FACULTY
OF MATHEMATICS
AND PHYSICS
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STOCHASTIC PROCESSES 2

Collection of solved exercises

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Contents

Introduction

This collection of solved exercises was created as a supporting material for the exercise classes for the course “NMSA409 Stochastic processes 2” at the Faculty of Mathematics and Physics, Charles University, Prague, during the winter semester 2017/2018.

The lecture notes for this course are available online [4]. Additional material can be found e.g. in the book by Peter J. Brockwell and Richard A. Davis [1].

1 Examples of stochastic processes

Example 1.1: The sequence $\{X_t, t \in \mathbb{Z}\}$ of uncorrelated random variables with zero mean and finite positive variance is called *white noise*. A sample realization is shown on Fig. 1.

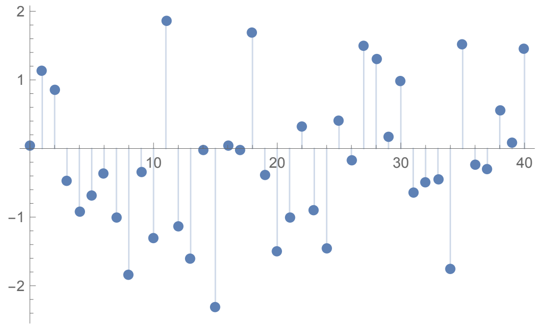

Figure 1: A sample realization of the white noise sequence with standard normal marginals.

Example 1.2: Let X be a random variable with a uniform distribution on the interval $(0, \pi)$. We define the sequence of random variables $\{Y_t, t \in \mathbb{N}\}$ by $Y_t = \cos(tX)$. Fig. 2 shows sample realizations of the sequence for different realizations of the random variable X.

In this example we consider the underlying probability space consisting of the interval $\Omega = (0, \pi)$ with the corresponding Borel σ-algebra and the appropriate multiple of the Lebesgue measure restricted to the interval. The mapping $X : \Omega \mapsto (0, \pi)$ is the identity. With this representation we can easily assign one sample realization of the sequence to each value of $\omega \in (0, \pi)$, see Fig. 2.

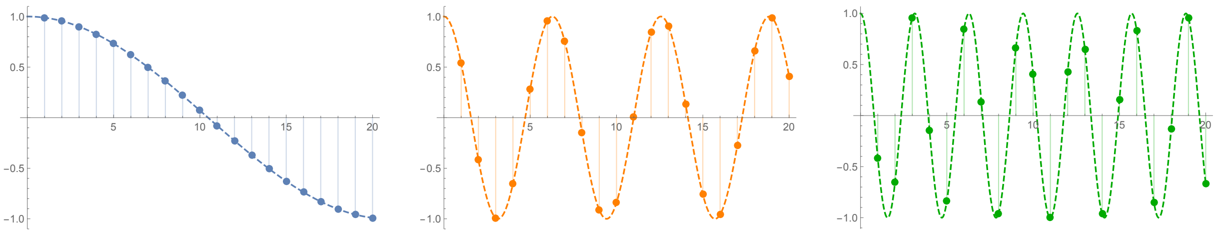

Figure 2: Sample realizations of the sequence from Example 1.2. The realizations correspond to the elementary events $\omega_1 = 0.15$ (left), $\omega_2 = 1$ (middle) and $\omega_3 = 2$ (right). Note that the curves $\cos(\omega_i t)$ are also plotted in dashed lines for clarity but only the values of the sequence in discrete times (shown by circles) are observed.

Example 1.3: Consider the stochastic process $X_t = \cos(t + B)$, $t \in \mathbb{R}$, where B is a random variable with a uniform distribution on the interval $(0, 2\pi)$. Fig. 3 shows sample realizations of the sequence for different realizations of the random variable B. Note that the realization of B does not influence the shape of the trajectories, only their shift along the real line.

In this example we consider the underlying probability space consisting of the interval $\Omega = (0, 2\pi)$ with the corresponding Borel σ-algebra and the appropriate multiple of the Lebesgue measure restricted to the interval. The mapping $B : \Omega \mapsto (0, 2\pi)$ is the identity. With this representation we can easily assign one sample realization of the sequence to each value of $\omega \in (0, \pi)$, see Fig. 3.

Example 1.4: The Wiener process $\{W_t, t \geq 0\}$ with parameter $\sigma^2 > 0$ (also called the Brownian motion process) is a Gaussian stochastic process with continuous trajectories, $W_0 = 0$ a.s. and independent increments with normal distribution (for any $0 \leq t < s$ the increment $W_s - W_t$ has normal distribution with zero mean and variance $\sigma^2(s - t)$). Fig. 4 shows sample realizations of the Wiener process for $\sigma^2 = 1$.

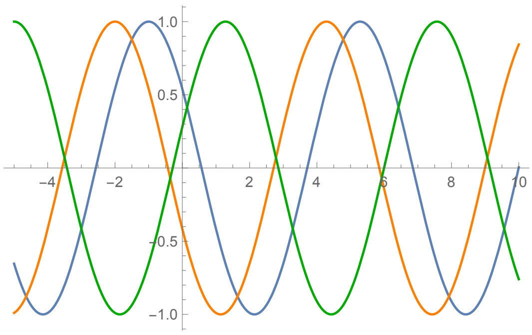

Figure 3: Sample realizations of the process from Example 1.3. The realizations correspond to the elementary events $\omega_1 = 1$ (blue), $\omega_2 = 2$ (red) and $\omega_3 = 3$ (green).

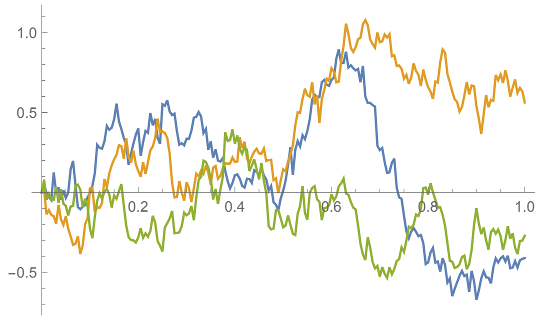

Figure 4: Sample realizations of the Wiener process with $\sigma^2 = 1$, see Example 1.4.

Example 1.5: Let $\{W_t, t \geq 0\}$ be the Wiener process with parameter $\sigma^2 > 0$. We define $B_t = W_t - tW_1, t \in [0, 1]$. The stochastic process $\{B_t, t \in [0, 1]\}$ is called the Brownian bridge. Fig. 5 shows sample realizations of the Brownian bridge for $\sigma^2 = 1$. Note that the process is constructed such that $B_1 = 0$ a.s.

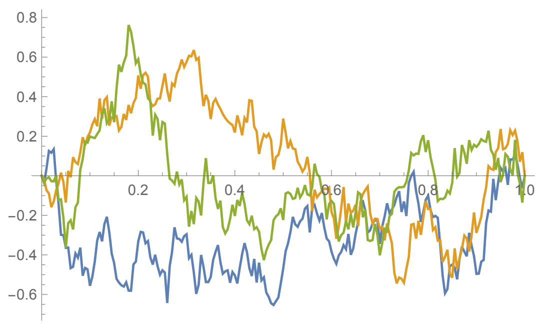

Figure 5: Sample realizations of the Brownian bridge, see Example 1.5.

Example 1.6: Let $\{W_t, t \geq 0\}$ be the Wiener process with parameter $\sigma^2 > 0$. We define the so-called Ornstein-Uhlenbeck process $\{U_t, t \geq 0\}$ by the formula $U_t = \mathrm{e}^{-\alpha t/2} W_{\exp\{\alpha t\}}, t \geq 0$, where $\alpha > 0$ is a parameter. Fig. 6 shows sample realizations of the Ornstein-Uhlenbeck process for $\sigma^2 = 1$ and $\alpha = 1$.

Example 1.7: The Poisson process $\{N_t, t \geq 0\}$ with intensity $\lambda > 0$ is the counting process with $N_0 = 0$ a.s. and independent increments with Poisson distribution (for any $0 \leq t < s$ the increment $N_s - N_t$ has Poisson distribution with the parameter $\lambda(s - t)$). Fig. 7 shows sample realizations of the Poisson process for $\lambda = 1$.

Example 1.8: Let $\{N_t, t \geq 0\}$ be the Poisson process with intensity $\lambda > 0$ and let a A be a random

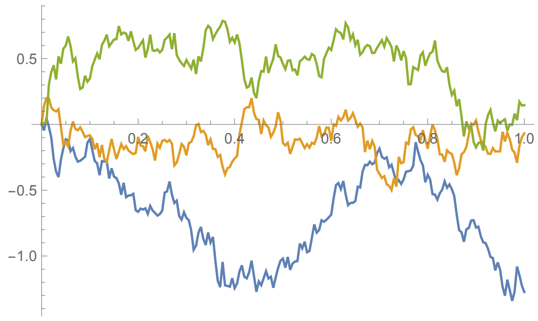

Figure 6: Sample realizations of the Ornstein-Uhlenbeck process, see Example 1.6.

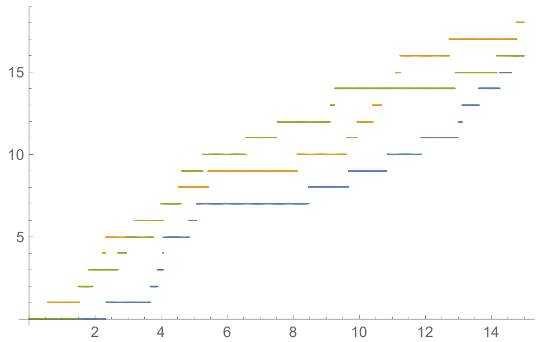

Figure 7: Sample realizations of the Poisson process with $\lambda = 1$, see Example 1.7.

variable with symmetric alternative distribution on $\{-1, 1\}$, i.e. $\mathbb{P}(A = 1) = \mathbb{P}(A = -1) = \frac{1}{2}$, independent of the process $\{N_t, t \geq 0\}$. We define $X_t = A(-1)^{N_t}$, $t \geq 0$. This is a process of switching between two values (A and $-A$) at random times given by the Poisson process. Fig. 8 shows two sample realizations of the process. The realization of A only determines whether the process starts at 1 or -1, i.e. the absolute value of the process is 1 for any $t \geq 0$.

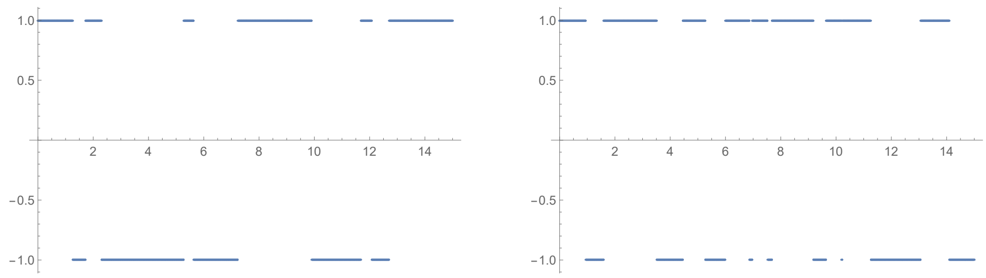

Figure 8: Sample realizations of the switching process from Example 1.8 with the symmetric alternative distribution of the random variable A.

We may also consider another distribution for the random variable A, for example the standard Gaussian $\mathcal{N}(0, 1)$ distribution. Note that in this case A has zero mean and unit variance just as above. Now the realization of A not only determines the sign of the initial value but also the absolute value of the process – it is equal to $|A|$ for any $t \geq 0$. Fig. 9 shows sample realizations of the process for different realizations of $A(\omega)$ and $\{N_t\}(\omega)$.

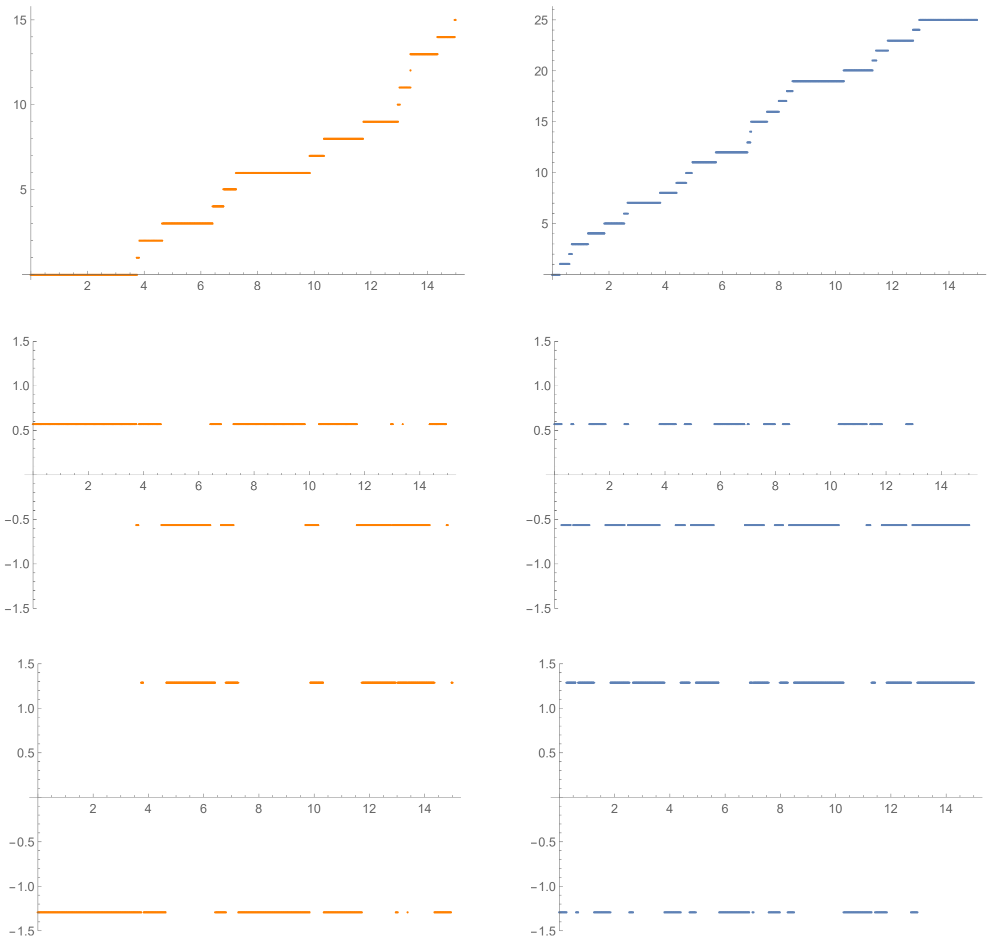

Figure 9: Sample realizations of the switching process from Example 1.8. In the first row two sample realizations $\{N_t\}(\omega)$ in the other two rows corresponding realizations of the process $\{X_t\}$ for $A(\omega) = 0.564$ and $A(\omega) = -1.291$.

