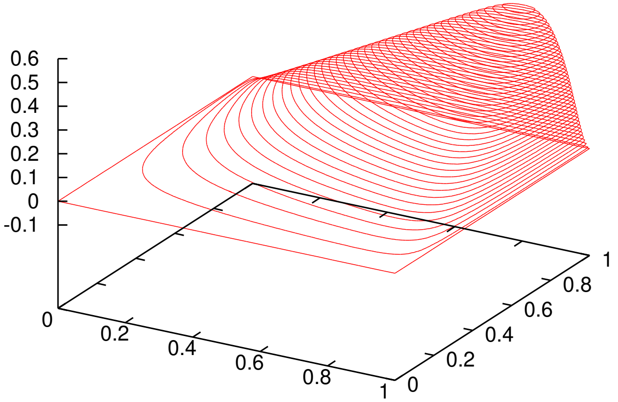


triangulation with 1056 elements



triangulation with 1056 elements

Numerical result, P_3 approximation



triangulation with 1056 elements