NMSA405: topic 1 – space of sequences of real numbers

Exercise 1.1: For vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ it is reasonable to define the L_1-distance (Manhattan distance, city-block distance) as $d(x,y) = \sum_{j=1}^n |x_j - y_j|$. For infinite sequences of real numbers $x = (x_1, x_2, \dots) \in \mathbb{R}^\mathbb{N}$ and $y = (y_1, y_2, \dots) \in \mathbb{R}^\mathbb{N}$, does it make sense to define the following "distances"?

$$d_1(x,y) = \sum_{j=1}^\infty |x_j - y_j|, \qquad d_2(x,y) = \sum_{j=1}^\infty \frac{|x_j - y_j|}{2^j}, \qquad d_3(x,y) = \sum_{j=1}^\infty \frac{\min\{|x_j - y_j|, 1\}}{2^j}$$

Definition: (D 1.3) For sequences of real numbers $x = (x_1, x_2, \dots) \in \mathbb{R}^\mathbb{N}$ and $y = (y_1, y_2, \dots) \in \mathbb{R}^\mathbb{N}$ we define

$$d(x,y) = \sum_{j=1}^\infty \frac{\min\{|x_j - y_j|, 1\}}{2^j}.$$

Recall: What properties does a metric have?

Exercise 1.2: (P 1.2a) Show that d defines a metric on $\mathbb{R}^\mathbb{N}$.

Exercise 1.3: (P 1.2b) Let $x^n = (x_1^n, x_2^n, \dots)$ be sequences of real numbers for $n \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$. Prove that

$$d(x^n, x) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{if and only if} \quad |x_j^n - x_j| \underset{n\to\infty}{\longrightarrow} 0 \ \text{ for all } \ j \in \mathbb{N}.$$

Recall: What is a complete separable metric space? What is a Cauchy sequence?

Exercise 1.4: (P 1.2c) Prove that $(\mathbb{R}^\mathbb{N}, d)$ is a complete separable metric space.

Definition: (D 1.5) Mapping $p : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is called *a finite permutation (of order n)*, if there is $n \in \mathbb{N}$ and a permutation $(k_1, \dots, k_n)$ of the elements of the set $\{1, \dots, n\}$ such that

$$p(x_1, \dots, x_n, x_{n+1}, \dots) = (x_{k_1}, \dots, x_{k_n}, x_{n+1}, \dots), \quad (x_1, x_2, \dots) \in \mathbb{R}^\mathbb{N}.$$

Recall: What properties does a homeomorphism have?

Exercise 1.5: (P 1.5a) Prove that any finite permutation p is a homeomorphism.

Definition: (D 1.6) Mapping $s : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ defined by

$$s(x_1, x_2, \dots) = (x_2, x_3, \dots), \quad (x_1, x_2, \dots) \in \mathbb{R}^\mathbb{N},$$

is called *shift*.

Recall: What properties does a continuous mapping have?

Exercise 1.6: (P 1.5b) Prove that the shift s is a continuous mapping.

Definition: (D 1.7) A set $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is called *terminal* if the following implication holds:

$x = (x_1, x_2, \ldots) \in T, y = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : y_k = x_k$ for all $k \in \mathbb{N}$ except of finitely many $\Rightarrow y \in T$.

We call $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ *n-terminal* if

$$x = (x_1, x_2, \ldots) \in T, y = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : y_k = x_k \text{ for } k > n \ \Rightarrow y \in T.$$

Exercise 1.7: Find examples of terminal and n-terminal sets of sequences.

Exercise 1.8: (P 1.5c) Prove that $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is n-terminal if and only if there is a $T_n \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ such that $T = \mathbb{R}^n \times T_n$.