2 Autocovariance function and stationarity

Definition 2.1: Let $\{X_t, t \in T\}$, where $T \subset \mathbb{R}$, be a stochastic process with finite second moments, i.e. $\mathbb{E}|X_t|^2 < \infty$ for all $t \in T$. (In general complex) function of two arguments defined on $T \times T$ by the formula

$$R(s,t) = \mathbb{E}(X_s - \mathbb{E}X_s)(\overline{X_t - \mathbb{E}X_t})$$

is called the *autocovariance function of the process* $\{X_t, t \in T\}$.

Definition 2.2: Let $\{X_t, t \in T\}$ be a stochastic process. We call the process

- *strictly stationary* if for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n \in T$ and $h > 0$ such that $t_1+h, \ldots, t_n + h \in T$ it holds that

$$\mathbb{P}(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n) = \mathbb{P}(X_{t_1+h} \le x_1, \ldots, X_{t_n+h} \le x_n),$$

- *weakly stationary* if the process has finite second moments, a constant mean value $\mathbb{E}X_t = \mu$ and its autocovariance function $R(s,t)$ depends only on $s-t$,
- *covariance stationary* if the process has finite second moments and its autocovariance function $R(s,t)$ depends on $s-t$ only,
- *process of uncorrelated random variables* if the process has finite second moments and for its autocovariance function it holds that $R(s,t) = 0$ for all $s \neq t$,
- *centered* if $\mathbb{E}X_t = 0$ for all $t \in T$,
- *Gaussian* if for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in T$ the vector $(X_{t_1}, \ldots, X_{t_n})^{\mathrm{T}}$ has n-dimensional normal distribution,
- *process with independent increments* if for all $t_1, \ldots, t_n \in T$ fulfilling $t_1 < \cdots < t_n$ the random variables $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,
- *process with stationary increments* if for all $s, t \in T$ fulfilling $s < t$ the distribution of increments $X_t - X_s$ depends on $t - s$ only.

Remark: The autocovariance function of a weakly stationary process is a function of one variable

$$R(t) = R(t,0) = R(t-0), \quad t \in T.$$

Theorem 2.1: The following implications hold:

a) strictly stationary with finite second moments $\Rightarrow$ weakly stationary,

b) weakly stationary and Gaussian $\Rightarrow$ strictly stationary,

c) weakly stationary $\Rightarrow$ covariance stationary,

d) process of uncorrelated random variables with (the same) finite second moment $\Rightarrow$ covariance stationary,

e) centered process of uncorrelated random variables with (the same) finite second moment $\Rightarrow$ weakly stationary.

Theorem 2.2: The autocovariance function has the following properties:

- it is non-negative on the diagonal: $R(t,t) \geq 0$,
- it is Hermitian: $R(s,t) = \overline{R(t,s)}$,
- it fulfills the Cauchy-Schwarz inequality: $|R(s,t)| \leq \sqrt{R(s,s)}\sqrt{R(t,t)}$,
- it is positive semidefinite: for all $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in T$ it holds that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} R(t_j, t_k) \geq 0.$$

Remark: The non-negative values on the diagonal and the Hermitian property follow from the positive semidefiniteness.

Theorem 2.3: For each positive semidefinite function R on $T \times T$ there is a stochastic process $\{X_t, t \in T\}$ with finite second moments such that R is its autocovariance function.

Corollary 2.4: Any complex valued function R on $T \times T$ is positive semidefinite if and only if it is an autocovariance function of some stochastic process.

Exercise 2.1: Let $X_t = a + bt + Y_t,\ t \in \mathbb{Z}$, where $a, b \in \mathbb{R},\ b \neq 0$ and $\{Y_t, t \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables with zero mean and finite positive variance σ^2.

a) Determine the autocovariance function of the sequence $\{X_t, t \in \mathbb{Z}\}$ and discuss its stationarity.

b) For $q \in \mathbb{N}$ we define random variables V_t by the formula

$$V_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j}, \quad t \in \mathbb{Z}.$$

Determine the autocovariance function of the sequence $\{V_t, t \in \mathbb{Z}\}$ and discuss its stationarity.

Solution:

a) We know from the assignment that $\mathbb{E}Y_t = 0$ and $\operatorname{var} Y_t = \sigma^2 > 0,\ t \in \mathbb{Z}$.

It follows from the linearity of expectation that

$$\mathbb{E}X_t = \mathbb{E}\left(a + bt + Y_t\right) = a + bt + \mathbb{E}Y_t = a + bt,\ t \in \mathbb{Z}.$$

Hence the mean value function is not constant and the sequence $\{X_t, t \in \mathbb{Z}\}$ is neither strictly nor weakly stationary.

The autocovariance function R_X can be then calculated as follows:

$$R_X(s,t) = \mathbb{E}\left(X_s - \mathbb{E}X_s\right)\left(X_t - \mathbb{E}X_t\right) = \mathbb{E}Y_sY_t,\ s,t \in \mathbb{Z}.$$

Since $\{Y_t, t \in \mathbb{Z}\}$ is a sequence of independent random variables it follows that $\mathbb{E}Y_sY_t = \mathbb{E}Y_s^2 = \sigma^2$ if and only if $s = t$, otherwise $\mathbb{E}Y_sY_t = \mathbb{E}Y_s\mathbb{E}Y_t = 0$.

The autocovariance function can be written as $R_X(s,t) = \sigma^2 \cdot \delta(s-t),\ s,t \in \mathbb{Z}$, and we conclude that the sequence $\{X_t, t \in \mathbb{Z}\}$ is covariance stationary.

Note that it could be also argued straight away that $R_X = R_Y$ since adding a constant to random variables does not affect the respective covariances. Thus the general observation is that changing the drift, i.e. the non-random part of the random process, does not change the autocovariance function.

b) The sequence $\{V_t, t \in \mathbb{Z}\}$ is a *moving average* type of sequence, i.e. its values are weighted averages of a certain number of values of an underlying sequence. In this particular case we take averages of $2q+1$ neighbouring values of the sequence $\{X_t, t \in \mathbb{Z}\}$ and the weights are all equal to $1/(2q+1)$.

We first calculate the mean value function of $\{V_t, t \in \mathbb{Z}\}$:

$$\begin{aligned}
\mathbb{E}V_t &= \frac{1}{2q+1}\sum_{j=-q}^{q} \mathbb{E}X_{t+j} = \frac{1}{2q+1}\sum_{j=-q}^{q} \left(a + b(t+j)\right) \\
&= \frac{1}{2q+1}\sum_{j=-q}^{q} (a+bt) + \frac{1}{2q+1}\sum_{j=-q}^{q} bj = a + bt,\ t \in \mathbb{Z}.
\end{aligned}$$

Hence the mean value function is not constant and the sequence $\{V_t, t \in \mathbb{Z}\}$ is neither strictly nor weakly stationary.

Before calculating the autocovariance function R_V we note that, for any $t \in \mathbb{Z}$,

$$\begin{aligned}
V_t - \mathbb{E}V_t &= \frac{1}{2q+1}\sum_{j=-q}^{q} (a + b(t+j) + Y_{t+j}) - \frac{1}{2q+1}\sum_{j=-q}^{q} (a + b(t+j)) \\
&= \frac{1}{2q+1}\sum_{j=-q}^{q} Y_{t+j}.
\end{aligned}$$

Now, for $s, t \in \mathbb{Z}$,

$$\begin{aligned}
R_V(s,t) = \mathbb{E}\left(V_s - \mathbb{E}V_s\right)\left(V_t - \mathbb{E}V_t\right) &= \mathbb{E}\left(\frac{1}{2q+1}\sum_{j=-q}^{q} Y_{s+j}\right)\left(\frac{1}{2q+1}\sum_{k=-q}^{q} Y_{t+k}\right) \\
&= \frac{1}{(2q+1)^2}\,\mathbb{E}\left[\sum_{j=-q}^{q}\sum_{k=-q}^{q} Y_{s+j}Y_{t+k}\right].
\end{aligned}$$

We recall that $\mathbb{E}Y_{s+j}Y_{t+k} = \sigma^2$ if and only if $s+j = t+k$, otherwise $\mathbb{E}Y_{s+j}Y_{t+k} = 0$. The problem of calculating R_V is now effectively reduced to the problem of counting how many times a pair of Y's with the same time index occurs in the double sum above. That is how many times $s+j = t+k$, which is the same as *const* $= s - t = k - j$ for $k, j \in \{-q, \ldots, q\}$. The maximum number of such pairs is $2q+1$, the minimum number is 0 – this is the case if s and t are at least $2q+1$ time units apart. Also note that if we increase the distance between s and t by one the number of such pairs is reduced by one (or stays equal to zero). It follows that

$$R_V(s,t) = \begin{cases} \frac{2q+1-|s-t|}{(2q+1)^2}\,\sigma^2, & |s-t| \le 2q, \\ 0, & |s-t| > 2q. \end{cases}$$

From this representation it is clear that R_V is a function of the difference of its arguments and hence the sequence $\{V_t, t \in \mathbb{Z}\}$ is covariance stationary. $\Diamond$

Exercise 2.2: Let X be a random variable with a uniform distribution on the interval $(0, \pi)$. Consider the sequence of random variables $\{Y_t, t \in \mathbb{N}\}$ where $Y_t = \cos(tX)$. Discuss the properties of such a random sequence.

Solution:

Sample realizations of the sequence $\{Y_t, t \in \mathbb{N}\}$ are shown in Example 1.2.

We first calculate the mean value function. For $t \in \mathbb{N}$ it holds that

$$\mathbb{E}Y_t = \mathbb{E}\cos(tX) = \int_0^{\pi} \frac{1}{\pi}\cos(tu)\,\mathrm{d}u = \frac{1}{\pi}\left[\frac{\sin(tu)}{t}\right]_{u=0}^{\pi} = 0. \tag{1}$$

In order to determine the autocovariance function R_Y we recall the well-known formula $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$. It follows that, for $s, t \in \mathbb{N}$,

$$\begin{aligned} R_Y(s,t) &= \mathbb{E}\left(Y_s - \mathbb{E}Y_s\right)\left(Y_t - \mathbb{E}Y_t\right) = \mathbb{E}Y_sY_t = \mathbb{E}\cos(sX)\cos(tX) \\ &= \mathbb{E}\frac{1}{2}\big(\cos(sX+tX) + \cos(sX-tX)\big) \\ &= \int_0^{\pi}\frac{1}{2\pi}\cos\big((s+t)u\big)\,\mathrm{d}u + \int_0^{\pi}\frac{1}{2\pi}\cos\big((s-t)u\big)\,\mathrm{d}u. \end{aligned}$$

The first integral is equal to 0 for any combination of $s, t \in \mathbb{N}$. To see this it is sufficient to take the same steps as in Eq. 1 with $s+t$ in place of t. Similarly, the second integral is equal to 0 for $s \neq t$. On the other hand, if $s = t \in \mathbb{N}$, the second integral reduces to

$$\int_0^{\pi}\frac{1}{2\pi}\cos\big((s-t)u\big)\,\mathrm{d}u = \int_0^{\pi}\frac{1}{2\pi}\,\mathrm{d}u = \frac{1}{2}.$$

Altogether, $R_Y(s,t) = \frac{1}{2}\cdot\delta(s-t), s, t \in \mathbb{N}$, and the sequence $\{Y_t, t \in \mathbb{N}\}$ is weakly stationary and hence also covariance stationary.

Concerning the strict stationarity, we have to decide whether we are going to prove or disprove it. The one-dimensional distributions are all the same since from the theorem about the transformation of probability density functions we get $Y_t \in (-1,1)$ with density $f_{Y_t}(y) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}$. Thus we have to consider (at least) two-dimensional distributions.

From the picture of sample trajectories on Fig. 2 we may get the idea that observing Y_1 very close to 1 constrains Y_2 to be also very close to 1, more than Y_2 constrains Y_3. We can formalize this idea as follows. Take $\epsilon > 0$ small, then

$$\begin{aligned} \mathbb{P}(Y_1 > 1-\epsilon, Y_2 > 1-\epsilon) &= \mathbb{P}\left[\{X \in (0,\alpha)\} \cap \left\{X \in \left(0, \frac{\alpha}{2}\right) \cup \left(\pi - \frac{\alpha}{2}, \pi\right)\right\}\right] = \mathbb{P}\left(X \in \left(0, \frac{\alpha}{2}\right)\right) = \frac{\alpha}{2\pi} \\ \mathbb{P}(Y_2 > 1-\epsilon, Y_3 > 1-\epsilon) &= \mathbb{P}\left[\left\{X \in \left(0, \frac{\alpha}{2}\right) \cup \left(\pi - \frac{\alpha}{2}, \pi\right)\right\} \cap \left\{X \in \left(0, \frac{\alpha}{3}\right) \cup \left(\frac{2}{3}\pi - \frac{\alpha}{3}, \frac{2}{3}\pi + \frac{\alpha}{3}\right)\right\}\right] \\ &= \frac{\alpha}{3\pi}, \end{aligned}$$

where $\alpha = \arccos(\epsilon) \neq 0$ thus the two expression are not equal and it follows that the process $\{Y_t, t \in \mathbb{N}\}$ is not strictly stationary.

$\diamondsuit$

Exercise 2.3: Consider the stochastic process $X_t = \cos(t+B)$, $t \in \mathbb{R}$, where B is a random variable with a uniform distribution on the interval $(0, 2\pi)$. Check whether the process is weakly stationary.

Solution:

Sample realizations of the process $\{X_t, t \in \mathbb{R}\}$ are shown in Example 1.3.

As before we start with calculating the mean value function and observe that the process $\{X_t, t \in \mathbb{R}\}$ is centered:

$$\mathbb{E}X_t = \mathbb{E}\cos(t+B) = \int_0^{2\pi} \frac{1}{2\pi}\cos(t+b)\,\mathrm{d}b = \frac{1}{2\pi}\Big[\sin(t+b)\Big]_{b=0}^{2\pi} = 0,\ t \in \mathbb{R}.$$

The last equality follows from the fact that sine is a 2π-periodic function.

Using again the formula $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$ it follows that for $s, t \in \mathbb{R}$

$$\begin{aligned} R_X(s,t) &= \mathbb{E}\,(X_s - \mathbb{E}X_s)\,(X_t - \mathbb{E}X_t) = \mathbb{E}X_sX_t \\ &= \mathbb{E}\cos(s+B)\cos(t+B) \\ &= \int_0^{2\pi} \frac{1}{2\pi}\cos(s+b)\cos(t+b)\,\mathrm{d}b \\ &= \int_0^{2\pi} \frac{1}{4\pi}\big(\cos(s+t+2b)+\cos(s-t)\big)\,\mathrm{d}b \\ &= \frac{1}{2}\cos(s-t). \end{aligned}$$

The last equality follows from the fact that cosine is a 2π-periodic function.

