NMSA405: exercise 3 – 0-1 laws, random walk

Theorem (Kolmogorov 0-1 law): Let $X = (X_1, X_2, \dots)$ be a random sequence of independent random variables. Then $\mathbb{P}(X \in T)$ equals either 0 or 1 for any terminal set T.

Theorem (Hewitt-Savage 0-1 law): Let $X = (X_1, X_2, \dots)$ be an iid random sequence. Then $\mathbb{P}(X \in S)$ equals either 0 or 1 for any symmetric set S.

Exercise 3.1: Let $X = (X_1, X_2, \dots)$ be a random sequence of independent random variables. Show that the event

$$\left[\sum_{n=1}^{\infty} X_n < \infty\right]$$

occurs with probability 0 or 1.

Definition 2.5: Let $X = (X_1, X_2, \dots)$ be an iid random sequence. We call the sequence of partial sums $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$ a *random walk.*

Exercise 3.2: Let $S = (S_1, S_2, \dots)$ be a random walk. Consider the event

$$A = [S_n = 0 \text{ for infinitely many } n].$$

Show that $\mathbb{P}(A)$ equals either 0 or 1.

Exercise 3.3: The following variants of the limit behaviour of the random walk $S = (S_1, S_2, \dots)$ are mutually exclusive:

- (i) $S_n = 0$ a.s. for all $n \in \mathbb{N}$,
- (ii) $S_n \underset{n\to\infty}{\longrightarrow} \infty$,
- (iii) $S_n \underset{n\to\infty}{\longrightarrow} -\infty$,
- (iv) $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$.

Prove that precisely one of these variants occurs with probability 1.

NMSA405: exercise 4 – stopping times

Definition: Let $X = (X_1, X_2, \dots)$ be a random sequence. The σ-algebra generated by the random vector $(X_1, \dots, X_n)$ is $\sigma(X_1, \dots, X_n) = \{[(X_1, \dots, X_n) \in B_n], B_n \in \mathcal{B}^n\}$ and the σ-algebra generated by the sequence X is $\sigma(X) = \{[X \in B], B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})\}$.

Exercise 4.1: (Proposition 2.1) Check that $\sigma(X_1, \dots, X_n)$ and $\sigma(X)$ are σ-algebras. Prove that

$$\sigma(X) = \sigma\left(\bigcup_{n=1}^{\infty} \sigma(X_1, \dots, X_n)\right).$$

Definition 2.1: Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ a non-decreasing sequence of σ-algebras. We call $(\mathcal{F}_n)$ a *filtration.* Denote $\mathcal{F}_\infty = \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_n\right)$. We call the random sequence $X = (X_1, X_2, \dots)$ *adapted* to the filtration $(\mathcal{F}_n)$, shortly $\mathcal{F}_n$*-adapted* if $\sigma(X_1, \dots, X_n) \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$. If $\sigma(X_1, \dots, X_n) = \mathcal{F}_n$ for all $n \in \mathbb{N}$ we call $(\mathcal{F}_n)$ the *canonical filtration* of the sequence X.

Exercise 4.2: (Proposition 2.2) Let $X = (X_1, X_2, \dots)$ be a random sequence and $S = (S_1, S_2, \dots)$ the sequence of its partial sums: $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Show that X and S have the same canonical filtration. Compare the canonical filtrations of the sequence X and the sequence $X^2 = (X_1^2, X_2^2, \dots)$.

Definition 2.3: The mapping $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time* with respect to the filtration $(\mathcal{F}_n)$ provided that $[T \leq n] \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Let $X = (X_1, X_2, \dots)$ be a random sequence. A stopping time $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time of the sequence* X if $[T \le n] \in \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.

Exercise 4.3: Show that T is a stopping time with respect to the filtration $(\mathcal{F}_n)$ if and only if the random sequence $X_n = \mathbf{1}\{T \le n\}$ is $\mathcal{F}_n$-adapted.

Definition 2.4: Let $(\mathcal{F}_n)$ be a filtration and T its stopping time. Then

$$\mathcal{F}_T = \{F \in \mathcal{F}_\infty : F \cap [T \le n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

is called the *stopping time σ-algebra.*

Exercise 4.4: Show that $\mathcal{F}_T$ defines a σ-algebra.

Exercise 4.5: (Proposition 2.3) Show that T is a stopping time with respect to the filtration $(\mathcal{F}_n)$ if and only if $[T = n] \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Further show that the following holds:

$$\mathcal{F}_T = \{F \in \mathcal{F}_\infty : F \cap [T = n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

Exercise 4.6: Consider a fixed $n_0 \in \mathbb{N}$ and $T = n_0$. Show that T is a stopping time with respect to any filtration $(\mathcal{F}_n)$ and determine the σ-algebra $\mathcal{F}_T$.

Definition: We define the mapping $X_T : \Omega \to \mathbb{R}$ as

$$X_T(\omega) = \begin{cases} X_{T(\omega)}(\omega) & \text{pro } T(\omega) < \infty, \\ 0 & \text{pro } T(\omega) = \infty. \end{cases}$$

Exercise 4.7: (Proposition 2.4) Let S and T be stopping times with respect to the filtration $(\mathcal{F}_n)$. Show that:

- a) T and X_T are $\mathcal{F}_T$-measurable random variables,
- b) $\min\{S, T\}$, $\max\{S, T\}$ and $S + T$ are stopping times with respect to the filtration $(\mathcal{F}_n)$,
- c) $\min\{T, n\}$ is a $\mathcal{F}_n$-measurable random variable for any $n \in \mathbb{N}$.

Exercise 4.8: Let $T_1, T_2, \dots$ be a sequence of stopping times with respect to the filtration $(\mathcal{F}_n)$. Show that $\sup_n T_n$ and $\inf_n T_n$ are also stopping times with respect to the filtration $(\mathcal{F}_n)$.

Exercise 4.9: (Proposition 2.5a) Let T be a stopping time with respect to the filtration $(\mathcal{F}_n)$. Consider the mapping $\lambda : \Omega \to \mathbb{N} \cup \{\infty\}$ which is $\mathcal{F}_T$-measurable and fulfills $\lambda \ge T$. Show that λ is a stopping time with respect to the filtration $(\mathcal{F}_n)$.

Exercise 4.10: (Proposition 2.5b) Let $X = (X_1, X_2, \dots)$ be a random sequence and T its stopping time. For $B \in \mathcal{B}(\mathbb{R})$ we define $\lambda = \min\{k > T : X_k \in B\}$, i.e. the first hitting time of the set B by the sequence X after the time T. Show that λ is a stopping time of the sequence X.

Exercise 4.11: Let $(S_1, S_2, \dots)$ be a symmetric simple random walk (with the step X_n taking on only the values 1 and -1 with equal probabilities). Determine whether the following random variables are stopping times of the sequence $X = (X_1, X_2, \dots)$:

- a) $T_N = \max\{n \le N : S_n = 0\}$ for $N \in \mathbb{N}$,
- b) $\lambda = \min\{n : S_n = 5\}$,
- c) $\nu = \min\{n : S_n < -3\}$,
- d) $\lambda + \nu$, $\min\{\lambda, \nu\} + 1$, $\max\{\lambda, \nu\}$, $\max\{\lambda, \nu\} - 1$, $2\lambda - 1$, λ^2.