Definition: (D 1.8) We use a particular notation for the following systems of sets:

- *n-symmetric sets*: $\mathcal{S}_n = \{S \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : p(S) = S$ for any finite permutation p of order $n\}$,
- *symmetric sets*: $\mathcal{S} = \{S \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : p(S) = S$ for any finite permutation $p\}$,
- *shift invariant sets*: $\mathcal{I} = \{I \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : s^{-1}I = I\}$,
- *n-terminal sets*: $\mathcal{T}_n = \{T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : T$ n-terminal$\}$,
- *terminal sets*: $\mathcal{T} = \{T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : T$ terminal$\}$.

Exercise 1.9: Find examples of symmetric, n-symmetric and shift invariant sets of sequences.

Exercise 1.10: (P 1.5d)

a) Show that $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ for all $n \in \mathbb{N}$ and $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$.
b) Show that $\mathcal{T}_{n+1} \subset \mathcal{T}_n$ for all $n \in \mathbb{N}$ and $\mathcal{T} = \cap_{n=1}^{\infty} \mathcal{T}_n$.
c) Prove that $\mathcal{I} \subset \mathcal{T}_n \subset \mathcal{S}_n$ for all $n \in \mathbb{N}$ and hence $\mathcal{I} \subset \mathcal{T} \subset \mathcal{S}$.
d) Show that the previous inclusions are strict, i.e. the sets are not equal. Provide examples!
e) Extra exercise: Check that $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{T}$ are σ-algebras.

Definition: (D 1.10) We call the set $B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ *finite-dimensional* if there are $n \in \mathbb{N}$ and $B_n \in \mathcal{B}(\mathbb{R}^n)$ such that $B = B_n \times \mathbb{R}^{\mathbb{N}}$.

Recall: What properties does an algebra (system of sets) have?

Exercise 1.11: (P 1.6) Denote by $\mathcal{A}$ the system of finite-dimensional sets from $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$. Prove that $\mathcal{A}$ is an algebra generating $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, i.e. it holds that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^{\mathbb{N}})$.

NMSA405: topic 2 – random sequences

Definition: (D 1.13) *Binary expansion* of the number $x \in (0, 1]$ is the sequence $x_1, x_2, \ldots$ of zeroes and ones such that it contains infinitely many ones and

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k}.$$

Binary expansion of the number 0 is the sequence of zeroes.

Exercise 2.1: (P 1.14) Prove that if X is a random variable with uniform distribution on the interval $[0, 1]$ and

$$X(\omega) = \sum_{k=1}^{\infty} \frac{X_k(\omega)}{2^k} \tag{1}$$

is its binary expansion then $X_1, X_2, \ldots$ is a sequence of independent random variables with Bernoulli distribution with parameter $1/2$.
Conversely, consider a sequence of independent random variables with Bernoulli distribution with parameter $1/2$ and define X using the equation (1). Prove that X has uniform distribution on the interval $[0, 1]$.

Exercise 2.2: Show that there is a random sequence $W_1, W_2, \ldots$ such that its increments W_1, $W_2 - W_1$, $W_3 - W_2, \ldots$ are independent random variables with standard normal distribution. Determine the distribution of the vector $(W_1, \ldots, W_n)$.

Definition: (D 1.14) We call the random sequence $X = (X_1, X_2, \ldots)$

- *iid* if the random variables X_j, $j \in \mathbb{N}$, are independent and identically distributed,
- *n-symmetric* if the distributions of $(X_1, \ldots, X_n, X_{n+1}, \ldots)$ and $(X_{k_1}, \ldots, X_{k_n}, X_{n+1}, \ldots)$ coincide for each finite permutation $(k_1, \ldots, k_n)$ of order $n \in \mathbb{N}$,
- *symmetric* if it is n-symmetric for each $n \in \mathbb{N}$,
- *stationary* if the distributions of $(X_1, \ldots, X_n, X_{n+1}, \ldots)$ and $(X_{n+1}, X_{n+2}, \ldots)$ coincide for each $n \in \mathbb{N}$.