We conclude that the process $\{X_t, t \in \mathbb{R}\}$ is weakly stationary. $\Diamond$

Remark: We can also consider strict stationarity of the process. From the picture of trajectories on Fig. 3 one gets the idea that the process could be strictly stationary. Let us check the definition. For $n \in \mathbb{N}, t_1, \dots, t_n \in \mathbb{R}$ and $h \in \mathbb{R}$ we can write

$$\begin{aligned} \mathbb{P}(X_{t_1+h} \le x_1, \dots, X_{t_n+h} \le x_n) &= \mathbb{P}(\cos(t_1+h+B) \le x_1, \dots, \cos(t_n+h+B) \le x_n) \\ &= \int_0^{2\pi} \frac{1}{2\pi}\mathbf{1}(\cos(t_1+h+b) \le x_1)\dots\mathbf{1}(\cos(t_n+h+b) \le x_n)\,\mathrm{d}b \\ &= \int_h^{2\pi+h} \frac{1}{2\pi}\mathbf{1}(\cos(t_1+\tilde{b}) \le x_1)\dots\mathbf{1}(\cos(t_n+\tilde{b}) \le x_n)\,\mathrm{d}\tilde{b} \\ &= \int_0^{2\pi} \frac{1}{2\pi}\mathbf{1}(\cos(t_1+\tilde{b}) \le x_1)\dots\mathbf{1}(\cos(t_n+\tilde{b}) \le x_n)\,\mathrm{d}\tilde{b} \\ &= \mathbb{P}(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n) \end{aligned}$$

where in the third equality we used the substitution $\tilde{b} = b+h$ in the integral. For the forth equality we observed that the integrand is a 2π-periodic function and as such it has the same value of the integral over any interval of length 2π. Thus the process $\{X_t, t \in \mathbb{R}\}$ is also strictly stationary.

Exercise 2.4: Let X be a random variable such that $\mathbb{E}X = 0$ and $\operatorname{var} X = \sigma^2 < \infty$. We define $X_t = (-1)^t X$, $t \in \mathbb{N}$. Discuss the properties of the sequence $\{X_t, t \in \mathbb{N}\}$.

Solution:

We note that the sequence is in fact of the form $(-X, X, -X, X, -X, \dots)$ and it has a periodic structure. The sequence is centered:

$$\mathbb{E}X_t = \mathbb{E}(-1)^t X = (-1)^t\mathbb{E}X = 0,\ t \in \mathbb{N}.$$

The autocovariance function R_X is then

$$R_X(s,t) = \mathbb{E}\,(X_s - \mathbb{E}X_s)\,(X_t - \mathbb{E}X_t) = \mathbb{E}X_sX_t = (-1)^{s+t}\mathbb{E}X^2 = (-1)^{s+t}\sigma^2.$$

It seems that the autocovariance function R_X is not a function of the difference of its arguments only. However, we note that $(-1)^{-2t} = 1$ and hence

$$R_X(s,t) = (-1)^{s+t}\sigma^2 = (-1)^{s+t}(-1)^{-2t}\sigma^2 = (-1)^{s-t}\sigma^2, \ s,t \in \mathbb{N}.$$

We conclude that the sequence $\{X_t, t \in \mathbb{N}\}$ is weakly stationary and hence covariance stationary.

For checking the strict stationarity we need to check that all finite dimensional distributions are translation invariant with respect to time. Starting with the one-dimensional distribution we need to check that

$$\mathbb{P}(X_{t_1} \in S) = \mathbb{P}(X(-1)^{t_1} \in S) = \mathbb{P}(X(-1)^{(t_1+h)} \in S) \quad \text{for any } t_1 \in \mathbb{N}, h \in \mathbb{N}, S \in \mathcal{B}.$$

The equation is trivially satisfied for h even and for h odd it is equivalent to

$$\mathbb{P}(X \in S) = \mathbb{P}(-X \in S) \quad \text{for any } S \in \mathcal{B}. \tag{2}$$

Thus if X does not have distribution symmetric w.r.t. 0, the sequence $\{X_t, t \in \mathbb{N}\}$ cannot be strictly stationary. On the other hand,

$$\begin{aligned}
&\mathbb{P}(X_{t_1} \in S_1, \dots, X_{t_n} \in S_n) = \mathbb{P}(X(-1)^{t_1} \in S_1, \dots, X(-1)^{t_n} \in S_n) \\
&= \mathbb{P}\left(X \in \bigcap_{t_i \ even} S_i, -X \in \bigcap_{t_j \ odd} S_j\right) = \mathbb{P}\left(X \in \bigcap_{t_i \ even} S_i \cap \bigcap_{t_j \ odd} -S_j\right) \\
&= \mathbb{P}\left(-X \in \bigcap_{t_i \ even} S_i \cap \bigcap_{t_j \ odd} -S_j\right),
\end{aligned}$$

the last equality following directly from (2). This is for any h equal either to

$$\mathbb{P}\left(X \in \bigcap_{t_i+h \ even} S_i \cap \bigcap_{t_j+h \ odd} -S_j\right) \quad \text{or} \quad \mathbb{P}\left(X \in \bigcap_{t_i+h \ even} -S_i \cap \bigcap_{t_j+h \ odd} S_j\right),$$

which are the same from (2) and are equal to $\mathbb{P}(X_{t_1+h} \in S_1, \dots, X_{t_n+h} \in S_n)$. Thus $\{X_t, t \in \mathbb{N}\}$ is strictly stationary if and only if X has distribution symmetric w.r.t. 0. $\Diamond$

Exercise 2.5: Determine the autocovariance function of the Poisson process $\{N_t, t \geq 0\}$ with intensity $\lambda > 0$ and discuss its stationarity.

Solution:

The basic properties were discussed in Example 1.7. In the following we will use the independence of increments and distributional properties of the process.

We recall that $\mathbb{E}N_t = \lambda t, t \geq 0$, and we see that the mean value function is not constant and hence the process is neither strictly nor weakly stationary. Also, $\operatorname{var} N_t = R_N(t,t) = \lambda t, t \geq 0$, and the process is not covariance stationary – otherwise the variance would be constant.

To make the calculation of the autocovariance function R_N simpler we start with considering the case $t \geq s \geq 0$. We also want to make use of the special property of the process at hand, i.e. the independence of its increments and their known distribution. Thus we write

$$\mathbb{E}N_tN_s = \mathbb{E}[(N_t - N_s) + (N_s - N_0)][N_s - N_0] = \mathbb{E}(N_t - N_s)\,\mathbb{E}(N_s - N_0) + \mathbb{E}(N_s - N_0)^2,$$

where the first equality follows from the fact that $N_0 = 0$ a.s. Further using the Poisson distribution of the increments we get

$$\mathbb{E}(N_s - N_0)^2 = \mathrm{var}(N_s - N_0) + [\mathbb{E}(N_s - N_0)]^2 = \lambda s + (\lambda s)^2 = \lambda s + \lambda^2 s^2,$$

and

$$\mathbb{E}(N_t - N_s)\,\mathbb{E}(N_s - N_0) = \lambda(t-s)\lambda(s-0) = \lambda^2 ts - \lambda^2 s^2.$$

The autocovariance function R_N is then

$$R_N(s,t) = \mathbb{E}N_s N_t - \mathbb{E}N_s\,\mathbb{E}N_t = \lambda s + \lambda^2 st - \lambda s \cdot \lambda t = \lambda s.$$

When considering $s \geq t \geq 0$, the role of s and t interchange and we obtain in this case that $R_N(s,t) = \lambda t$. We conclude that

$$R_N(s,t) = \lambda \min\{s,t\},\ s,t \geq 0.$$

Finally, we remark that $R_N(s,t) = \lambda \min\{s,t\}$ is not a function of the difference of its arguments. To see this, consider e.g. $\lambda = R_N(1,2) \neq R_N(2,3) = 2\lambda$. $\Diamond$

Remark: A remark to the above stated solution is due here. The solution is perfectly correct, but it is not the most effective/fastest one. If we write $R_N(t,s)$ by means of the covariance instead of using literally Definition 2.1 we get

$$\begin{aligned} R_N(t,s) &= \mathrm{cov}(N_t, N_s) = \mathrm{cov}((N_t - N_s) + (N_s - N_0), N_s - N_0) \\ &= \mathrm{cov}(N_t - N_s, N_s - N_0) + \mathrm{var}(N_s - N_0) = 0 + \lambda s. \end{aligned}$$

Here we used the assumption $t \geq s \geq 0$ in the second equality and independence of increments and information about their mean value and variance in the last equality.

Remark: The Wiener process $\{W_t, t \geq 0\}$ with parameter $\sigma^2 > 0$ has the same form of the autocovariance function as the Poisson process discussed in the previous exercise: $R_W(s,t) = \sigma^2 \min\{s,t\},\ s,t \geq 0$. The key point here is again the independence of the increments of the process. The calculation uses the same trick as above, i.e. rewrite all the terms using increments.

Remark: As discussed in the remark above, the Wiener process has the same form of the autocovariance function as the Poisson process, despite the former process having continuous trajectories and the latter process having piecewise constant trajectories with jumps. This illustrates the fact that the distribution of a stochastic process with finite second moments is *not* fully determined by its autocovariance function.

Exercise 2.6: Let $\{N_t, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$ and let A be a real-valued random variable with zero mean and unit variance, independent of the process $\{N_t, t \geq 0\}$. We define $X_t = A(-1)^{N_t}, t \geq 0$. Determine the autocovariance function of $\{X_t, t \geq 0\}$.

Solution:

Sample realizations of the process $\{X_t, t \geq 0\}$ are shown in Example 1.8.

First we determine the mean value function of the process $\{X_t, t \geq 0\}$. Using the independence of A and N_t we write

$$\mathbb{E}X_t = \mathbb{E}A(-1)^{N_t} = \mathbb{E}A\,\mathbb{E}(-1)^{N_t} = 0,\ t \geq 0,$$

since $\mathbb{E}A = 0$ and $\mathbb{E}(-1)^{N_t} < \infty$ – it is an expectation of a bounded random variable.

Now we write

$$R_X(s,t) = \mathbb{E}\left(X_s - \mathbb{E}X_s\right)\left(X_t - \mathbb{E}X_t\right) = \mathbb{E}X_s X_t = \mathbb{E}A^2(-1)^{N_s+N_t} = \mathbb{E}A^2\,\mathbb{E}(-1)^{N_s+N_t}.$$

Recalling that $\mathbb{E}A^2 = 1$ and considering now the case $s \geq t \geq 0$ we get similarly to the Exercise 2.4 that

$$R_X(s,t) = \mathbb{E}(-1)^{N_s+N_t} \cdot (-1)^{-2N_t} = \mathbb{E}(-1)^{N_s-N_t}.$$

Since the random variable $N_s - N_t$ has Poisson distribution with parameter $\lambda(s-t)$ we finish the calculation as follows:

$$R_X(s,t) = \mathbb{E}(-1)^{N_s-N_t} = \sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{-\lambda(s-t)}\frac{\left[\lambda(s-t)\right]^k}{k!} = \mathrm{e}^{-\lambda(s-t)}\,\mathrm{e}^{-\lambda(s-t)} = \mathrm{e}^{-2\lambda(s-t)}.$$

When considering $t > s \geq 0$, the role of s and t interchange and we obtain in this case that $R_X(s,t) = \mathrm{e}^{-2\lambda(t-s)}$. We conclude that

$$R_X(s,t) = \mathrm{e}^{-2\lambda|s-t|},\ s,t \geq 0.$$

Finally, we remark that $R_X(s,t)$ is a function of the difference of its arguments. Hence the process is weakly and covariance stationary and we can write

$$R_X(t) = \mathrm{e}^{-2\lambda|t|},\ t \in \mathbb{R}.$$

$\Diamond$

Remark: Compare the processes in Exercise 2.4 and 2.6: the construction is similar but the switching times are changed from natural numbers in Exercise 2.4 to the random times of the events in the independent Poisson process in Exercise 2.6. This dilutes the correlation in the process and the autocorrelation function changes from $|r(s,t)| = 1$ in Exercise 2.4 to $r(s,t) = \mathrm{e}^{-2\lambda|s-t|}$ in Exercise 2.6.

Remark: Observe that if the random variable A has a discrete distribution then the process $\{X_t, t \geq 0\}$ is a Markov chain with continuous time. In particular, the switching process with A with symmetric alternative distribution on $\{-1,1\}$ is a Markov chain starting from the stationary distribution. Thus it is a strictly stationary process (prove this by deriving the intensity matrix Q of $\{X_t, t \geq 0\}$ and finding its stationary distribution).

Remark: Also for non-discretely distributed A the process $X_t = A(-1)^{N_t}, t \geq 0$, is strictly stationary if and only if A has distribution symmetric w.r.t. 0. The necessity of the condition follows in the same way as in Exercise 2.4. For the sufficiency the proof follows similar lines but we need to condition on the realization of the Poisson process $\{N_t\}$.

Exercise 2.7: Let $\{W_t, t \geq 0\}$ be a Wiener process. We define $B_t = W_t - tW_1, t \in [0,1]$. The stochastic process $\{B_t, t \in [0,1]\}$ is called the *Brownian bridge*. Determine the autocovariance function of $\{B_t, t \in [0,1]\}$ and discuss its stationarity.

Solution:

Sample realizations of the process $\{B_t, t \in [0,1]\}$ are shown in Example 1.5.

Since the Wiener process is centered, it holds for a $t \in [0,1]$ that $\mathbb{E}B_t = \mathbb{E}W_t - t\mathbb{E}W_1 = 0$. We already know the autocovariance function of the Wiener process $R_W(s,t) = \sigma^2 \min\{s,t\}, s,t \geq 0$ and we can write for $s,t \in [0,1]$:

$$\begin{aligned} R_B(s,t) &= \mathbb{E}B_sB_t = \mathbb{E}(W_s - sW_1)(W_t - tW_1) = \mathbb{E}W_sW_t - s\mathbb{E}W_1W_t - t\mathbb{E}W_sW_1 + st\mathbb{E}W_1^2 \\ &= R_W(s,t) - sR_W(1,t) - tR_W(s,1) + stR_W(1,1) \\ &= \sigma^2 \left(\min\{s,t\} - st - st + st\right) = \sigma^2 \left(\min\{s,t\} - st\right). \end{aligned}$$

Looking at the formula above, it seems that the function $R_B(s,t)$ is *not* a function of the difference of its arguments. However, for claiming that the process is not weakly (or covariance) stationary we need to prove it.

