Space-time DGM for time-dependent PDEs

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Lecture 6

Model problem

- model heat equation,
- Let $\Omega \subset \mathbb{R}^d$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \mathbb{R}$ such that

Time-dependent heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon \, \Delta u &= g \qquad \text{in } Q_T, \\ u &= u_D \qquad \text{on } \partial\Omega_D, \ t \in (0, T), \\ \nabla(u) \cdot \boldsymbol{n} &= g_N \qquad \text{on } \partial\Omega_N, \ t \in (0, T), \\ u(x, 0) &= u^0(x) \qquad x \in \Omega, \end{aligned} \tag{1}$$

where $\varepsilon > 0$.

Suitable assumptions on g, u_D, g_N and u^0 guarantee the existence and uniqueness of the weak solution.

Semi-discrete solution (see pres4.pdf)

Definition

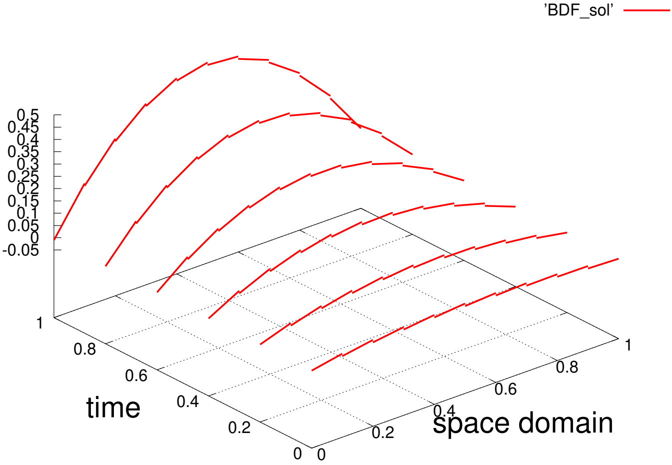
We say that u_h is a DGFE solution iff

$$
\begin{aligned}
&\text{a)} \quad u_h \in C^1(0,T;S_{hp}), && (2)\\
&\text{b)} \quad (u_h'(t),\varphi_h) + A_h(u_h(t),\varphi_h) = \ell_h(\varphi_h) \quad \forall \varphi_h \in S_{hp},\ t\in(0,T) \\
&\text{c)} \quad u_h(0) = \Pi_{hp}u^0,
\end{aligned}
$$

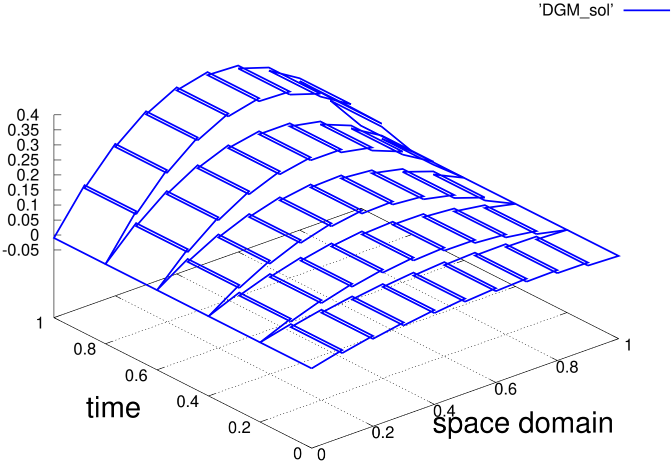
where $\Pi_{hp}u^0$ is a projection of IC in S_{hp}

- system of ODEs,
- number of equations = $\dim S_{hp}$
- (semi)-implicit ODE solver advantageous,

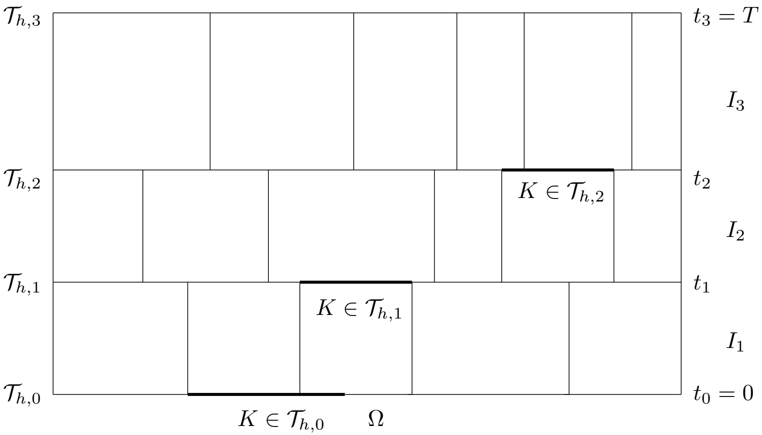
Approximation by BDF–DGM - 1D illustration



Approximation by space-time DGM - 1D illustration



Space-time discretization (1D illustration)



Space-time DG functional spaces

Time partition

- $0 = t_0 < \cdots < t_r = T$
- $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$, $m = 1, \dots, r$.