Exercise 2.3: Show that the following statements are equivalent:

a) random sequence $X = (X_1, X_2, \ldots)$ is stationary,
b) X and $s(X)$ have the same distribution,
c) random vectors $(X_1, \ldots, X_{n-1})$ and $(X_2, \ldots, X_n)$ have the same distribution for each $n \in \mathbb{N}$.

Exercise 2.4: Prove the following assertions.

a) Each iid sequence is symmetric.
b) Each symmetric sequence is stationary.
c) Each $(n + 1)$-symmetric sequence is n-symmetric for any $n \in \mathbb{N}$.

d) Let $X = (X_1, X_2, \ldots)$ be an iid random sequence and $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ Borel-measurable mapping such that $f \circ s = s \circ f$ (f and the shift commute). Prove that in such a case $f(X) = (Y_1, Y_2, \ldots)$ is stationary. Does this assertion hold if we instead assumed only stationarity of X?

Exercise 2.5: Give an example of

a) a symmetric sequence which is not iid,
b) a stationary sequence which is not symmetric,
c) n-symmetric sequence which is not $(n+1)$-symmetric.

NMSA405: topic 3 – 0-1 laws, random walk

Theorem (Kolmogorov 0-1 law): Let $X = (X_1, X_2, \ldots)$ be a random sequence of independent random variables. Then $\mathbb{P}(X \in T)$ equals either 0 or 1 for any terminal set T.

Theorem (Hewitt-Savage 0-1 law): Let $X = (X_1, X_2, \ldots)$ be an iid random sequence. Then
$\mathbb{P}(X \in S)$ equals either 0 or 1 for any symmetric set S.

Exercise 3.1: Let $X = (X_1, X_2, \ldots)$ be a random sequence of independent random variables. Show that the event

$$\left[\sum_{n=1}^{\infty} X_n < \infty\right]$$

occurs with probability 0 or 1.

Definition: (D 2.5) Let $X = (X_1, X_2, \ldots)$ be an iid random sequence. We call the sequence of partial sums $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$ a *random walk*.

Exercise 3.2: Let $S = (S_1, S_2, \ldots)$ be a random walk. Consider the event

$$A = [S_n = 0 \text{ for infinitely many } n].$$

Show that $\mathbb{P}(A)$ equals either 0 or 1.

Exercise 3.3: The following variants of the limit behaviour of the random walk $S = (S_1, S_2, \ldots)$ are mutually exclusive:

- (i) $S_n = 0$ a.s. for all $n \in \mathbb{N}$,
- (ii) $S_n \underset{n\to\infty}{\longrightarrow} \infty$,
- (iii) $S_n \underset{n\to\infty}{\longrightarrow} -\infty$,
- (iv) $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$.

Prove that precisely one of these variants occurs with probability 1.

NMSA405: topic 4 – stopping times

Definition: Let $X = (X_1, X_2, \dots)$ be a random sequence. The σ-algebra generated by the random vector $(X_1, \dots, X_n)$ is $\sigma(X_1, \dots, X_n) = \{[(X_1, \dots, X_n) \in B_n], B_n \in \mathcal{B}^n\}$ and the σ-algebra generated by the sequence X is $\sigma(X) = \{[X \in B], B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})\}$.

Exercise 4.1: (P 2.1) Check that $\sigma(X_1, \dots, X_n)$ and $\sigma(X)$ are σ-algebras. Prove that

$$\sigma(X) = \sigma\left(\bigcup_{n=1}^{\infty} \sigma(X_1, \dots, X_n)\right).$$

Definition: (D 2.1) Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}$ a non-decreasing sequence of σ-algebras. We call $(\mathcal{F}_n)$ a *filtration*. Denote $\mathcal{F}_\infty = \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_n\right)$. We call the random sequence $X = (X_1, X_2, \dots)$ *adapted* to the filtration $(\mathcal{F}_n)$, shortly $\mathcal{F}_n$*-adapted* if $\sigma(X_1, \dots, X_n) \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$. If $\sigma(X_1, \dots, X_n) = \mathcal{F}_n$ for all $n \in \mathbb{N}$ we call $(\mathcal{F}_n)$ the *canonical filtration* of the sequence X.