Consider e.g. the following two cases: $R_B(1,1) = 0, R_B(1/2,1/2) = \sigma^2/4$. We see that the variance of the process is not constant and $R_B(s,t)$ is not a function of the difference of its arguments. We conclude that the Brownian bridge is neither weakly nor covariance stationary. It follows that it also is not strictly stationary – the process has finite second moments but is not weakly stationary. $\diamondsuit$

Exercise 2.8: Let $\{W_t, t \geq 0\}$ be a Wiener process. We define the so-called *Ornstein-Uhlenbeck process* $\{U_t, t \geq 0\}$ by the formula $U_t = \mathrm{e}^{-\alpha t/2} W_{\exp\{\alpha t\}}, t \geq 0$, where $\alpha > 0$ is a parameter. Decide whether $\{U_t, t \geq 0\}$ is weakly (strictly) stationary and determine its autocovariance function.

Solution:

Sample realizations of the process $\{U_t, t \geq 0\}$ are shown in Example 1.6.

We first note that, just as the Wiener process, the Ornstein-Uhlenbeck process is a Gaussian process (all finite-dimensional distributions are Gaussian since the transformation of the values of the Wiener process is linear and the transformation of the times does not affect Gaussianity of finite-dimensional distributions).

It is easy to see that the Ornstein-Uhlenbeck process is centered: for any $t \geq 0$ it holds that $\mathbb{E}U_t = \mathrm{e}^{-\alpha t/2}\mathbb{E}W_{\exp\{\alpha t\}} = 0$.

Now we can calculate the autocovariance function. For $s,t \geq 0$ we write

$$R_U(s,t) = \mathbb{E}U_tU_s = \mathrm{e}^{-\alpha s/2}\mathrm{e}^{-\alpha t/2}\mathbb{E}W_{\exp\{\alpha s\}}W_{\exp\{\alpha t\}} = \sigma^2\mathrm{e}^{-\alpha s/2}\mathrm{e}^{-\alpha t/2}\min\left\{\exp\{\alpha s\}, \exp\{\alpha t\}\right\}.$$

Looking at the formula above we are tempted to conclude that the Ornstein-Uhlenbeck process is not weakly or covariance stationary. However, when trying to prove it, we fail at finding examples such that $R_U(s,t) \neq R_U(s+h,t+h)$. We inspect the following two cases separately and take advantage of the monotonicity of the exponential function:

$$\begin{aligned} s \leq t: &\quad R_U(s,t) = \sigma^2\mathrm{e}^{-\alpha(t-s)/2}, \\ s > t: &\quad R_U(s,t) = \sigma^2\mathrm{e}^{-\alpha(s-t)/2}. \end{aligned}$$

Thus we can write for $s,t \geq 0$: $R_U(s,t) = \sigma^2\mathrm{e}^{-\alpha|s-t|/2} = R_U(s-t)$. We conclude that in fact the process is weakly (and covariance) stationary. It is also strictly stationary because it is Gaussian. We have seen that the mere fact that the autocovariance function does not look like a function of the difference of its arguments does not mean that there is no other expression for the autocovariance function that would clearly show the contrary. $\diamondsuit$

Exercise 2.9: Check if the following functions are autocovariance functions of a stochastic process with index set $T = \mathbb{R}$:

a) $R(s,t) = \mathrm{e}^{\mathrm{i}\omega(t-s)}$, for a given $\omega > 0$,

b) $R(s,t) = st$,

c) $R(s,t) = s + t$,

d) $R(s,t) = \sin^2(t-s)$.

Solution:

Recalling Corollary 2.4, positive semidefiniteness of a function $R(s,t)$, $s,t \in T$, is equivalent to $R(s,t)$, $s,t \in T$, being an autocovariance function of a stochastic process with index set T.

a) We try to check the positive semidefiniteness property and write for $n \in N, c_1, \dots, c_n \in \mathbb{C}$ and $t_1, \dots, t_n \in \mathbb{R}$:

$$\begin{aligned}\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} R(t_j, t_k) &= \sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} \mathrm{e}^{\mathrm{i}\omega(t_j - t_k)} = \sum_{j=1}^n \sum_{k=1}^n c_j \mathrm{e}^{\mathrm{i}\omega(t_j)} \overline{c_k \mathrm{e}^{\mathrm{i}\omega(t_k)}} \\ &= \left(\sum_{j=1}^n c_j \mathrm{e}^{\mathrm{i}\omega(t_j)}\right) \overline{\left(\sum_{k=1}^n c_k \mathrm{e}^{\mathrm{i}\omega(t_k)}\right)} = \left|\sum_{j=1}^n c_j \mathrm{e}^{\mathrm{i}\omega(t_j)}\right|^2 \geq 0.\end{aligned}$$

We conclude that the function $R(s,t) = \mathrm{e}^{\mathrm{i}\omega(s-t)}$, $s,t \in \mathbb{R}$, is the autocovariance function of a stochastic process with index set $\mathbb{R}$.

b) Similarly as above, we write for $n \in N, c_1, \dots, c_n \in \mathbb{C}$ and $t_1, \dots, t_n \in \mathbb{R}$:

$$\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} R(t_j, t_k) = \sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} t_j t_k = \left(\sum_{j=1}^n c_j t_j\right) \overline{\left(\sum_{k=1}^n c_k t_k\right)} = \left|\sum_{j=1}^n c_j t_j\right|^2 \geq 0.$$

We conclude that the function $R(s,t) = st$, $s,t \in \mathbb{R}$, is the autocovariance function of a stochastic process with index set $\mathbb{R}$.

c) Unlike the two functions above, $R(s,t) = s + t$, $s,t \in \mathbb{R}$, is not of a product form and we might get suspiscious that it is in fact *not* positive semidefinite and that it is not useful to try using the same approach as above.

We could try using Corollary 2.4 and find a combination of $n \in N, c_1, \dots, c_n \in \mathbb{C}$ and $t_1, \dots, t_n \in \mathbb{R}$ such that $\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} R(t_j, t_k) < 0$. However, this approach can be very laborious with no guarantee of success. On the other hand, we may look at Theorem 2.2 (properties of autocovariance functions) and check if some other, easier to check property is not violated.

For example, the variance $R(t,t)$, $t \in \mathbb{R}$, must be non-negative. In this case $R(t,t) = 2t$, $t \in \mathbb{R}$, and we see that for $t < 0$ the variance is negative: $R(t,t) < 0$. We conclude that $R(s,t) = s + t$, $s,t \in \mathbb{R}$, is not the autocovariance function of a stochastic process with index set $\mathbb{R}$.

What if we restrict the index set to $T = [0, \infty)$? In this case the variance is non-negative, $R(t,t) = 2t \geq 0$, $t \in [0, \infty)$, and we have to find a different argument. The Hermitian property is clearly fulfilled and we focus on the Cauchy-Schwarz inequality. We might try to check the inequality for a couple of values of s and t and we notice that the inequality does not hold e.g. for $s = 1, t = 2$. We conclude that $R(s,t) = s + t$, $s,t \in [0, \infty)$, is not the autocovariance function of a stochastic process with index set $[0, \infty)$.

d) Checking positive semidefiniteness of this function from definition does not seem to be an easy task. Thus we first try to check the properties from Theorem 2.2. $R(s,s) = \sin^2(s-s) = 0$ thus R is nonnegative on the diagonal and it is obviously Hermitian. But having zero on the whole diagonal is a very special property. It is also suspicious since there are other times for which $R(s,t)$ is non zero, e.g. $R(0,\pi/2) = 1$. Realizing this we see that the Cauchy-Schwartz inequality is not fulfilled since e.g.

$$|R(0,\pi/2)| = 1 > 0 = \sqrt{R(0,0)}\sqrt{R(\pi/2,\pi/2)}.$$

We conclude that $R(s,t) = \sin^2(t-s)$ is not the autocovariance function of a stochastic process with index set $\mathbb{R}$. $\diamondsuit$

Further exercises

Exercise 2.10: Prove parts c) d) e) of Theorem 2.1.

Exercise 2.11: Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables. Prove that the process is strictly stationary. Is it also weakly stationary?

Exercise 2.12: Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of uncorrelated random variables with zero mean and (the same) finite positive variance (so-called *white noise*). Prove that it is weakly stationary. Is it also strictly stationary?

Exercise 2.13: Let $X_0 = 0$, $X_t = Y_1 + \cdots + Y_t$ for $t = 1, 2, \ldots$, where $Y_1, Y_2, \ldots$ are independent identically distributed discrete random variables with zero mean and finite positive variance. Show that $\{X_t, t \in \mathbb{N}_0\}$ is a Markov chain. Determine its autocovariance function. What can we say about the properties of such a random sequence?

Exercise 2.14: Let $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ be uncorrelated weakly stationary processes, i.e. for all $s, t \in T$ the random variables X_s and Y_t are uncorrelated. Show that in such a case also the process $\{Z_t, t \in T\}$ with $Z_t = X_t + Y_t$ is weakly stationary.

Exercise 2.15: Let Y_t, $t \in \mathbb{Z}$, be independent random variables with the standard normal distribution (so-called *Gaussian white noise*). For all $t \in \mathbb{Z}$ we define $X_t = a + bY_t + cY_{t-1}$ where a, b, c are real constants. Discuss the stationarity of the sequence $\{X_t, t \in \mathbb{Z}\}$.

Exercise 2.16: Show that any positive semidefinite function is non-negative on the diagonal and Hermitian.

Exercise 2.17: Let $\{X_t, t \in T\}$ be a centered Gaussian stationary process. Let $Y_t = X_t^2$, $t \in T$. Determine the mean value and the autocovariance function of $\{Y_t, t \in T\}$ and discuss its stationarity.

Hint: Use the formula for the moments of the joint normal distribution $(X_1, X_2, X_3, X_4)^{\mathrm{T}}$ with zero mean: $\mathbb{E}X_1X_2X_3X_4 = \mathbb{E}X_1X_2\,\mathbb{E}X_3X_4 + \mathbb{E}X_1X_3\,\mathbb{E}X_2X_4 + \mathbb{E}X_1X_4\,\mathbb{E}X_2X_3$.

Exercise 2.18: Determine the autocovariance function of the Wiener process $\{W_t, t \geq 0\}$. For $0 \leq t_1 < t_2 < \cdots < t_n$ determine the variance matrix of the random vector $(W_{t_1}, \ldots, W_{t_n})^{\mathrm{T}}$.

3 L_2-properties of stochastic processes

Definition 3.1: We say that a sequence of random variables X_n such that $\mathbb{E}|X_n|^2 < \infty$ *converges in L_2 (or in the mean square)* to a random variable X, if $\mathbb{E}|X_n - X|^2 \to 0$ for $n \to \infty$. In that case we write $X = \text{l.i.m.}\, X_n$.

Let $T \subset \mathbb{R}$ be an open interval and consider a stochastic process $\{X_t, t \in T\}$ with continuous time and finite second moments.

Definition 3.2: We call the process $\{X_t, t \in T\}$ *L_2-continuous (mean square continuous)* at the point $t_0 \in T$ if $\mathbb{E}|X_t - X_{t_0}|^2 \to 0$ for $t \to t_0$. The process is L_2-continuous if it is L_2-continuous at all points $t \in T$.

Theorem 3.1: A stochastic process $\{X_t, t \in T\}$ is L_2-continuous if and only if its mean value $\mathbb{E}X_t$ is a continuous function on T and its autocovariance function $R_X(s,t)$ is continuous at points $[s,t]$ for which $s = t$.

Corollary 3.1: Centered weakly stationary process is L_2-continuous if and only if its autocovariance function $R(t)$ is continuous at point 0.

Definition 3.3: We call the process $\{X_t, t \in T\}$ *L_2-differentiable (mean square differentiable)* at the point $t_0 \in T$ if there is a random variable X'_{t_0} such that

$$\lim_{h \to 0} \mathbb{E} \left| \frac{X_{t_0+h} - X_{t_0}}{h} - X'_{t_0} \right|^2 = 0.$$

The random variable X'_{t_0} is called the *derivative in the L_2 (mean square) sense* of the process $\{X_t, t \in T\}$ at the point t_0. The process is L_2-differentiable if it is L_2-differentiable at all points $t \in T$.

Theorem 3.2: A stochastic process $\{X_t, t \in T\}$ is L_2-differentiable if and only if its mean value $\mathbb{E}X_t$ is differentiable and the second-order generalized partial derivative of the autocovariance function $R(s,t)$ exists and is finite at points $[s,t]$ for which $s = t$, i.e. there is a finite limit

$$\lim_{h,h' \to 0} \frac{1}{hh'} \big[R_X(t+h, t+h') - R_X(t, t+h') - R_X(t+h, t) + R_X(t,t)\big].$$

Remark: A sufficient condition for the existence of the second-order generalized partial derivative is the existence and continuity of the second-order partial derivatives $\frac{\partial^2 R(s,t)}{\partial s \partial t}$ and $\frac{\partial^2 R(s,t)}{\partial t \partial s}$.

Remark: Any L_2-differentiable process is also L_2-continuous.

Definition 3.4: Let $T = [a,b]$ be a bounded closed interval. For any $n \in \mathbb{N}$ let $D_n = \{t_{n,0}, \ldots, t_{n,n}\}$ be a division of the interval $[a,b]$ where $a = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = b$. We define the partial sums I_n of the centered stochastic process $\{X_t, t \in T\}$ by the formula

$$I_n = \sum_{i=0}^{n-1} X_{t_{n,i}} (t_{n,i+1} - t_{n,i}), \quad n \in \mathbb{N}.$$

If there is a random variable I such that $\mathbb{E}|I_n - I|^2 \to 0$ for $n \to \infty$ and for each division of the interval $[a, b]$ such that $\max_{0 \le i \le n-1}(t_{n,i+1} - t_{n,i}) \to 0$ we call it the *Riemann integral* of the process $\{X_t, t \in T\}$ and denote it by $I = \int_a^b X_t \,\mathrm{d}t$. For a non-centered process with the mean value $\mathbb{E}X_t$ we define the Riemann integral as

$$\int_a^b X_t \,\mathrm{d}t = \int_a^b (X_t - \mathbb{E}X_t) \,\mathrm{d}t + \int_a^b \mathbb{E}X_t \,\mathrm{d}t,$$

if the centered process $\{X_t - \mathbb{E}X_t, t \in T\}$ has a Riemann integral and the Riemann integral $\int_a^b \mathbb{E}X_t \,\mathrm{d}t$ exists and is finite.

Theorem 3.3: A stochastic process $\{X_t, t \in [a, b]\}$ where $[a, b]$ is a bounded closed interval is Riemann-integrable if the Riemann integrals $\int_a^b \mathbb{E}X_t \,\mathrm{d}t$ and $\int_a^b \int_a^b R_X(s, t) \,\mathrm{d}s \,\mathrm{d}t$ exist and are finite.

