Spectral decomposition of the autocovariance function

Theorem 4.1: A complex function $R(t)$, $t \in \mathbb{Z}$, is an autocovariance function of a weakly stationary random sequence if and only if

$$R(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}F(\lambda), \ t \in \mathbb{Z}, \tag{1}$$

where F is a bounded right-continuous non-decreasing function on $[-\pi, \pi]$ such that $F(-\pi) = 0$.

The function F is determined uniquely and it is called the *spectral distribution function of a random sequence.* If F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$ we call its density f the *spectral density.* It follows that $F(\lambda) = \int_{-\pi}^{\lambda} f(x) \, \mathrm{d}x$, $f = F'$ and

$$R(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda) \, \mathrm{d}\lambda, \ t \in \mathbb{Z}. \tag{2}$$

If F is piecewise constant with jumps at points $\lambda_i \in (-\pi, \pi]$ of the magnitudes $a_i > 0$ then

$$R(t) = \sum_i a_i \mathrm{e}^{\mathrm{i}t\lambda_i}, \ t \in \mathbb{Z}.$$

Theorem 4.2: A complex function $R(t)$, $t \in \mathbb{R}$, is an autocovariance function of a centered weakly stationary L_2-continuous stochastic process if and only if

$$R(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}F(\lambda), \ t \in \mathbb{R},$$

where F is a right-continuous non-decreasing function such that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = R(0) < \infty$.

The function F is determined uniquely and it is called the *spectral distribution function of an L_2-continuous stochastic process.* If F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$ we call its density f the *spectral density.* It follows that $F(\lambda) = \int_{-\infty}^{\lambda} f(x) \, \mathrm{d}x$, $f = F'$ and

$$R(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda) \, \mathrm{d}\lambda, \ t \in \mathbb{R}. \tag{3}$$

If F is piecewise constant with jumps at points $\lambda_i \in \mathbb{R}$ of the magnitudes $a_i > 0$ then

$$R(t) = \sum_i a_i \mathrm{e}^{\mathrm{i}t\lambda_i}, \ t \in \mathbb{R}.$$

Theorem 4.3: Let $\{X_t, t \in \mathbb{Z}\}$ be a weakly stationary sequence with the autocovariance function $R(t)$ such that $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$. Then the spectral density of the sequence $\{X_t, t \in \mathbb{Z}\}$ exists and is given by

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} R(t), \ \lambda \in [-\pi, \pi]. \tag{4}$$

Theorem 4.4: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary L_2-continuous process with the autocovariance function $R(t)$ such that $\int_{-\infty}^{\infty} |R(t)| \, \mathrm{d}t < \infty$. Then the spectral density of the process $\{X_t, t \in \mathbb{R}\}$ exists and is given by

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} R(t) \, \mathrm{d}t, \ \lambda \in \mathbb{R}. \tag{5}$$

Exercise 4.3 alt: Determine the autocovariance function of the centered weakly stationary process $\{X_t, t \in \mathbb{R}\}$ with the spectral distribution function

$$F_X(\lambda) = \begin{cases} 0, & \lambda \leq -b, \\ (\lambda + b)a, & -b \leq \lambda < b, \\ 3ab, & \lambda \geq b, \end{cases}$$

where $a > 0$ and $b > 0$ are constants.

Exercise 4.6: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \cos t, \quad t \in \mathbb{R}.$$

Determine the spectral distribution function of the process.

Exercise 4.7: Determine the spectral density of the weakly stationary sequence $\{X_t, t \in \mathbb{Z}\}$ with the autocovariance function

$$R_X(t) = \begin{cases} \frac{16}{15} \cdot \frac{1}{2^{|t|}} & \text{for even values of } t, \\ 0 & \text{for odd values of } t. \end{cases}$$

Exercise 4.8: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \exp\{\kappa(\mathrm{e}^{\mathrm{i}t} - 1)\}, \quad t \in \mathbb{R},$$

where $\kappa > 0$. Determine the spectral distribution function of the process.

Exercise 4.9: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \frac{1}{1 - \mathrm{i}t}, \quad t \in \mathbb{R}.$$

Determine the spectral density of the process.

Exercise 4.13: Consider a real-valued centered random variable Y with finite positive variance σ^2 and a random sequence defined as $X_t = (-1)^t Y$, $t \in \mathbb{Z}$. Decide whether the spectral density of this sequence exists. If it does, find a formula for it.

Exercise 4.14: The *elementary process* $\{X_t, t \in \mathbb{R}\}$ is defined as $X_t = Y\mathrm{e}^{\mathrm{i}\omega t}, t \in \mathbb{R}$, where $\omega \in \mathbb{R}$ is a constant and Y is a (complex) random variable such that $\mathbb{E}Y = 0$ and $\mathbb{E}|Y|^2 = \sigma^2 < \infty$. Discuss the stationarity of the process $\{X_t, t \in \mathbb{R}\}$ and determine its spectral density.

Exercise 4.15: Let $\{N_t, t \geq 0\}$ be a Poisson process with the intensity $\lambda > 0$ and let A be a real-valued random variable with zero mean and variance 1, independent of the process $\{N_t, t \geq 0\}$. Define $X_t = A(-1)^{N_t}$, $t \geq 0$. Determine the spectral distribution function and the spectral density (if it exists) of the process $\{X_t, t \geq 0\}$.