Exercise 4.2: (P 2.2) Let $X = (X_1, X_2, \dots)$ be a random sequence and $S = (S_1, S_2, \dots)$ the sequence of its partial sums: $S_n = X_1 + \dots + X_n$, $n \in \mathbb{N}$. Show that X and S have the same canonical filtration. Compare the canonical filtrations of the sequence X and the sequence $X^2 = (X_1^2, X_2^2, \dots)$.

Definition: (D 2.3) The mapping $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time* with respect to the filtration $(\mathcal{F}_n)$ provided that $[T \le n] \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Let $X = (X_1, X_2, \dots)$ be a random sequence. A stopping time $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time of the sequence* X if $[T \le n] \in \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.

Exercise 4.3: Show that T is a stopping time with respect to the filtration $(\mathcal{F}_n)$ if and only if the random sequence $X_n = \mathbf{1}\{T \le n\}$ is $\mathcal{F}_n$-adapted.

Definition: (D 2.4) Let $(\mathcal{F}_n)$ be a filtration and T its stopping time. Then

$$\mathcal{F}_T = \{F \in \mathcal{F}_\infty : F \cap [T \le n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

is called the *stopping time σ-algebra.*

Exercise 4.4: Show that $\mathcal{F}_T$ defines a σ-algebra.

Exercise 4.5: (P 2.3) Show that T is a stopping time with respect to the filtration $(\mathcal{F}_n)$ if and only if $[T = n] \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Further show that the following holds:

$$\mathcal{F}_T = \{F \in \mathcal{F}_\infty : F \cap [T = n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

Exercise 4.6: Consider a fixed $n_0 \in \mathbb{N}$ and $T = n_0$. Show that T is a stopping time with respect to any filtration $(\mathcal{F}_n)$ and determine the σ-algebra $\mathcal{F}_T$.

Definition: We define the mapping $X_T : \Omega \to \mathbb{R}$ as

$$X_T(\omega) = \begin{cases} X_{T(\omega)}(\omega) & \text{pro } T(\omega) < \infty, \\ 0 & \text{pro } T(\omega) = \infty. \end{cases}$$

Exercise 4.7: (P 2.4) Let S and T be stopping times with respect to the filtration $(\mathcal{F}_n)$ and let the sequence X be $\mathcal{F}_n$-adapted. Show that:

- a) T and X_T are $\mathcal{F}_T$-measurable random variables,
- b) $\min\{S,T\}$, $\max\{S,T\}$ and $S+T$ are stopping times with respect to the filtration $(\mathcal{F}_n)$,
- c) $\min\{T,n\}$ is a $\mathcal{F}_n$-measurable random variable for any $n \in \mathbb{N}$.

Exercise 4.8: Let $T_1, T_2, \dots$ be a sequence of stopping times with respect to the filtration $(\mathcal{F}_n)$. Show that $\sup_n T_n$ and $\inf_n T_n$ are also stopping times with respect to the filtration $(\mathcal{F}_n)$.

Exercise 4.9: (P 2.5a) Let T be a stopping time with respect to the filtration $(\mathcal{F}_n)$. Consider the mapping $\lambda : \Omega \to \mathbb{N} \cup \{\infty\}$ which is $\mathcal{F}_T$-measurable and fulfills $\lambda \geq T$. Show that λ is a stopping time with respect to the filtration $(\mathcal{F}_n)$.

Exercise 4.10: (P 2.5b) Let $X = (X_1, X_2, \dots)$ be a random sequence and T its stopping time. For $B \in \mathcal{B}(\mathbb{R})$ we define $\lambda = \min\{k > T : X_k \in B\}$, i.e. the first hitting time of the set B by the sequence X after the time T. Show that λ is a stopping time of the sequence X.