Theorem 3.4: [3, p.447] Let $M \subset \mathbb{R}^n$ be a bounded set, f be a real function on $\mathbb{R}^n$, bounded on M. Then the Riemann integral $\int_M f(x) \,\mathrm{d}x$ exists if and only if both following conditions are fulfilled:
a) the boundary of M has Lebesgue measure 0,
b) the set of inner points of M in which f is not continuous has Lebesgue measure 0.

Theorem 3.5: [3, p.440] Let $M \subset \mathbb{R}^n$ be a bounded set and let the Riemann integral $\int_M f(x) \,\mathrm{d}x$ exist. Then also the Lebesgue integral $\int_M f(x) \,\mathrm{d}x$ exists and both integrals are equal.

Exercise 3.1: Consider a stochastic process $X_t = \cos(t + B)$, $t \in \mathbb{R}$, where B is a random variable with the uniform distribution on the interval $(0, 2\pi)$. Is this process L_2-continuous and L_2-differentiable? Is it Riemann-integrable on a bounded closed interval $[a, b]$?

Solution:

This stochastic process has been already discussed in Exercise 2.3 and we recall that $\mathbb{E}X_t = 0,\ t \in \mathbb{R}$, and $R_X(s, t) = \frac{1}{2}\cos(s - t),\ s, t \in \mathbb{R}$.

Clearly the mean value $\mathbb{E}X_t$ is a continuous function on $\mathbb{R}$ and the autocovariance function $R_X(s, t)$ is a continuous function on $\mathbb{R}^2$ – it is a composition of two continuous functions. Specifically, $R_X(s, t)$ is continuous at points $[s, t]$ for which $s = t$. Using Theorem 3.1 we now obtain L_2-continuity of the stochastic process $\{X_t, t \in \mathbb{R}\}$.

Considering L_2-differentiability of the process, we use Theorem 3.2 and the following Remark about the existence and continuity of the second-order partial derivatives of $R_X(s, t)$. The partial derivatives can be calculated easily:

$$\frac{\partial R_X(s,t)}{\partial t} = \frac{1}{2}\sin(s - t), \qquad \frac{\partial R_X(s,t)}{\partial t \partial s} = \frac{1}{2}\cos(s - t);$$
$$\frac{\partial R_X(s,t)}{\partial s} = -\frac{1}{2}\sin(s - t), \qquad \frac{\partial R_X(s,t)}{\partial s \partial t} = \frac{1}{2}\cos(s - t).$$

Hence, the process $\{X_t, t \in \mathbb{R}\}$ is L_2-differentiable.

For deciding about the Riemann integrability of the process on a given bounded interval $[a, b]$ we use Theorem 3.3. The mean value function is constant and hence integrable. The autocovariance function $R_X(s, t)$ is a continuous function on $\mathbb{R}^2$ and hence the Riemann integral $\int_a^b \int_a^b R_X(s, t) \,\mathrm{d}s \,\mathrm{d}t$ exists and is finite. Hence the process $\{X_t, t \in \mathbb{R}\}$ is Riemann integrable on the interval $[a, b]$. $\Diamond$

Remark: We have proved that the L_2-derivative and L_2-integral of the process exist. A natural question follows – how do they look like? For the process at hand the answer is quite easy since the derivative and the integral with respect to time exist for each trajectory separately. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the underlying probability space, e.g. $\Omega = (0, 2\pi)$, $\mathcal{A}$ the corresponding Borel sigma-algebra and $\mathbb{P}$ the multiple of the Lebesgue measure on $(0, 2\pi)$. For $\omega \in \Omega$ the realization of B is $B(\omega)$ and we can write

$$\frac{\partial \cos\big(t + B(\omega)\big)}{\partial t} = -\sin\big(t + B(\omega)\big) \text{ and } \int_c^d \cos\big(t + B(\omega)\big)\,\mathrm{d}t = \sin\big(d + B(\omega)\big) - \sin\big(c + B(\omega)\big).$$

Thus the candidate for the L_2-derivative is the process $Y_t = -\sin(t+B)$. Now we need to make sure that the limit from the Definition 3.2,

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}\left|\frac{\cos(t_0+h+B) - \cos(t_0+B)}{h} + \sin(t_0+B)\right|^2 \\ = \lim_{h\to 0}\int_0^{2\pi} \frac{1}{2\pi}\left(\frac{\cos(t_0+h+b) - \cos(t_0+b)}{h} + \sin(t_0+b)\right)^2 \mathrm{d}\,b, \quad (3)\end{aligned}$$

is equal to 0. Let us write

$$\begin{aligned}&\mathbb{E}\left|\frac{\cos(t_0+h+B) - \cos(t_0+B)}{h} + \sin(t_0+B)\right|^2 \\
&= \mathbb{E}\left|\frac{\cos(t_0+B)\cos h - \sin(t_0+B)\sin h - \cos(t_0+B)}{h} + \sin(t_0+B)\right|^2 \\
&= \mathbb{E}\left(\cos(t_0+B)\frac{\cos h - 1}{h} - \sin(t_0+B)\left(\frac{\sin h}{h} - 1\right)\right)^2 \\
&\le \mathbb{E}\left(\cos(t_0+B)\frac{\cos h - 1}{h}\right)^2 + \mathbb{E}\left(\sin(t_0+B)\frac{\sin h - h}{h}\right)^2 \\
&+ 2\mathbb{E}\left|\cos(t_0+B)\frac{\cos h - 1}{h}\sin(t_0+B)\frac{\sin h - h}{h}\right| \\
&\le 4\max\left\{\mathbb{E}\left|\cos(t_0+B)\frac{\cos h - 1}{h}\right|^2, \mathbb{E}\left|\sin(t_0+B)\frac{\sin h - h}{h}\right|^2\right\} \\
&= 4\max\left\{\left(\frac{\cos h - 1}{h}\right)^2 \mathbb{E}\cos^2(t_0+B), \left(\frac{\sin h}{h} - 1\right)^2 \mathbb{E}\sin^2(t_0+B)\right\},\end{aligned}$$

where we used the formula for $\cos\alpha\cos\beta$ and the Cauchy-Schwartz inequality. The last expression obviously goes to 0 for $h \to 0$ and our candidate process $\{Y_t, t \in \mathbb{R}\}$ is the L_2-derivative of the process $\{X_t, t \in \mathbb{R}\}$.

We could also show that (3) equals 0 in a faster way by using the Lebesgue Theorem to change the order of $\mathbb{E}$ and lim. The inner limit is 0 for any $b = B(\omega), \omega \in \Omega$ and there exists an integrable majorant for $\frac{1}{2\pi}\left(\frac{\cos(t_0+h+B)-\cos(t_0+B)}{h} + \sin(t_0+B)\right)$ since cos is a Lipschitz function and sin is a bounded function.

For the Riemann integral $\int_c^d \cos(t+B)\,\mathrm{d}t$ we have to compute the limit of

$$\begin{aligned}\mathbb{E}\left|\sum_{i=0}^{n-1}\cos(t_{n,i}+B)(t_{n,i+1}-t_{n,i}) - \sin(d+B) + \sin(c+B)\right|^2 \\ = \int_0^{2\pi}\frac{1}{2\pi}\left|\sum_{i=0}^{n-1}\cos(t_{n,i}+b)(t_{n,i+1}-t_{n,i}) - \sin(d+b) + \sin(c+b)\right|^2 \mathrm{d}\,b \quad (4)\end{aligned}$$

for any division of the interval $[c, d]$ whose norm goes to 0. However we again know from the nonrandom case (i.e. for each nonrandom trajectory separately) that

$$h_n(b) = \frac{1}{2\pi} \sum_{i=0}^{n-1} \cos(t_{n,i} + b)(t_{n,i+1} - t_{n,i}) - \sin(d + b) + \sin(c + b)$$

goes to 0 for any fixed $b = B(\omega), \omega \in \Omega, b \in (0, 2\pi)$, resp., and any division of the interval $[c, d]$ whose norm goes to 0. To be able to apply the Lebesgue Theorem we need to find an integrable majorant for $h_n^2(b)$. A safe upper bound for any $b \in (0, 2\pi)$ is e.g. $h_n^2(b) \leq (d - c + 2)^2/4\pi^2$ which is a finite constant and thus integrable. The Lebesgue Theorem then gives that (4) converges to 0 for any division of the interval $[c, d]$ whose norm goes to 0 and thus our candidate random variable $\sin(d + B) - \sin(c + B)$ really is the L_2 integral $\int_c^d X_t \, \mathrm{d}t$.

Exercise 3.2: Consider the Wiener process with parameter $\sigma^2 > 0$. Determine the L_2-properties of the process, including the Riemann-integrability on $[a, b]$, $a < b \in \mathbb{R}^+$.

Solution:

We recall that the Wiener process $\{W_t, t \geq 0\}$ is a centered process with the autocovariance function $R_W(s, t) = \sigma^2 \min\{s, t\},\ s, t \geq 0$ (see the remark below Exercise 2.5).

The function $\min\{s, t\},\ s, t \in \mathbb{R}$, can be equivalently expressed in terms of continuous functions: $\min\{s, t\} = \frac{1}{2}(s + t - |s - t|)\,,\ s, t \in \mathbb{R}$. Hence the autocovariance function R_W is a continuous function of s and t. Specifically, it is continuous at points $[s, t]$ for which $s = t$ and using Theorem 3.1 we obtain the L_2-continuity of the Wiener process.

L_2-continuity of the Wiener process can be also established by a direct computation:

$$\mathbb{E}|W_t - W_{t_0}|^2 = \operatorname{var}(W_t - W_{t_0}) = \sigma^2 |t - t_0| \to 0,\ t \to t_0.$$

Here we took advantage of the fact that the increments of the Wiener process are (Gaussian) random variables with zero mean and variance depending linearly on the time lag.

The mean value function of the Wiener process is differentiable (constant) and thus only the properties of the autocovariance function need to be discussed in detail in order to assess L_2-differentiability of the process.

We note that the autocovariance function $R_W(s, t) = \sigma^2 \min\{s, t\},\ s, t \geq 0$, is not smooth at points $[s, t]$ for which $s = t$. Recalling Theorem 3.2 we might get suspicious that the Wiener process is in fact *not* L_2-differentiable and that we should first try showing that the second-order generalized partial derivative of the autocovariance function $R_W(s, t)$ does not exist or is not finite at points $[s, t]$ for which $s = t$. We fix a point $t > 0$ and focus on the special case $h = h' > 0$:

$$\begin{aligned}
&\lim_{h \to 0+} \frac{1}{h^2} \big[R_W(t + h, t + h) - R_W(t, t + h) - R_W(t + h, t) + R_W(t, t)\big] \\
&= \lim_{h \to 0+} \frac{1}{h^2} \big[\sigma^2 (t + h) - \sigma^2 t - \sigma^2 t + \sigma^2 t\big] \\
&= \lim_{h \to 0+} \frac{\sigma^2 h}{h^2} = +\infty.
\end{aligned}$$

The special case with $h = h' < 0$ can be handled in a similar way to obtain $-\infty$ as a result. We conclude that, at point (t, t), $t > 0$, the second-order generalized partial derivative of $R_W(s, t)$ does not exist (limits along the diagonal from above and from below are not equal). The Wiener process is not L_2-differentiable at point t and thus it is not L_2-differentiable.

For deciding about the Riemann integrability of the process on a given bounded interval $[a, b], 0 \leq a < b$, we use Theorem 3.3. The mean value function is constant and hence integrable. The function

$\min\{s,t\}$ is a continuous function on $\mathbb{R}^2$ and hence the Riemann integral $\int_a^b \int_a^b R_W(s,t)\,\mathrm{d}s\,\mathrm{d}t$ exists and is finite. Hence the process $\{W_t, t > 0\}$ is Riemann integrable on the interval $[a,b]$. $\diamondsuit$

Exercise 3.3: *Integrated Wiener process* is defined as

$$X_t = \int_0^t W_\tau \,\mathrm{d}\tau, \quad t \geq 0.$$

Using the properties of the Wiener process and L_2-convergence prove that $X_t \sim N(0, v_t^2)$ for all $t \geq 0$ where $v_t^2 = \frac{1}{3}\sigma^2 t^3$ and σ^2 is the parameter of the Wiener process W_t. Use the fact that the L_2-limit of a sequence of Gaussian random variables is a Gaussian random variable.

Solution:

For the Wiener process we cannot use the method from the Remark after Exercise 3.1. We are not able to suggest a candidate for the integral in L_2 by integrating the process "‘by trajectories"’. But we may try to use directly the definition of the integral in L_2

$$X_t = \text{l.i.m.} \sum_{i=0}^{n-1} W_{t_{n,i}}(t_{n,i+1} - t_{n,i}) = \text{l.i.m.}\, I_n,$$

for any division of the interval $[0,t]$ whose norm goes to 0.

W_t is a process with independent increments and we know their distribution (centered Gaussian with variance $\sigma^2(\tau - s)$ for $W_\tau - W_s$). Thus we need to rewrite the approximating sums I_n using the increments of W_τ. We write

$$\begin{aligned} I_n &= \sum_{i=0}^{n-1} W_{t_{n,i}}(t_{n,i+1} - t_{n,i}) = \sum_{i=0}^{n-1}\sum_{j=1}^{i}(W_{t_{n,j}} - W_{t_{n,j-1}})(t_{n,i+1} - t_{n,i}) \\ &= \sum_{j=1}^{n-1}\sum_{i=j}^{n-1}(W_{t_{n,j}} - W_{t_{n,j-1}})(t_{n,i+1} - t_{n,i}) = \sum_{j=1}^{n-1}(W_{t_{n,j}} - W_{t_{n,j-1}})(t - t_{n,j}), \end{aligned}$$

where in the third equality we changed the order of summation over i and j. The last expression is sum of independent centered Gaussian random variables with variances $\sigma^2(t_{n,j} - t_{n,j-1})(t - t_{n,j})^2$. Thus

$$I_n \sim N\left(0, \sum_{j=1}^{n-1} \sigma^2 (t - t_{n,j})^2 (t_{n,j} - t_{n,j-1})\right)$$

We know from the assignment that L_2 limit of Gaussian random variables is again a Gaussian random variable and from the continuity of the inner product in L_2 we know that the limit of the mean values and variances of I_n is equal to the mean and variance of l.i.m. I_n. Further we observe that for any division of the interval $[0,t]$ whose norm goes to 0

$$\lim \sum_{j=1}^{n-1} \sigma^2 (t - t_{n,j})^2 (t_{n,j} - t_{n,j-1}) = \int_0^t \sigma^2 (t-\tau)^2 \mathrm{d}\,\tau = \sigma^2 \frac{t^3}{3}.$$

Thus together we obtain $X_t = \text{l.i.m.}\, I_n \sim N\left(0, \sigma^2 \frac{t^3}{3}\right)$. $\diamondsuit$

Exercise 3.4: Let $\{X_t, t \in \mathbb{R}\}$ be a centered, weakly stationary stochastic process with the autocovariance function

$$R_X(t) = \exp\{\lambda(\mathrm{e}^{\mathrm{i}t} - 1)\}, \quad t \in \mathbb{R},$$

where $\lambda > 0$. Determine the L_2 properties of the process, including Riemann-integrability.

