Ergodicity

Definition 8.1: We say that a stationary sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ is *mean square ergodic* or it follows the law of large numbers in $L_2(\Omega, \mathcal{A}, P)$ if, as $n \to \infty$,

$$\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t \to \mu \qquad \text{in the mean square.} \tag{1}$$

If $\{X_t, t \in \mathbb{Z}\}$ is a sequence that is mean square ergodic then

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{P} \mu,$$

i.e. $\{X_t, t \in \mathbb{Z}\}$ satisfies the weak law of large numbers for stationary sequences.

Definition 8.2: A stationary mean square continuous process $\{X_t, t \in \mathbb{R}\}$ with mean μ is *mean square ergodic* if, as $\tau \to \infty$,

$$\overline{X}_\tau = \frac{1}{\tau}\int_0^\tau X_t \,\mathrm{d}t \to \mu \qquad \text{in the mean square.} \tag{2}$$

Remark: The above described convergences imply that the empirical average (1) or the integral (2) are weakly consistent estimates of the mean value μ of the random sequence or process $\{X_t\}$, respectively.

Theorem 8.1: A stationary random sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ and autocovariance function R is mean square ergodic if and only if

$$\frac{1}{n}\sum_{t=1}^{n} R(t) \to 0 \quad \text{as } n \to \infty.$$

If the sequence is real-valued and moreover satisfies $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$ then $n \operatorname{var}(\overline{X}_n) \to \sum_{t=-\infty}^{\infty} R(t)$.

Theorem 8.2: A stationary, mean square continuous process $\{X_t, t \in \mathbb{R}\}$ is mean square ergodic if and only if its autocovariance function satisfies the condition

$$\frac{1}{\tau}\int_0^\tau R(t)\,\mathrm{d}t \to 0 \quad \text{as } \tau \to \infty.$$

If the process is real-valued and moreover satisfies $\int_{-\infty}^{\infty} |R(t)|\,\mathrm{d}t < \infty$ then $\tau \operatorname{var}(\overline{X}_\tau) \to \int_{-\infty}^{\infty} R(t)\,\mathrm{d}t$.

Exercise 8.1: Are the AR models from Exercises 6.3–6.5 mean square ergodic? And what about the ARMA(2,1) model from Exercise 6.6?

Exercise 8.4: Let $\{X_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with autocovariance function $R(t) = \cos(\pi t), t \in \mathbb{Z}$. Is $\{X_t, t \in \mathbb{Z}\}$ mean square ergodic?

Exercise 8.5: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with autocovariance function $R(t) = \cos t, t \in \mathbb{R}$. Is $\{X_t, t \in \mathbb{R}\}$ mean square ergodic?

Prediction based on infinite history

Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence. $H_{-\infty}^n = \mathcal{H}\{\dots, X_{n-1}, X_n\}$ denotes the Hilbert space generated by the random variables $\{X_t, t \leq n\}$, i.e. by the history of the process $\{X_t, t \in \mathbb{Z}\}$ up to time n.

Prediction $\widehat{X}_{n+h}(n)$ of X_{n+h} (where $h \in \mathbb{N}$) based on the infinite history $X_n, X_{n-1}, \dots$ is the orthogonal projection of the random variable X_{n+h} into the space $H_{-\infty}^n$. We denote $\widehat{X}_{n+h}(n) = P_{H_{-\infty}^n}(X_{n+h})$.

Prediction error (*residual variance*) is defined as $\mathbb{E}|X_{n+h} - \widehat{X}_{n+h}(n)|^2$.

Consider a causal and invertible ARMA(m, n) sequence. Invertibility implies that

$$X_{n+1} = -\sum_{j=1}^{\infty} d_j X_{n+1-j} + Y_{n+1}, \quad n \in \mathbb{Z}.$$

Causality implies that $X_n \in \mathcal{H}\{\dots, Y_{n-1}, Y_n\} \perp Y_{n+1}$. Thus the random variable Y_{n+1} is uncorrelated with X_n, and similarly with any other $X_{n-k}, k \in \mathbb{N}$. From linearity and continuity of the inner product we get $Y_{n+1} \perp H_{-\infty}^n$. Furthermore, $-\sum_{j=1}^{\infty} d_j X_{n+1-j} \in H_{-\infty}^n$. Thus the best linear prediction of X_{n+1} based on the whole history X_n, X_{n-1}, ... is the projection

$$\widehat{X}_{n+1} = -\sum_{j=1}^{\infty} d_j X_{n+1-j}, \tag{3}$$

and the prediction error is

$$\mathbb{E}|X_{n+1} - \widehat{X}_{n+1}|^2 = \mathbb{E}|Y_{n+1}|^2 = \sigma^2.$$

Exercise 7.3: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is WN$(0, \sigma^2)$. Determine $\widehat{X}_{n+h}$ for $h \in \mathbb{N}$ and compute the prediction error.

Exercise 7.4: Consider the AR(2) model defined by

$$X_t - 0.4X_{t-1} + 0.04X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. Find the prediction of X_{n+1}, X_{n+2} and X_{n+3} based on the history $X_n, X_{n-1}, \dots$. Compute the prediction errors.

Exercise 7.6: Consider the ARMA(2,1) model defined by

$$X_t - 0.5X_{t-1} + 0.04X_{t-2} = Y_t + 0.25Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. Find the prediction of X_{n+1} and X_{n+2} based on the history X_n, X_{n-1}, ... Compute the prediction errors.

Prediction based on finite history

Concerning the prediction based on the finite history we denote $H_1^n = \mathcal{H}\{X_1, \ldots, X_n\}$ the Hilbert space generated by the random variables $X_1, \ldots, X_n$.

The best linear prediction of X_{n+h} (for $h \in \mathbb{N}$) is the orthogonal projection into the space H_1^n, i.e. $\widehat{X}_{n+h}(n) = \sum_{j=1}^n c_j X_j \in H_1^n$ such that $X_{n+h} - \widehat{X}_{n+h}(n) \perp H_1^n$.

Prediction error (*residual variance*) is again defined as $\mathbb{E}|X_{n+h} - \widehat{X}_{n+h}(n)|^2$.

Exercise 7.9: Consider a stationary AR(2) process $\{X_t, t \in \mathbb{Z}\}$ defined by the equation

$$X_t + \frac{1}{3}X_{t-1} + \frac{1}{3}X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Assume that you have observed the values of the process
a) $X_0 = X_1 = 1$,
b) $X_0 = 1$,
c) $X_1 = 1$,
d) $X_0 = X_1 = X_{-1} = 1$.
Predict the value of X_2 and compute the prediction error.