Exercise 4.11: Let $(S_1, S_2, \dots)$ be a symmetric simple random walk (with the step X_n taking on only the values 1 and -1 with equal probabilities). Determine whether the following random variables are stopping times of the sequence $X = (X_1, X_2, \dots)$:

- a) $T_N = \max\{n \leq N : S_n = 0\}$ for $N \in \mathbb{N}$,
- b) $\lambda = \min\{n : S_n = 5\}$,
- c) $\nu = \min\{n : S_n < -3\}$,
- d) $\lambda + \nu$, $\min\{\lambda, \nu\} + 1$, $\max\{\lambda, \nu\}$, $\max\{\lambda, \nu\} - 1$, $2\lambda - 1$, λ^2.

NMSA405: topic 5 – symmetric simple random walk

Definition: (D 2.6) Let $X_1, X_2, \dots$ be an iid random sequence with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. We call the corresponding random walk (S_n) the *symmetric simple random walk*.

Exercise 5.1: (P 2.9) (Reflection principle) Let (S_n) be a symmetric simple random walk. Consider the stopping time T, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Denote

$$S_k^r = 2S_{\min\{k,T\}} - S_k, \quad k \in \mathbb{N}.$$

Then

$$(S_1^r, S_2^r, \dots) \stackrel{d}{=} (S_1, S_2, \dots).$$

Exercise 5.2: (P 2.10) (Maxima of the symmetric simple random walk) For a symmetric simple random walk (S_n) denote $M_n = \max_{k=1,\dots,n} S_k$, $n \in \mathbb{N}$. Consider the stopping time T, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Then

$$\mathbb{P}(T \leq n) = \mathbb{P}(M_n \geq a) = 2\mathbb{P}(S_n \geq a) - \mathbb{P}(S_n = a) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(M_n \geq a) = 1.$$

NMSA405: topic 6 – martingales

Definition: (D 2.10) Let $\{\mathcal{F}_n\}$ be a filtration and let $X = (X_1, X_2, \ldots)$ be a sequence of integrable random variables. We say that X is an $\mathcal{F}_n$-*martingale* if it is $\mathcal{F}_n$-adapted and $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ a.s. for all $n \in \mathbb{N}$. If $\{\mathcal{F}_n\}$ is the canonical filtration of X, we call X simply a *martingale* and it satisfies $\mathbb{E}[X_{n+1}|X_1, X_2, \ldots, X_n] = X_n$ a.s. for all $n \in \mathbb{N}$. If the equality sign is replaced by $\geq$, X is called $\mathcal{F}_n$-*submartingale* or *submartingale*, respectively. If the equality sign is replaced by $\leq$, X is called $\mathcal{F}_n$-*supermartingale* or *supermartingale*, respectively.

Exercise 6.1: (P 2.18) Let (X_n) be a sequence of independent integrable random variables. Denote $S_n = X_1 + \ldots + X_n$ for $n \in \mathbb{N}$.

- c) If $\mathbb{E}X_n = 1$ for all $n \in \mathbb{N}$ then $Z_n = \prod_{j=1}^n X_j$ is a martingale.
- d) If $\mathbb{P}(X_n = -1) = q$ and $\mathbb{P}(X_n = 1) = p$ where $p \in (0,1)$ and $p+q = 1$ then $Y_n = (q/p)^{S_n}$ is a martingale.

Exercise 6.2: Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, a finite measure $\mu \ll \lambda$ on $([0,1], \mathcal{B}([0,1]))$ and an increasing sequence of sets $\{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = 1\}$ such that

$$\max_{k \in \{0,1,\ldots,k_n-1\}} |t_{k+1}^n - t_k^n| \to 0.$$

Denote $B_k^n = [t_k^n, t_{k+1}^n)$ and

$$D_n(x) = \frac{\mu(B_k^n)}{\lambda(B_k^n)}, \quad x \in B_k^n.$$

Show that (D_n) is an $(\mathcal{F}_n)$-martingale where $\mathcal{F}_n = \sigma(B_1^n, \ldots, B_{k_n}^n)$. What is the a.s. limit of D_n for $n \to \infty$?