Solution:

First we remark that the function $R_X(t) = \exp\{\lambda(\mathrm{e}^{\mathrm{i}t} - 1)\}, \quad t \in \mathbb{R}$, is the characteristic function of the Poisson distribution. Thus it is positive semidefinite and it is an autocovariance function of a stochastic process.

Recalling Corollary 3.1 it is enough to show that $R_X(t)$ is continuous at point 0. This is clearly the case for this process ($R_X(t)$ is a composition of continuous functions) and we conclude that the process $\{X_t, t \in \mathbb{R}\}$ is L_2-continuous.

The process is centered and hence the mean value function is differentiable (constant) and only the properties of the autocovariance function need to be discussed in detail in order to assess L_2-differentiability of the process. To work with the partial derivatives we write the autocovariance function of the (weakly stationary) process as a function of two variables:

$$R_X(s,t) = \exp\{\lambda(\mathrm{e}^{\mathrm{i}(t-s)} - 1)\}, \quad t \in \mathbb{R}.$$

Looking at the formula above we see no reason the autocovariance should not be smooth and we suppose the second-order partial derivatives exist and are continuous (see the Remark below Theorem 3.2). We calculate the derivatives:

$$\begin{aligned}
\frac{\partial R_X(s,t)}{\partial t} =& R_X(s,t) \cdot \mathrm{i}\lambda \exp\{\mathrm{i}t - \mathrm{i}s\}, \\
\frac{\partial R_X(s,t)}{\partial t \partial s} =& \left[R_X(s,t) \cdot (-\mathrm{i})\lambda \exp\{\mathrm{i}t - \mathrm{i}s\}\right]\left[\mathrm{i}\lambda \exp\{\mathrm{i}t - \mathrm{i}s\}\right] + \left[R_X(s,t)\right]\left[\mathrm{i}\lambda \exp\{\mathrm{i}t - \mathrm{i}s\}(-\mathrm{i})\right] \\
=& R_X(s,t)\left[\lambda^2 \exp\{2(\mathrm{i}t - \mathrm{i}s)\} + \lambda \exp\{\mathrm{i}t - \mathrm{i}s\}\right].
\end{aligned}$$

Clearly the partial derivative $\frac{\partial R_X(s,t)}{\partial t \partial s}$ exists and is continuous. It is easy to check that $\frac{\partial R_X(s,t)}{\partial t \partial s} = \frac{\partial R_X(s,t)}{\partial s \partial t}$ and hence both the second-order partial derivatives exist and are continuous. The Remark below Theorem 3.2 now gives us L_2-differentiability of the process $\{X_t, t \in \mathbb{R}\}$.

For deciding about the Riemann integrability of the process on a given bounded interval $[a,b]$ we use Theorem 3.3. The mean value function is constant and hence integrable. The autocovariance function $R_X(s,t)$ is a continuous function on $\mathbb{R}^2$ and hence the Riemann integral $\int_a^b \int_a^b R_X(s,t)\,\mathrm{d}s\,\mathrm{d}t$ exists and is finite. Hence the process $\{X_t, t \in \mathbb{R}\}$ is Riemann integrable on the interval $[a,b]$. $\Diamond$

Exercise 3.5: Let $\{N_t, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$ and let A be random variable with symmetric alternative distribution on $\{-1, 1\}$, i.e. $\mathbb{P}(A = 1) = \mathbb{P}(A = -1) = \frac{1}{2}$, independent of the process $\{N_t, t \geq 0\}$. We define $X_t = A(-1)^{N_t}$, $t \geq 0$. Is the process $\{X_t, t \geq 0\}$ L_2-continuous?

Solution:

In the Exercise 2.6 we have shown that the process $\{X_t, t \geq 0\}$ is centered and weakly stationary. Its autocovariance function can be written as $R_X(t) = \mathrm{e}^{-2\lambda|t|}$, $t \in \mathbb{R}$.

Recalling Corollary 3.1 it is enough to show that $R_X(t)$ is continuous at point 0. This is clearly the case for this process ($R_X(t)$ is a composition of continuous functions) and we conclude that the process $\{X_t, t \geq 0\}$ is L_2-continuous.

This Exercise illustrates the fact that L_2-continuity is not equivalent to the continuity of trajectories – the process $\{X_t, t \geq 0\}$ is L_2-continuous, yet its trajectories are piecewise constant with jumps. $\Diamond$

Further exercises

Exercise 3.6: Let $\{X_t, t \in \mathbb{R}\}$ be a process of independent identically distributed random variables with a mean value μ and a finite variance $\sigma^2 > 0$. What are the L_2 properties of such a process (including Riemann-integrability)?

Exercise 3.7: Consider the Poisson process with intensity λ. Determine the L_2 properties of the process, including Riemann-integrability.

Exercise 3.8: Determine the L_2 properties, including Riemann-integrability, of the Ornstein-Uhlenbeck process $\{U_t, t \geq 0\}$, defined by the formula

$$U_t = \mathrm{e}^{-\alpha t/2} W_{\exp\{\alpha t\}}, \quad t \geq 0$$

where $\alpha > 0$ is a positive parameter and $\{W_t, t \geq 0\}$ is a Wiener process.

Exercise 3.9: Let $\{W_t, t \geq 0\}$ be a Wiener process. We define $B_t = W_t - tW_1$, $t \in [0,1]$. The stochastic process $\{B_t, t \in [0,1]\}$ is called the *Brownian bridge*. Determine whether the process $\{B_t, t \in (0,1)\}$ is L_2-continuous and L_2-differentiable. Does the Riemann integral $\int_0^1 B_t \, \mathrm{d}t$ exist?

4 Spectral decomposition of the autocovariance function

Theorem 4.1: A complex function $R(t)$, $t \in \mathbb{Z}$, is an autocovariance function of a weakly stationary random sequence if and only if

$$R(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}F(\lambda),\ t \in \mathbb{Z}, \tag{5}$$

where F is a bounded right-continuous non-decreasing function on $[-\pi, \pi]$ such that $F(-\pi) = 0$.

The function F is determined uniquely and it is called the *spectral distribution function of a random sequence*. If F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$ we call its density f the *spectral density*. It follows that $F(\lambda) = \int_{-\pi}^{\lambda} f(x)\,\mathrm{d}x$, $f = F'$ and

$$R(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda)\,\mathrm{d}\lambda,\ t \in \mathbb{Z}. \tag{6}$$

If F is piecewise constant with jumps at points $\lambda_i \in (-\pi, \pi]$ of the magnitudes $a_i > 0$ then

$$R(t) = \sum_j a_j \mathrm{e}^{\mathrm{i}t\lambda_j},\ t \in \mathbb{Z}.$$

Theorem 4.2: A complex function $R(t)$, $t \in \mathbb{R}$, is an autocovariance function of a centered weakly stationary L_2-continuous stochastic process if and only if

$$R(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}F(\lambda),\ t \in \mathbb{R},$$

where F is a right-continuous non-decreasing function such that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = R(0) < \infty$.

The function F is determined uniquely and it is called the *spectral distribution function of an L_2-continuous stochastic process*. If F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$ we call its density f the *spectral density*. It follows that $F(\lambda) = \int_{-\infty}^{\lambda} f(x)\,\mathrm{d}x$, $f = F'$ and

$$R(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda)\,\mathrm{d}\lambda,\ t \in \mathbb{R}. \tag{7}$$

If F is piecewise constant with jumps at points $\lambda_i \in \mathbb{R}$ of the magnitudes $a_i > 0$ then

$$R(t) = \sum_j a_j \mathrm{e}^{\mathrm{i}t\lambda_j},\ t \in \mathbb{R}.$$

Theorem 4.3: Let $\{X_t, t \in \mathbb{Z}\}$ be a weakly stationary sequence with the autocovariance function $R(t)$ such that $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$. Then the spectral density of the sequence $\{X_t, t \in \mathbb{Z}\}$ exists and is given by

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} R(t),\ \lambda \in [-\pi, \pi]. \tag{8}$$

Theorem 4.4: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary L_2-continuous process with the autocovariance function $R(t)$ such that $\int_{-\infty}^{\infty} |R(t)|\,\mathrm{d}t < \infty$. Then the spectral density of the process $\{X_t, t \in \mathbb{R}\}$ exists and is given by

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} R(t)\,\mathrm{d}t,\ \lambda \in \mathbb{R}. \tag{9}$$

Exercise 4.1: Determine the autocovariance function of a weakly stationary sequence with the spectral density

$$f(\lambda) = a\cos\frac{\lambda}{2},\ \lambda \in [-\pi, \pi],$$

where $a > 0$ is a constant.

Solution:

The autocovariance function can be computed using the formula (6) in Theorem 4.1 above. The spectral density $f(\lambda)$ is symmetric with respect to point 0 in this case and hence the autocovariance function will be a real function. Equation (6) then reduces to $R(t) = \int_{-\pi}^{\pi} \cos(t\lambda) f(\lambda)\,\mathrm{d}\lambda,\ t \in \mathbb{Z}$. We may then write

$$R(t) = \int_{-\pi}^{\pi} \cos(t\lambda)\cdot a\cos\frac{\lambda}{2}\,\mathrm{d}\lambda = \frac{a}{2}\int_{-\pi}^{\pi}\Big(\cos\big(\lambda(t+1/2)\big) + \cos\big(\lambda(t-1/2)\big)\Big)\,\mathrm{d}\lambda,\ t \in \mathbb{Z}.$$

In the last step we used the formula $\cos\alpha\cdot\cos\beta = \frac{1}{2}\big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\big)$. Now it is not difficult to compute the integral above and we obtain the result

$$R(t) = (-1)^{t+1}\frac{a}{t^2 - 1/4},\ t \in \mathbb{Z}.$$

We used the following facts to obtain the simple form of the result above: $\sin(x \pm y) = \sin x\,\cos y \pm \cos x\,\sin y$; $\cos(\pi t) = (-1)^t, t \in \mathbb{Z}$.

We stress here that the computation above is correct also for $t = 0$. This is not always the case (see the Exercise 4.3) as the factor $\mathrm{e}^{\mathrm{i}t\lambda}$ in Equation (6) is constant 1 for $t = 0$. However, we might try to check this by specifically calculating

$$R(0) = \int_{-\pi}^{\pi} a\cos\frac{\lambda}{2}\,\mathrm{d}\lambda = \left[2a\sin\frac{\lambda}{2}\right]_{-\pi}^{\pi} = 2a\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = 4a.$$

This result conforms to the formula we obtained above and we checked independently the result for $t = 0$. ◊

Exercise 4.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a weakly stationary sequence with the spectral density

$$f(\lambda) = a\cos\frac{\lambda}{2}, \quad \lambda \in [-\pi, \pi],$$

where $a > 0$ is a constant. Determine the autocovariance function of the sequence $\{Y_t, t \in \mathbb{Z}\}$ defined by $Y_t = \frac{1}{5}X_{t-5}, t \in \mathbb{Z}$.

Solution:

The autocovariance function of $\{X_t, t \in \mathbb{Z}\}$ was determined in the previous exercise. Now we try to express $R_Y(s,t)$ by means of $R_X(s,t)$:

$$\begin{aligned} R_Y(s,t) &= \operatorname{cov}\left(\frac{1}{5}X_{s-5}, \frac{1}{5}, X_{t-5}\right) = \frac{1}{25}\operatorname{cov}(X_{s-5}, X_{t-5}) = \frac{1}{25}R_X(s-5, t-5) \\ &= \frac{1}{25}R_X((t-5)-(s-5)) = \frac{1}{25}R_X(t-s). \end{aligned}$$

Thus the random sequence is again weakly stationary and $R_Y(t) = \frac{1}{25}(-1)^{t+1}\frac{a}{t^2-1/4},\ t \in \mathbb{Z}$. ◊

Exercise 4.3: Determine the autocovariance function of the centered weakly stationary process $\{X_t, t \in \mathbb{R}\}$ with the spectral distribution function

$$F_X(\lambda) = \begin{cases} 0, & \lambda \leq -b, \\ (\lambda + b)a, & -b \leq \lambda \leq b, \\ 2ab, & \lambda \geq b, \end{cases}$$

where $a > 0$ and $b > 0$ are constants.

Solution:

Computing the autocovariance function using the Lebesgue-Stieltjes integral in formula (7) is not convenient. However, we can calculate the spectral density and change the problem into integration with respect to the Lebesgue measure.

The spectral density is obtained from the spectral distribution function by differentiation (see Theorem 4.2):

$$f_X(\lambda) = \frac{\mathrm{d}F_X(\lambda)}{\mathrm{d}\lambda} = \begin{cases} a, & \lambda \in (-b, b), \\ 0, & \text{otherwise.} \end{cases}$$

We then calculate, for $t \in \mathbb{R}, t \neq 0$:

$$\begin{aligned} R_X(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\lambda = \int_{-b}^{b} a\mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\lambda = \left[\frac{a\mathrm{e}^{\mathrm{i}t\lambda}}{\mathrm{i}t}\right]_{\lambda=-b}^{b} \\ &= \frac{a}{\mathrm{i}t}\left(\mathrm{e}^{\mathrm{i}tb} - \mathrm{e}^{-\mathrm{i}tb}\right) = \frac{2a}{t} \cdot \frac{\mathrm{e}^{\mathrm{i}tb} - \mathrm{e}^{-\mathrm{i}tb}}{2\mathrm{i}} = \frac{2a}{t}\sin(bt). \end{aligned}$$

Note that the computation above is only correct for $t \neq 0$ (otherwise the primitive function is not correct). The case $t = 0$ needs to be treated separately:

$$R_X(0) = \int_{-b}^{b} a\,\mathrm{d}\lambda = 2ab.$$

Above we have determined the values of the autocovariance function $R_X(t)$ for all values of $t \in \mathbb{R}$. Note that, even though we need two different formulas for $t = 0$ and $t \neq 0$, the autocovariance function is continuous at point 0 (in fact it is continuous everywhere). This is because the limit of $\frac{1}{bt}\sin(bt)$ for $t \to 0$ is 1. It follows that the process $\{X_t, t \in \mathbb{R}\}$ is L_2-continuous. $\Diamond$

Exercise 4.4: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the spectral distribution function

$$F_X(\lambda) = \begin{cases} 0, & \lambda \leq -b, \\ (\lambda + b)a, & -b \leq \lambda \leq b, \\ 2ab, & \lambda \geq b, \end{cases}$$

where $a > 0$ and $b > 0$ are constants. Define the random process $\{Y_t, t > 0\}$ by $Y_t = \frac{1}{t}X_{t^2}, t \in \mathbb{R}^+$. Determine the autocovariance function of $\{Y_t, t \in > 0\}$ and decide about its weak stationarity.

