NMSA409, topic 5: Linear models of time series

MA(n): *The moving average sequence of order n* is defined by

$$X_t = b_0 Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and $b_0, b_1, \ldots, b_n$ are real- or complex-valued constants, $b_0 \neq 0$, $b_n \neq 0$. It is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 (b_t \overline{b_0} + \cdots + b_n \overline{b_{n-t}}) & \text{for } 0 \leq t \leq n, \\ \sigma^2 (b_0 \overline{b_{|t|}} + \cdots + b_{n-|t|} \overline{b_n}) & \text{for } -n \leq t \leq 0, \\ 0 & \text{for } |t| > n, \end{cases}$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{n} b_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

MA(∞): *The causal linear process* is a random sequence defined by

$$X_t = \sum_{j=0}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}, \qquad (\star)$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $c_0, c_1, \ldots$ is a sequence of (real- or complex-valued) constants such that $\sum_{j=0}^{\infty} |c_j| < \infty$ (this condition implies that the sum converges absolutely almost surely). $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 \sum_{k=0}^{\infty} c_{k+t} \overline{c_k} & \text{for } t \geq 0, \\ \sigma^2 \sum_{k=0}^{\infty} c_k \overline{c_{k+|t|}} & \text{for } t \leq 0, \end{cases} \qquad (\circ)$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

AR(m): *The autoregressive sequence of order m* is defined by

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \ldots, a_m$ are real-valued constants, $a_m \neq 0$. If all the roots of the polynomial $1 + a_1 z + \cdots + a_m z^m$ lie outside the unit circle in $\mathbb{C}$ (which is equivalent to all the roots of $z^m + a_1 z^{m-1} + \cdots + a_m$ lying inside the unit circle) then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process $(\star)$ with coefficients c_j determined by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1}{1 + a_1 z + \cdots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in $(\star)$ into the defining formula and by comparing the coefficients at the respective terms Y_{t-j} on both sides. The autocovariance function is given by $(\circ)$ and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

ARMA(m, n): This model is defined by the equation

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \ldots, a_m, b_1, \ldots, b_n$ are real-valued constants, $a_m \neq 0$, $b_n \neq 0$. Suppose that the polynomials $1 + a_1 z + \cdots + a_m z^m$ and $1 + b_1 z + \cdots + b_n z^n$ have no common roots and all the roots of the polynomial $1 + a_1 z + \dots + a_m z^m$ lie outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process ($\star$) with coefficients c_j given by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1 + b_1 z + \dots + b_n z^n}{1 + a_1 z + \dots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in ($\star$) into the defining formula and by comparing the coefficients at the respective terms Y_{t-j} on both sides. The autocovariance function is given by ($\circ$) and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|1 + b_1 \mathrm{e}^{-\mathrm{i}\lambda} + \dots + b_n \mathrm{e}^{-\mathrm{i}n\lambda}|^2}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \dots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

Exercise 5.1: Determine the autocovariance function and the spectral density of the sequence

$$X_t = Y_t + \theta Y_{t-2}, \quad t \in \mathbb{Z},$$

where $\theta \in \mathbb{C}$ a $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$.

Exercise 5.2: Determine the autocovariance function and the spectral density of the sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t = Y_t + c_1 Y_{t-1} + c_2 Y_{t-2}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and c_1, c_2 are the coefficients in the equation $z^2 + c_1 z + c_2 = 0$ with roots $z_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $z_2 = \rho \mathrm{e}^{-\mathrm{i}\theta}$, where $\rho > 0$, $\theta \in (0, \pi)$.

Exercise 5.3: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by

$$X_t - 0{,}7 X_{t-1} + 0{,}1 X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Express the random sequence $\{X_t, t \in \mathbb{Z}\}$ as a causal linear process and compute its autocovariance function and spectral density.

Exercise 5.4: Solve the Yule-Walker equations and determine the autocovariance function of the random sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t - 0{,}4 X_{t-1} + 0{,}04 X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise.

Exercise 5.5: Solve the Yule-Walker equations and determine the autocovariance function of the random sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t - 1{,}4 X_{t-1} + 0{,}98 X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise.

Exercise 5.6: Let $\{X_t, t \in \mathbb{Z}\}$ be an AR(2) random sequence defined by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} = Y_t, \quad t \in \mathbb{Z}.$$

Determine for which values of a_1 and a_2 is $\{X_t, t \in \mathbb{Z}\}$ a causal linear process. Express the variance of $\{X_t, t \in \mathbb{Z}\}$ by means of a_1 and a_2 and the white noise variance σ^2.

Exercise 5.7: Let $\{X_t, t \in \mathbb{Z}\}$ be an ARMA(2,1) random sequence defined by

$$X_t - X_{t-1} + \frac{1}{4} X_{t-2} = Y_t + Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the $\mathrm{MA}(\infty)$ representation of X_t and compute its autocovariance function (both from the linear process representation and from the Yule-Walker equations) and spectral density.