Exercise 6.3: Let Y be an integrable random variable and let $(\mathcal{F}_n)$ be a filtration. Consider the sequence $X_n = \mathbb{E}[Y \mid \mathcal{F}_n]$, $n \in \mathbb{N}$, and show that (X_n) is a $\mathcal{F}_n$-martingale.

Exercise 6.4: (Pólya urn model) Consider an urn which at time $n = 0$ contains b black and w white balls, $b, w \in \mathbb{N}$. At each time $n \in \mathbb{N}$ we draw a ball from the urn at random, write down its color and put it back together with $\Delta \in \mathbb{N}$ new balls of the same color. Denote X_n the relative frequency of the white balls in the urn at time n (i.e. the ratio of the number of white balls to the number of all balls in the urn at the given time). Show that (X_n) is a martingale. Consider also the case with $\Delta = 0$ or $\Delta = -1$.

Exercise 6.5: A deck of cards contains a black and b red cards. The deck has been shuffled randomly and we start drawing the cards from the top one after another. Denote X_n the relative number of black cards after drawing n cards where $n \in \{0, \ldots, a+b-1\}$. Let $X_n = X_{a+b-1}$ for $n \geq a+b$. Show that (X_n) is a martingale.

Exercise 6.6: Let (X_n) be a sequence of random variables such that the probability density function $f_n : \mathbb{R}^n \to (0, \infty)$ of the random vector $(X_1, \ldots, X_n)$ is positive on $\mathbb{R}^n$. Suppose we are given a consistent system of probability density functions (g_n), i.e. $g_n : \mathbb{R}^n \to [0, \infty)$ fulfills

$\int_{\mathbb{R}^n} g_n(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}} g_{n+1}(x,y)\,\mathrm{d}y = g_n(x)$ for almost all $x \in \mathbb{R}^n$. We define the *likelihood ratio*

$$S_n = \frac{g_n(X_1,\dots,X_n)}{f_n(X_1,\dots,X_n)}, \quad n \in \mathbb{N}.$$

Show that (S_n) is a martingale.

Exercise 6.7: Let $(\mathcal{F}_n)$ be a filtration on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and (Q_n) a consistent system of $\mathcal{F}_n$-probability measures, i.e. $Q_{n+1}|_{\mathcal{F}_n} = Q_n$ for $n \in \mathbb{N}$, such that $Q_n \ll \mathbb{P}|_{\mathcal{F}_n}$. We define $X_n = \frac{\mathrm{d}Q_n}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_n}}$. Show that (X_n) is a $\mathcal{F}_n$-martingale.

Exercise 6.8: Let $X_n : (\Omega,\mathcal{F}) \to (S_n,\mathcal{S}_n)$, $n \in \mathbb{N}$, be a sequence of random variables. Let $\mathbb{P}$ be a probability measure on $(\Omega,\mathcal{F})$ and (ν_n) a consistent system of probability distributions such that $\nu_n \ll P_{X_1,\dots,X_n} =: \mu_n$. Similarly as above show that the likelihood ratio $T_n = \frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}(X_1,\dots,X_n)$ between $H_1 : (X_1,\dots,X_n)^{\mathrm{T}} \sim \nu_n$ and $H_0 : (X_1,\dots,X_n)^{\mathrm{T}} \sim \mu_n$ is a $\sigma(X_1,\dots,X_n)$-martingale under the null hypothesis H_0.

Exercise 6.9: Let (X_n) be an iid random sequence. Let $\alpha \in \mathbb{R}$ be such that $\beta = \ln \mathbb{E}e^{\alpha X_1} \in \mathbb{R}$. We define $Z_n = \exp\{\alpha S_n - \beta n\}$ where $S_n = X_1 + \ldots + X_n$. Show that (Z_n) is a martingale.