Solution:

The autocovariance function of $\{X_t, t \in \mathbb{R}\}$ was determined in the previous exercise. Now we try to express $R_Y(s,t)$ by means of $R_X(s,t)$.

$$R_Y(s,t) = \operatorname{cov}\left(\frac{1}{s}X_{s^2}, \frac{1}{t}X_{t^2}\right) = \frac{1}{st}\operatorname{cov}(X_{s^2}, X_{t^2}) = \frac{1}{st}R_X(s^2, t^2).$$

Thus together we obtain

$$R_Y(s,t) = \begin{cases} \frac{2a}{st}\frac{\sin(b(s^2-t^2))}{s^2-t^2}, & s \neq t > 0, \\ \frac{2ab}{st} = \frac{2ab}{s^2}, & s = t > 0. \end{cases}$$

Since $R_Y(s,s) = \frac{2ab}{s^2}$ is not a constant function of s the process $\{Y_t, t \in \mathbb{R}^+\}$ is not weakly stationary. $\Diamond$

Exercise 4.5: Determine the spectral distribution function and the spectral density (if it exists) of the Ornstein-Uhlenbeck process $\{U_t, t \geq 0\}$ defined by the formula

$$U_t = \mathrm{e}^{-\alpha t/2} W_{\exp\{\alpha t\}}, \quad t \geq 0,$$

where $\alpha > 0$ is a parameter and $\{W_t, t \geq 0\}$ is a Wiener process.

Solution:

We recall that (see Exercise 2.8) the Ornstein-Uhlenbeck process is a centered weakly stationary process with the autocovariance function $R_U(t) = \sigma^2 \mathrm{e}^{-\alpha|t|/2}, t \in \mathbb{R}$, where σ^2 is a parameter of the underlying Wiener process. The process is also L_2-continuous – it is centered and its autocovariance function $R_U(t)$ is continuous at point 0 (see also the comments in the solution of Exercise 3.5 which apply also here).

Note that although the process itself is defined on $T = [0, \infty)$, its autocovariance function is defined on the whole $\mathbb{R}$. This is because the argument t of $R_U(t)$ in fact plays the role of the difference of two points in $[0, \infty)$ which can be any real number. Thus the relevant theorems about the spectral decomposition of the autocovariance function of a weakly stationary stochastic process can be applied here.

The spectral distribution function of a weakly stationary L_2-continuous process always exists (see Theorem 4.2). However, there is no routine approach for determining it. On the other hand, the spectral density might not exist for certain processes but we have a standard tool for looking for it – the criterion and inverse formula in Theorem 4.4.

Note that the Ornstein-Uhlenbeck process fulfills the assumptions of Theorem 4.4 and thus we may look at the criterion:

$$\int_{-\infty}^{\infty} |R_U(t)| \,\mathrm{d}t = 2\int_0^{\infty} \sigma^2 \mathrm{e}^{-\alpha t/2} \,\mathrm{d}t < \infty.$$

Of course it is not difficult to compute the value of the integral above but the precise value is not important here – it only matters that the integral is finite. Hence the spectral density of the process exists and can be determined using the inverse formula from Theorem 4.4:

$$\begin{aligned} f_U(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda} R_U(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\sigma^2 \mathrm{e}^{-\alpha|t|/2}\,\mathrm{d}t \\ &= \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \big(\cos(t\lambda) - \mathrm{i}\sin(t\lambda)\big)\mathrm{e}^{-\alpha|t|/2}\,\mathrm{d}t = \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \cos(t\lambda)\mathrm{e}^{-\alpha|t|/2}\,\mathrm{d}t \\ &= \frac{\sigma^2}{\pi}\int_0^{\infty} \cos(t\lambda)\mathrm{e}^{-\alpha t/2}\,\mathrm{d}t = \frac{2\sigma^2}{\pi}\cdot\frac{\alpha}{\alpha^2 + 4\lambda^2},\ \lambda \in \mathbb{R}. \end{aligned}$$

On the second line of the calculation above we used symmetry arguments to realize that the imaginary part of the integral is 0 and we may neglect the term $\mathrm{i}\sin(t\lambda)$ in the integrand. Later we used a different symmetry argument to change the integration domain from $(-\infty, \infty)$ to $(0, \infty)$. The last step requires double application of integration by parts.

The spectral distribution function can now be recovered from the spectral density by integration:

$$F_U(\lambda) = \int_{-\infty}^{\lambda} f_U(x)\,\mathrm{d}x = \frac{\sigma^2}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{2\lambda}{\alpha}\right)\right),\ \lambda \in \mathbb{R}.$$

$\Diamond$

Exercise 4.6: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \cos t, \quad t \in \mathbb{R}.$$

Determine the spectral distribution function of the process.

Solution:

We first note that the process is L_2-continuous (its autocovariance function $R_X(t)$ is continuous at point 0) and we can use the relevant theorems above. If we try to use the inverse formula in Theorem 4.4, we run into trouble:

$$\int_{-\infty}^{\infty} |R_X(t)|\,\mathrm{d}t = \int_{-\infty}^{\infty} |\cos t|\,\mathrm{d}t = \infty,$$

i.e. the criterion is not fulfilled and the inverse formula (9) cannot be used. We stress, however, that this does not imply non-existence of the spectral density! At this point we simply cannot tell if the spectral density exists or not.

On the other hand, the spectral distribution function F_X exists – the assumptions of Theorem 4.2 are fulfilled. We may hence write, for any $t \in \mathbb{R}$,

$$\cos t = R_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}F(\lambda).$$

How can the Lebesgue-Stieltjes integration of the complex exponential function result in the cosine function? The answer becomes clear if we realize that, in fact, for any $t \in \mathbb{R}$,

$$\cos t = \frac{1}{2}\mathrm{e}^{\mathrm{i}t} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}t},$$

i.e. only two discrete contributions of the complex exponential function make up the cosine function. The spectral distribution function must be piecewise constant with two jumps corresponding to the two terms in the formula above. The positions of the jumps are determined by the arguments of the complex exponential functions (-1 and 1) and their size is determined by the corresponding weights (1/2 and 1/2). It can be said that the autocovariance function has a discrete spectrum. We conclude that

$$F_X(\lambda) = \begin{cases} 0, & \lambda < -1, \\ 1/2, & -1 \le \lambda < 1, \\ 1, & 1 \le \lambda. \end{cases}$$

Now that we know the form of the spectral distribution function we can conclude that the spectral density of the process does not exist – the spectral distribution function has jumps and there is no (non-negative, measurable) function f_X such that $F_X(\lambda) = \int_{-\infty}^{\lambda} f_X(x)\,\mathrm{d}x,\ \lambda \in \mathbb{R}$. $\Diamond$

Exercise 4.7: Determine the spectral density of the weakly stationary sequence $\{X_t, t \in \mathbb{Z}\}$ with the autocovariance function

$$R_X(t) = \begin{cases} \frac{16}{15} \cdot \frac{1}{2^{|t|}} & \text{for even values of } t, \\ 0 & \text{for odd values of } t. \end{cases}$$

Solution:

Theorem 4.3 provides an inverse formula for determining the spectral density of a weakly stationary random sequence. The assumption of the theorem is fulfilled:

$$\sum_{t=-\infty}^{\infty} |R_X(t)| \leq \frac{16}{15} \cdot 2 \sum_{t=0}^{\infty} \frac{1}{2^t} < \infty.$$

Hence we may use the inverse formula (8) and write for $\lambda \in [-\pi, \pi]$, noting that only every other value of the autocovariance function is non-zero:

$$\begin{aligned}
f_X(\lambda) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\lambda} R_X(k) \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\mathrm{e}^{-\mathrm{i}2k\lambda} \frac{16}{15} \cdot \frac{1}{2^{2k}} + \mathrm{e}^{\mathrm{i}2k\lambda} \frac{16}{15} \cdot \frac{1}{2^{2k}} \right) - \frac{1}{2\pi} \cdot \frac{16}{15} \\
&= -\frac{8}{15\pi} + \frac{8}{15\pi} \sum_{k=0}^{\infty} \left(\frac{\mathrm{e}^{-2\mathrm{i}\lambda}}{4} \right)^k + \frac{8}{15\pi} \sum_{k=0}^{\infty} \left(\frac{\mathrm{e}^{2\mathrm{i}\lambda}}{4} \right)^k \\
&= \frac{8}{15\pi} \left[-1 + \frac{1}{1 - \frac{\mathrm{e}^{-2\mathrm{i}\lambda}}{4}} + \frac{1}{1 - \frac{\mathrm{e}^{2\mathrm{i}\lambda}}{4}} \right] \\
&= \frac{8}{15\pi} \left[-1 + \frac{4(8 - 2\cos(2\lambda))}{(4 - \mathrm{e}^{-2\mathrm{i}\lambda})(4 - \mathrm{e}^{2\mathrm{i}\lambda})} \right] \\
&= \frac{8}{15\pi} \cdot \frac{32 - 8\cos(2\lambda) - 17 + 8\cos(2\lambda)}{17 - 8\cos(2\lambda)} \\
&= \frac{8}{\pi} \cdot \frac{1}{17 - 8\cos(2\lambda)}.
\end{aligned}$$

Finally, note that the spectral density is a non-negative function – it must be due to its link to the spectral distribution function. ◊

Exercise 4.8: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \exp\{\kappa(\mathrm{e}^{\mathrm{i}t} - 1)\}, \quad t \in \mathbb{R},$$

where $\kappa > 0$. Determine the spectral distribution function of the process.

Solution:

We first note that the process $\{X_t, t \in \mathbb{R}\}$ is L_2-continuous – its autocovariance function $R_X(t)$ is continuous at point 0. Theorem 4.2 then tells us that the spectral distribution function exists.

We may try to use Theorem 4.4 for determining the spectral density (if it exists) and then obtain the spectral distribution function by integration. However, the criterion in Theorem 4.4 is not fulfilled:

$$\int_{-\infty}^{\infty} |R_X(t)| \,\mathrm{d}t = \int_{-\infty}^{\infty} |\exp\{\kappa \mathrm{e}^{\mathrm{i}t}\}| \,\mathrm{e}^{-\kappa} \,\mathrm{d}t = \infty.$$

In the last step we took advantage of the fact that for any $\kappa > 0$ there is a constant $c > 0$ such that $|\exp\{\kappa \mathrm{e}^{\mathrm{i}t}\}| \geq c$ for any $t \in \mathbb{R}$. The inverse formula (9) cannot be used. On the other hand we do not know if the spectral density exists or not.

At this point we should notice that the autocovariance function $R_X(t)$ is in fact the characteristic function of the Poisson distribution with parameter $\kappa > 0$. In the following we denote Z a random variable with such distribution. For any $t \in \mathbb{R}$ we may write:

$$R_X(t) = \exp\{\kappa(\mathrm{e}^{\mathrm{i}t} - 1)\} = \mathbb{E}\mathrm{e}^{\mathrm{i}tZ} = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}tk} \cdot \frac{\kappa^k}{k!}\mathrm{e}^{-\kappa}.$$

With this in mind we see that the spectral distribution function $F_X(\lambda)$ is piecewise constant with jumps at points $\lambda = k, k = 0, 1, 2, \ldots$. The size of the jump at point k is $\mathrm{e}^{-\kappa}\frac{\kappa^k}{k!}$. This is an example of a spectral distribution function with countably many jumps. ♢

Exercise 4.9: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R_X(t) = \frac{1}{1 - \mathrm{i}t}, \quad t \in \mathbb{R}.$$

Determine the spectral density of the process.

Solution:

We first note that the process $\{X_t, t \in \mathbb{R}\}$ is L_2-continuous – its autocovariance function $R_X(t)$ is continuous at point 0. Theorem 4.2 then tells us that the spectral distribution function exists.

When we attempt to use the inverse formula in Theorem 4.4 we first need to determine $|R_X(t)|, t \in \mathbb{R}$:

$$|R_X(t)|^2 = \frac{1}{1-\mathrm{i}t}\overline{\frac{1}{1-\mathrm{i}t}} = \frac{1}{1-\mathrm{i}t}\frac{1}{1+\mathrm{i}t} = \frac{1}{(1-\mathrm{i}t)(1+\mathrm{i}t)} = \frac{1}{1+t^2},$$
$$|R_X(t)| = \frac{1}{\sqrt{1+t^2}}.$$

We now see that the assumption of Theorem 4.4 is not fulfilled and the inverse formula cannot be used:

$$\int_{-\infty}^{\infty} |R_X(t)| \,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+t^2}} \,\mathrm{d}t = \infty.$$

This does not mean the spectral density does not exist, we simply do not know at this point. However, we might notice that the autocovariance function $R_X(t)$ is in fact the characteristic function of the exponential distribution with mean 1. We denote Z a random variable with such distribution and write:

$$R_X(t) = \frac{1}{1-\mathrm{i}t} = \mathbb{E}\mathrm{e}^{\mathrm{i}tZ} = \int_0^{\infty} \mathrm{e}^{\mathrm{i}tx} \cdot \mathrm{e}^{-x} \,\mathrm{d}x.$$

Clearly the spectral density $f_X(\lambda)$ of the process $\{X_t, t \in \mathbb{R}\}$ exists and is given by

$$f_X(\lambda) = \begin{cases} \mathrm{e}^{-\lambda}, & \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This is an example of the situation where the spectral density of the process exists but cannot be determined by the inverse formula. ♢

Further exercises

Exercise 4.10: The centered weakly stationary process $\{X_t, t \in \mathbb{R}\}$ has the spectral density

$$f(\lambda) = c^2 \mathbf{1}\{\lambda_0 \le |\lambda| \le 2\lambda_0\}, \quad \lambda \in \mathbb{R},$$

where c and λ_0 are positive constants. Determine the autocovariance function of the process.

Exercise 4.11: Let $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ be independent weakly stationary sequences with the spectral densities f_X and f_Y. Consider the sequence $Z_t = X_t + Y_t$, $t \in \mathbb{Z}$. Show that the sequence $\{Z_t, t \in \mathbb{Z}\}$ has the spectral density of the form $f_Z = f_X + f_Y$.

Exercise 4.12: Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of uncorrelated centered random variables with finite positive variance σ^2. Determine the spectral density of the sequence.

Exercise 4.13: Consider a real-valued centered random variable Y with finite positive variance σ^2 and a random sequence defined as $X_t = (-1)^t Y$, $t \in \mathbb{Z}$. Decide whether the spectral density of this sequence exists. If it does, find a formula for it.

Exercise 4.14: The *elementary process* $\{X_t, t \in \mathbb{R}\}$ is defined as $X_t = Y e^{i\omega t}, t \in \mathbb{R}$, where $\omega \in \mathbb{R}$ is a constant and Y is a (complex) random variable such that $\mathbb{E}Y = 0$ and $\mathbb{E}|Y|^2 = \sigma^2 < \infty$. Discuss the stationarity of the process $\{X_t, t \in \mathbb{R}\}$ and determine its spectral density.