Time limits

- let φ be a function defined on $\bigcup_{m=1}^r I_m$,
- $\varphi|_m^\pm := \lim_{t \to t_m \pm} \varphi(t)$,
- $\{\varphi\}_m := \varphi|_m^+ - \varphi|_m^-$, $m = 1, \dots, r$.

DG space

$$S_{h,\tau}^{p,q} := \Big\{ \varphi \in L^2(Q_T);\ \varphi(x,t)\big|_{I_m} = \sum_{i=0}^{q} t^i\, \varphi_{m,i}(x) \qquad (3)$$
$$\text{with } \varphi_{m,i} \in S_{h,m}^p,\ i = 0, \dots, q, \quad m = 1, \dots, r \Big\}.$$

$$(u_h'(t), \varphi_h(t)) + A_h(u_h(t), \varphi_h(t)) = \ell_h(\varphi_h(t)), \quad \varphi_h \in S_{h,\tau}^{p,q}$$

integration over I_m

$$\int_{I_m} (u_h'(t), \varphi_h(t)) \, \mathrm{d}t + \int_{I_m} (A_h(u_h(t), \varphi_h(t)) - \ell_h(\varphi_h)) \, \mathrm{d}t = 0.$$

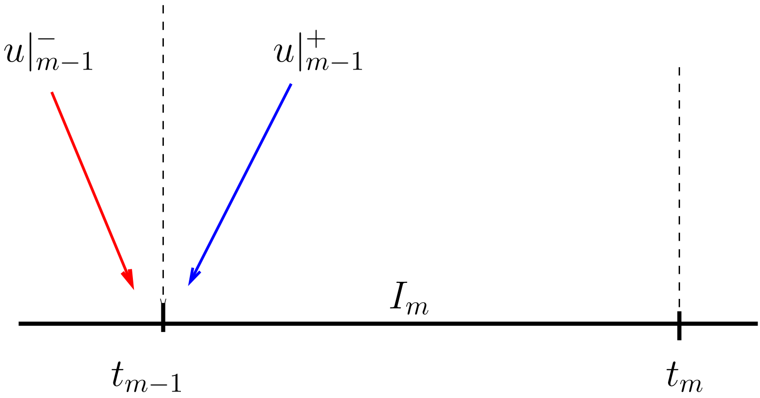
Manipulation with the first term (integration by parts):

$$\int_{I_m} (u', \varphi) \, \mathrm{d}t = -\int_{I_m} (u, \varphi') \, \mathrm{d}t + (u|_m^-, \varphi|_m^-) - (u|_{m-1}^+, \varphi|_{m-1}^+).$$

$$(u|_{m-1}^+, \varphi|_{m-1}^+) = (u|_{m-1}^-, \varphi|_{m-1}^+) \text{ (valid for } u \text{ being exact)}$$

reverse integration by parts:

$$\int_{I_m} (u', \varphi) \, \mathrm{d}t = \int_{I_m} (u', \varphi) \, \mathrm{d}t + \left(\{u\}_{m-1}, \varphi|_{m-1}^+\right)$$


$u|_{m-1}^{-}$
$u|_{m-1}^{+}$
I_m
t_{m-1}
t_m

Space-time DG discretization (2)

Approximate STDG solution

We say that $u_h \in S^{p,q}_{h,\tau}$ is the STDG solution of (1) is

$$\int_{I_m} \left((u_h', \varphi) + A_h(u_h, \varphi) - \ell(\varphi) \right) \mathrm{d}t + \left(\{u_h\}_{m-1}, \varphi|^{+}_{m-1} \right) = 0$$

$$\forall\, \varphi \in S^{p,q}_{h,\tau}, \quad m = 1, \dots, r, \quad \text{with} \quad u_h|^{-}_{0} := \Pi_h u^0, \qquad (4)$$

where $u_h' = \partial u_h / \partial t$.

- system of linear (nonlinear) algebraic equations
- # eqs. $= \#\mathcal{T}_h (p+1)(p+2)/2\ (q+1)$ (in 2D)
- the mesh can be different on each time level t_m

Theorem

There exists a unique approximate solution

Proof: coercivity of the left-hand side can be shown

Numerical analysis

A priori error estimate

$$\|(u-u_h)|_m^-\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2}\sum_{j=1}^{m}\int_{I_j} \|\!|u-u_h|\!\|_j^2 \,\mathrm{d}t \tag{5}$$

$$\leq C\varepsilon \left(h^{2(\mu-1)} |u|^2_{C([0,T];H^\mu(\Omega))} + \tau^{2(q+\gamma)} \|u\|^2_{H^{q+1}(0,T;H^1(\Omega))} \right),$$

$\gamma = 0$ for a general u_D, $\gamma = 1$ for u_D polynomial w.r.t. t

A priori error estimate (2)

$$\sup_{t\in I_m} \|u(t) - u_h(t)\|^2_{L^2(\Omega)} \tag{6}$$

$$\leq C \left(h^{2(\mu-1)} |u|^2_{C([0,T];H^\mu(\Omega))} + \tau_m^{2(q+1)} |u|^2_{W^{q+1,\infty}(0,T;L^2(\Omega))} \right),$$