Exercise 6.10: Let (X_n) be a sequence of independent integrable random variables with zero mean. We define $M_n = \sum_{k=1}^n \prod_{i=1}^k X_i$ for $n \in \mathbb{N}$. Show that (M_n) is a martingale.

NMSA405: topic 7 – Doob decomposition

Definition: (D 2.11) Let $\{\mathcal{F}_n\}$ be a filtration. The random sequence $I_1, I_2, \ldots$ is $\mathcal{F}_n$-*predictable* if I_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$, where we put $\mathcal{F}_0 = \{\emptyset, \Omega\}$, i.e. I_1 is a constant.

Theorem (Doob decomposition theorem): Let $\{S_n\}$ be an $\mathcal{F}$-submartingale. Then there exists an $\mathcal{F}_n$-martingale $\{M_n\}$ and a non-decreasing $\mathcal{F}_n$-predictable sequence $\{I_n\}$ so that $S_n = M_n + I_n, n \in \mathbb{N}$. The summands M_n and I_n are a.s. uniquely determined under the additional condition $I_1 = 0$. The sequence $\{I_n\}$ is called the *compensator* of $\{S_n\}$.

Exercise 7.1: Let (X_n) be an iid random sequence with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = \sigma^2 \in (0,\infty)$ and $\mathbb{E}\exp\{X_1\} = \gamma < \infty$. Consider the corresponding random walk (S_n). Show that the following sequences are submartingales and determine their compensators:

- a) S_n^2,
- b) $V_n = X_1^2 + \ldots + X_n^2$,
- c) $\exp\{S_n\}$.

Exercise 7.2: Let (X_n) be a $\mathcal{F}_n$-martingale such that $X_n \in L_2$. Show that

$$I_n = \sum_{k=1}^{n} \operatorname{var}(X_k \mid \mathcal{F}_{k-1})$$

is the compensator of the sequence $Z_n = X_n^2$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

NMSA405: topic 8 – optional sampling theorem

Theorem (Optional sampling theorem): Let $X_1, X_2, \dots$ be an $\mathcal{F}_n$-martingale and let $T_1 \le T_2 \le \dots$ be a.s. finite $\mathcal{F}_n$-stopping times. If

$$X_{T_k} \in L_1 \quad \text{and} \quad \lim_{n\to\infty} \int_{[T_k>n]} |X_n| \, \mathrm{d}\mathbb{P} = 0$$

for all $k \in \mathbb{N}$, then $(X_{T_1}, X_{T_2}, \dots)$ is an $\mathcal{F}_{T_n}$-martingale.

Exercise 8.1: Let (X_n) be a sequence of iid random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ and let $S_n = \sum_{k=1}^n 2^{k-1} X_k$, $n \in \mathbb{N}$. Consider the first hitting time T of the sequence (S_n) of the set $\{1\}$. Then for (S_n) and T the optional sampling theorem does not hold. Show that $\mathbb{E}S_1 \neq \mathbb{E}S_T$ and the condition $\lim_{n\to\infty} \int_{[T>n]} |S_n| \, \mathrm{d}\mathbb{P} = 0$ is not fulfilled.

Exercise 8.2: (remark to the Theorem 3.5) Let (X_n) be a $\mathcal{F}_n$-martingale and $T < \infty$ a.s. be a $\mathcal{F}_n$-stopping time. Show that the condition

$$\exists\, 0 < c < \infty : T > n \Longrightarrow |X_n| \le c \quad \text{a.s.}$$

does not imply the condition

$$X_T \in L_1 \quad \text{and} \quad \int_{[T>n]} |X_n| \, \mathrm{d}\mathbb{P} \xrightarrow[n\to\infty]{} 0$$

from the Theorem 3.3.
Hint: Consider the sequence $X_n = \sum_{k=1}^n 3^k Y_k$ where (Y_k) is a sequence of iid random variables with the uniform distribution on $\{-1, 0, 1\}$.

