NMTP438, topic 1: random fields on a lattice

1. Show that a Markov chain $\{Z_1, \dots, Z_n\}$ is a Markov random field with respect to the relation $i \sim j \Leftrightarrow |i-j| \leq 1$. Prove that the converse implication holds as follows: if $\{Z_1, \dots, Z_n\}$ is a Markov random field with a probability density function satisfying $p(\boldsymbol{z}) > 0$ for all $\boldsymbol{z} = (z_1, \dots, z_n)^T$ then it is a Markov chain.

2. Assume that $p(\boldsymbol{z}) > 0$ for all $\boldsymbol{z} \in S^L$. Show that $\boldsymbol{z}_A$ and $\boldsymbol{z}_B$ are conditionally independent given $\boldsymbol{z}_C$ if and only if

$$p(\boldsymbol{z}_A \mid \boldsymbol{z}_B \boldsymbol{z}_C) = p(\boldsymbol{z}_A \mid \boldsymbol{z}_C) \quad \text{for } \nu\text{-almost all } \boldsymbol{z},$$

which holds if and only if

$$p(\boldsymbol{z}_B \mid \boldsymbol{z}_A \boldsymbol{z}_C) = p(\boldsymbol{z}_B \mid \boldsymbol{z}_C) \quad \text{for } \nu\text{-almost all } \boldsymbol{z}.$$

3. Local characteristics do not determine the joint distribution. Consider a lattice with two lattice points $L = \{i, j\}$ and assume that $Z_i \mid Z_j = z_j$ has an exponential distribution with rate z_j and $Z_j \mid Z_i = z_i$ has an exponential distribution with rate z_i. Show that these conditional distributions do not correspond to any probability distribution, i.e. a (proper) joint probability density function of the vector $(Z_i, Z_j)^T$ does not exist.

4. Show that any Gibbs random field satisfies

$$p(\boldsymbol{z}_A \mid \boldsymbol{z}_{-A}) = p(\boldsymbol{z}_A \mid \boldsymbol{z}_{\partial A})$$

for any $A \subseteq L$ and $\boldsymbol{z} \in S^L$. The symbol ∂A denotes the set of neighbours of the set A, i.e. $\partial A = (\cup_{i \in A} \partial i) \setminus A$.

5. Let $S = \mathbb{N}_0$ and L be a finite lattice in $\mathbb{R}^d$. Show that if $\beta_{ij} \geq 0$ for all $i, j \in L$ such that $i \sim j$, $i \neq j$, then the constant

$$\sum_{z \in S^L} \exp\left(-\sum_{i \in L} (\log z_i! + \beta_i z_i) - \sum_{\{i,j\} \in \mathcal{C}} \beta_{ij} z_i z_j \right)$$

is finite. On the other hand, it is infinite if $\beta_{ij} < 0$ for any $i, j \in L$: $i \sim j$, $i \neq j$.

Hint: In the first case consider the configurations with $\max z_i = k$ (there are $(k+1)^n - k^n$ of those). In the second case consider the configurations with $z_i = z_j = k$ and $z_l = 0$ for $l \in L \setminus \{i, j\}$.

6. Let the random field $\{Z_i : i \in L\}$ be given by the joint probability density function

$$p(\boldsymbol{z}) = \frac{\sqrt{\det \boldsymbol{Q}}}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \sum_{i,j \in L} q_{ij}(z_i - \mu_i)(z_j - \mu_j) \right\}, \quad \boldsymbol{z} \in \mathbb{R}^L.$$

Determine the probability density function of the conditional distribution of $Z_i \mid \boldsymbol{Z}_{-i}$ for $i \in L$.

7. Let the random variable Z_1 have a normal distribution $N(0, \frac{1}{1-\varphi^2})$ with $|\varphi| < 1$. Consider a first-order autoregressive sequence $\{Z_1, \dots, Z_n\}$ defined by the formula

$$Z_t = \varphi Z_{t-1} + \varepsilon_t, \quad t = 2, \dots, n,$$

where $\{\varepsilon_2, \dots, \varepsilon_n\}$ is a sequence of independent identically distributed random variables with the $N(0,1)$ distribution and the sequence is independent of Z_1. Determine the variance matrix $\boldsymbol{\Sigma}$ of the vector $(Z_1, \dots, Z_n)^T$ and the matrix $\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1}$. Show that $\{Z_1, \dots, Z_n\}$ is a Gaussian Markov random field with respect to the relation $i \sim j \Leftrightarrow |i - j| \leq 1$.

8. Let $\{Z_i : i \in L\}$ be a Gaussian Markov random field with the precision matrix $\boldsymbol{Q}$. Show that

$$\operatorname{corr}(Z_i, Z_j \mid \boldsymbol{Z}_{-\{i,j\}}) = -\frac{q_{ij}}{\sqrt{q_{ii} q_{jj}}}, \quad i \neq j.$$

9. Consider random variables X_1, X_2 having only values 0 or 1. We specify the conditional distributions using the logistic regression models:

$$\operatorname{logit} \mathbb{P}(X_1 = 1 \mid X_2) = \alpha_0 + \alpha_1 X_2, \quad \operatorname{logit} \mathbb{P}(X_2 = 1 \mid X_1) = \beta_0 + \beta_1 X_1,$$

where $\operatorname{logit} p = \log \frac{p}{1-p}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are real-valued parameters. Determine the joint distribution of the random vector $(X_1, X_2)^T$ using the Brook lemma.