NMSA405: topic 9 – random walks

Definition: Let (X_n) be an iid random sequence such that $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p$ where $p \in [0, 1]$. We call the corresponding random walk (S_n) a *(simple) discrete random walk*. If $p = 1/2$ we get the symmetric simple random walk.

Exercise 9.1: Consider the stopping time $T^B = \min\{n \in \mathbb{N} : S_n \notin B\}$ defined as the first exit time of the discrete random walk S_n from the bounded set $B \in \mathcal{B}(\mathbb{R})$ and the stopping time $T_a = \min\{n \in \mathbb{N} : S_n = a\}$ defined as the first hitting time of the random walk S_n of the set $\{a\}$ for $a \in \mathbb{Z}$. Show that

1. $T^B < \infty$ a.s.,
2. $T_a < \infty$ a.s. if $p = 1/2$.

Exercise 9.2: Show that the discrete random walk fulfills

(i) $S_n \xrightarrow[n\to\infty]{} \infty$ a.s. $\Longleftrightarrow p > 1/2$,

(ii) $S_n \xrightarrow[n\to\infty]{} -\infty$ a.s. $\Longleftrightarrow p < 1/2$,

(iii) $\limsup_{n\to\infty} S_n = \infty$ a.s., $\liminf_{n\to\infty} S_n = -\infty$ a.s. $\Longleftrightarrow p = 1/2$.

Exercise 9.3: Consider a discrete symmetric random walk (S_n). For $a, b \in \mathbb{Z}$, $a < 0$, $b > 0$, we define $T_{a,b} = \min\{n \in \mathbb{N} : S_n \notin (a, b)\}$ as the first exit time of S_n from the interval (a, b). Show that in that case

$$\mathbb{P}(S_{T_{a,b}} = a) = \frac{b}{b-a} \quad \text{and} \quad \mathbb{E}T_{a,b} = -ab.$$

Corollary:
(i) $\mathbb{E}T^B < \infty$ for any bounded set $B \in \mathcal{B}(\mathbb{R})$, (ii) $\mathbb{E}T_b = \infty$ for any $b \in \mathbb{Z}$, $b \neq 0$.

Exercise 9.4: Let (S_n) be a symmetric simple random walk and let $A < 0 < B$ be independent integrable random variables, independent of (S_n). Denote $T = \min\{n \in \mathbb{N} : S_n \notin (A, B)\}$. Show that in that case

$$\mathbb{P}(S_T = A) = \mathbb{E}\frac{B}{B-A} \quad \text{and} \quad \mathbb{E}T = -\mathbb{E}A \cdot \mathbb{E}B < \infty.$$

NMSA405: topic 10 – convergence theorems

Exercise 10.1: Give an example of a martingale which converges to the random variable $X_\infty \in L_1$ almost surely but not in L_1.

Exercise 10.2: Let (Y_n) be a sequence of independent random variables such that

$$\mathbb{P}(Y_n = 2^n - 1) = 2^{-n}, \quad \mathbb{P}(Y_n = -1) = 1 - 2^{-n}, \quad n \in \mathbb{N}.$$

Check that $X_n = \sum_{k=1}^n Y_k$ is a martingale. Show that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} -\infty$ and hence the assumptions of the martingale convergence theorems cannot be fulfilled.

Exercise 10.3: (martingale proof of the Kolmogorov 0-1 law) Let $X = (X_1, X_2, \dots)$ be a sequence of independent random variables and $F = [X \in T]$ where $T \in \mathcal{T}$ is a terminal set. Show that

$$\forall n \in \mathbb{N} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] = \mathbb{P}(F) \quad \text{a.s.} \qquad \text{and at the same time} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] \xrightarrow[n\to\infty]{\text{a.s.}} \mathbf{1}_F.$$

From this conclude that $\mathbb{P}(F)$ is either 0 or 1.