Exercise 4.15: Let $\{N_t, t \geq 0\}$ be a Poisson process with the intensity $\lambda > 0$ and let A be a real-valued random variable with zero mean and variance 1, independent of the process $\{N_t, t \geq 0\}$. Define $X_t = A(-1)^{N_t}$, $t \geq 0$. Determine the spectral distribution function and the spectral density (if it exists) of the process $\{X_t, t \geq 0\}$.

Exercise 4.16: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with the autocovariance function

$$R(t) = c \exp\{-at^2\}, \quad t \in \mathbb{R},$$

where a and c are positive constants. Determine the spectral density of the process.

5 Spectral representation of stochastic processes

Definition 5.1: Let $\{X_t, t \in T\}$, T an interval, be a (generally complex-valued) second order process on $(\Omega, \mathcal{A}, P)$. We say that $\{X_t, t \in T\}$ is an *orthogonal increment process*, if for any $t_1, \dots, t_4 \in T$ such that $(t_1, t_2] \cap (t_3, t_4] = \emptyset$,

$$\mathbb{E}(X_{t_2} - X_{t_1})(\overline{X}_{t_4} - \overline{X}_{t_3}) = 0.$$

Theorem 5.1: Let $\{Z_\lambda, \lambda \in [a, b]\}$ be a centered mean square right-continuous process with orthogonal increments, $[a, b]$ a bounded interval. Then there exists a unique non-decreasing right-continuous function F such that

$$\begin{aligned} F(\lambda) &= 0, & \lambda &\leq a, \\ &= F(b), & \lambda &\geq b, \\ F(\lambda_2) - F(\lambda_1) &= \mathbb{E}|Z_{\lambda_2} - Z_{\lambda_1}|^2, & A &\leq \lambda_1 < \lambda_2 \leq b. \end{aligned}$$

We call F the *distribution function associated with the orthogonal increment process* $\{Z_\lambda, \lambda \in [a, b]\}$.

Let $\{Z_\lambda, \lambda \in [a, b]\}$ be a centered mean square right-continuous process with orthogonal increments and the associated distribution function F. Let $f \in L_2(F)$ be a measurable function. Let us denote $I(f) = \int_a^b f(\lambda) \, \mathrm{d}Z(\lambda)$ the integral of f with respect to the orthogonal increment process $\{Z_\lambda, \lambda \in [a, b]\}$ as it was defined in the lecture notes [4] in Chapter 7.2.

Theorem 5.2: Let $\{Z_\lambda, \lambda \in [a, b]\}$ be a centered mean square right-continuous process with orthogonal increments and the associated distribution function F. Then the integral $I(f)$ has the following properties.

1. Let $f \in L_2(F)$. Then $\mathbb{E}I(f) = \mathbb{E}\int_a^b f(\lambda) \, \mathrm{d}Z(\lambda) = 0$.
2. Let $f, g \in L_2(F), \alpha, \beta \in \mathbb{C}$ be constants. Then $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.
3. Let $f, g \in L_2(F)$. Then
$$\mathbb{E}I(f)\overline{I(g)} = \int_a^b f(\lambda)\overline{g(\lambda)} \, \mathrm{d}F(\lambda).$$
4. Let $\{f_n, n \in \mathbb{N}\}$ and f be functions in $L_2(f)$, respectively. Then, as $n \to \infty$,
$$f_n \to f \text{ in } L_2(f) \Longleftrightarrow I(f_n) \to I(f) \text{ in } L_2(\Omega, \mathcal{A}, P).$$

Theorem 5.3: Let $X_t, t \in \mathbb{Z}$, be random variables such that

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z(\lambda),$$

where $\{Z(\lambda), \lambda \in [-\pi, \pi]\}$ is a centered mean square right-continuous process with orthogonal increments on $[\pi, \pi]$ and associated distribution function F. Then $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary sequence with the spectral distribution function F.

Theorem 5.4: Let $\{X_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with the spectral distribution function F. Then there exists a centered orthogonal increment process $\{Z(\lambda), \lambda \in [-\pi, \pi]\}$ such that

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{d}Z(\lambda),$$

and

$$\mathbb{E}|Z(\lambda) - Z(-\pi)|^2 = F(\lambda), \quad -\pi \le \lambda \le \pi.$$

Remark: Similar spectral decomposition exists also for centered weakly stationary mean square continuous processes.

Exercise 5.1: Let $\{Z_\lambda, \lambda \in [-\pi, \pi]\}$ be a centered mean square right-continuous process with orthogonal increments such that

$$\mathbb{E}|Z_{\lambda_2} - Z_{\lambda_1}|^2 = a(\lambda_2 - \lambda_1), \quad -\pi \le \lambda_1 \le \lambda_2 \le \pi, \tag{10}$$

for some $a > 0$. Show that the random sequence $\{Y_t, t \in \mathbb{Z}\}$ defined as

$$Y_t = \int_{-\pi}^{\pi} \left(\pi - \frac{|\lambda|}{2}\right)^{-1} \mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{d}Z(\lambda), \quad t \in \mathbb{Z},$$

is weakly stationary. Determine its variance and find its spectral density.

Solution:

First we see from (10) and Theorem 5.1 that the distribution function associated with the orthogonal increment process $\{Z_\lambda, \lambda \in [-\pi, \pi]\}$ is

$$F_Z(\lambda) = \mathbb{E}|Z_\lambda - Z_{-\pi}|^2 = a(\lambda + \pi), \quad \lambda \in [-\pi, \pi].$$

The function F_Z is differentiable on $(-\pi, \pi)$ and $F_Z'(\lambda) = a, \lambda \in (\pi, \pi)$.

The function $g(\lambda) = \left(\pi - \frac{|\lambda|}{2}\right)^{-1} \mathrm{e}^{\mathrm{i}t\lambda}$ is in $L_2(F_Z)$ for any $t \in \mathbb{Z}$ and Y_t is defined as the integral of g with respect to the orthogonal increment process $\{Z_\lambda, \lambda \in [-\pi, \pi]\}$. Thus from Theorem 5.2, part 1, we have

$$\mathbb{E}\, Y_t = 0, \quad t \in \mathbb{Z},$$

i.e. the random sequence $\{Y_t, t \in \mathbb{Z}\}$ is centered. Furthermore, from Theorem 5.2, part 3, and by using the particular form of $\mathrm{d}F_Z(\lambda)$ we get

$$R_Y(s,t) = \mathbb{E} Y_s \overline{Y}_t = \int_{-\pi}^{\pi} \left(\pi - \frac{|\lambda|}{2}\right)^{-2} \mathrm{e}^{\mathrm{i}s\lambda} \mathrm{e}^{-\mathrm{i}t\lambda} \,\mathrm{d}F_Z(\lambda) = a \int_{-\pi}^{\pi} \left(\pi - \frac{|\lambda|}{2}\right)^{-2} \mathrm{e}^{\mathrm{i}(s-t)\lambda} \,\mathrm{d}\lambda, \quad s, t \in \mathbb{Z}. \tag{11}$$

This, obviously, is a function of $(s - t)$ only, thus $\{Y_t, t \in \mathbb{Z}\}$ is weakly stationary. For the special case $s = t$ we compute the variance of Y_t for $t \in \mathbb{Z}$:

$$\begin{aligned}
\operatorname{var} Y_t = R_Y(t,t) = R_Y(0) = \mathbb{E}|Y_t|^2 &= a \int_{-\pi}^{\pi} \left(\pi - \frac{|\lambda|}{2}\right)^{-2} \mathrm{d}\lambda \\
&= 2a \int_0^{\pi} \left(\pi - \frac{\lambda}{2}\right)^{-2} \mathrm{d}\lambda = 2a \int_{\pi}^{\frac{\pi}{2}} \frac{1}{x^2} \cdot (-2) \,\mathrm{d}x = 4a \int_{\frac{\pi}{2}}^{\pi} x^{-2} \,\mathrm{d}x = \frac{4a}{\pi},
\end{aligned}$$

where in the sixth equality we used the substitution $x = \pi - \frac{\lambda}{2}$.

To find the spectral density let us rewrite R_Y given in (11) as a function of one variable only:

$$R_Y(t) = R_Y(t,0) = \int_{-\pi}^{\pi} a\left(\pi - \frac{|\lambda|}{2}\right)^{-2} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\lambda, \quad t \in \mathbb{Z}.$$

Obviously, from the uniqueness of spectral density (Theorem 4.1) the function $a\left(\pi - \frac{|\lambda|}{2}\right)^{-2}$ is the spectral density of the random sequence $\{Y_t, t \in \mathbb{Z}\}$. $\Diamond$

Exercise 5.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with spectral decomposition

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}Z(\lambda), \quad t \in \mathbb{Z},$$

where $\{Z_\lambda, \lambda \in [-\pi.\pi]\}$ is a centered right-continuous process with orthogonal increments and associated distribution function F_Z. Let $\{Y_t, t \in \mathbb{Z}\}$ be a random sequence defined by

$$Y_t - \phi Y_{t-1} = X_t, \quad t \in \mathbb{Z}, \tag{12}$$

where $\phi \in \mathbb{C}$ is a constant with $|\phi| < 1$. Find a function ψ such that

$$Y_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\psi(\lambda)\,\mathrm{d}Z(\lambda), \quad t \in \mathbb{Z}. \tag{13}$$

Prove that $\{Y_t, t \in \mathbb{Z}\}$ is weakly stationary centered random sequence. Determine $\mathbb{E}Y_{s+t}\overline{Y}_s$ and $\mathbb{E}Y_{s+t}\overline{X}_s$ as integrals w.r.t. F_Z. Calculate them for the special choice

$$F_Z(\lambda) = \begin{cases} 0, & \lambda \le -\pi, \\ \frac{\sigma^2}{2\pi}(\lambda + \pi), & \lambda \in [-\pi, \pi], \\ \sigma^2, & \lambda \ge \pi. \end{cases}$$

Solution:

To find ψ we have to combine the desired representation (13) with the definition of Y_t . By plugging (13) into (12) and using linearity of the integral we get

$$\begin{aligned} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\psi(\lambda)\,\mathrm{d}Z(\lambda) - \phi\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(t-1)\lambda}\psi(\lambda)\,\mathrm{d}Z(\lambda) &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}Z(\lambda), \\ \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\left(\psi(\lambda) - \frac{\phi}{\mathrm{e}^{\mathrm{i}\lambda}}\psi(\lambda)\right)\mathrm{d}Z(\lambda) &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}Z(\lambda), \end{aligned} \tag{14}$$

which must hold for all $t \in \mathbb{Z}$. Now if we find $\psi(\lambda)$ such that

$$\psi(\lambda) - \frac{\phi}{\mathrm{e}^{\mathrm{i}\lambda}}\psi(\lambda) = 1, \text{ for all } \lambda \in [-\pi, \pi], \tag{15}$$

then (14) will be satisfied for all $t \in \mathbb{Z}$. Since $\left|\frac{\phi}{\mathrm{e}^{\mathrm{i}\lambda}}\right| < 1$ we have $1 - \frac{\phi}{\mathrm{e}^{\mathrm{i}\lambda}} \neq 0$ and we get from (15) directly the solution

$$\psi(\lambda) = \frac{1}{1 - \phi\,\mathrm{e}^{-\mathrm{i}\lambda}}, \quad \lambda \in [-\pi, \pi].$$

Since

$$\left|\mathrm{e}^{\mathrm{i}t\lambda}\frac{1}{1 - \phi\,\mathrm{e}^{-\mathrm{i}\lambda}}\right| \le \frac{1}{1 - |\phi|} < K < \infty \quad \text{for all } \lambda \in [-\pi, \pi],$$

for some constant $K > 0$, the function $\mathrm{e}^{it\lambda}\psi(\lambda)$ must be in $L_2(F_Z)$ for any associated distribution function F_Z on $[-\pi, \pi]$ and we may use Theorem 5.2. Thus we obtain

$$\begin{aligned}
\mathbb{E}Y_t &= \mathbb{E}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\psi(\lambda)\,\mathrm{d}Z(\lambda) = 0,\\
\mathbb{E}Y_{s+t}\overline{Y}_s &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(s+t)\lambda}\psi(\lambda)\overline{\mathrm{e}^{\mathrm{i}s\lambda}\psi(\lambda)}\,\mathrm{d}F_Z(\lambda) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}|\psi(\lambda)|^2\,\mathrm{d}F_Z(\lambda),\\
\mathbb{E}Y_{s+t}\overline{X}_s &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(s+t)\lambda}\psi(\lambda)\overline{\mathrm{e}^{\mathrm{i}s\lambda}}\,\mathrm{d}F_Z(\lambda) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}\psi(\lambda)\,\mathrm{d}F_Z(\lambda).
\end{aligned}$$

We see that the sequence $\{Y_t, t \in \mathbb{Z}\}$ is centered and weakly stationary (second equation).

For the special choice of F_Z from the assignment we get the derivative $f_Z(\lambda) = \frac{\sigma^2}{2\pi}\mathbf{1}(\lambda \in (-\pi, \pi))$. This means that $\{X_t, t \in \mathbb{Z}\}$ is a white noise since the associated distribution function F_Z is in fact the spectral distribution function of $X_t, t \in \mathbb{Z}$ (Theorem 5.3). Then

$$R_Y(t) = \mathbb{E}Y_{s+t}\overline{Y}_s = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda}|\psi(\lambda)|^2\frac{\sigma^2}{2\pi}\,\mathrm{d}\lambda, \tag{16}$$

and

$$\frac{\sigma^2}{2\pi}|\psi(\lambda)|^2 = \frac{\sigma^2}{2\pi}\left|\frac{1}{1-\phi\mathrm{e}^{-\mathrm{i}\lambda}}\right|^2 = \frac{\sigma^2}{2\pi}\frac{1}{1+|\phi|^2-2Re(\phi\mathrm{e}^{-\mathrm{i}\lambda})}, \quad \lambda \in [-\pi, \pi],$$

is the spectral density of $\{Y_t, t \in \mathbb{Z}\}$. ♢

Remark: In the previous exercise it would not be easy to compute the last integral (16) to get $R_Y(t)$ in a closed form. But the random sequence $\{Y_t, t \in \mathbb{Z}\}$ from the previous exercise is an AR(1) sequence and there are easier methods to compute $R(t)$ – see Exercise 6.2.

6 Linear models of time series

6.1 ARMA models

MA(n): *The moving average sequence of order n* is defined by

$$X_t = b_0 Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and $b_0, b_1, \dots, b_n$ are real- or complex-valued constants, $b_0 \neq 0$, $b_n \neq 0$. It is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 \sum_{k=0}^{n-t} b_{k+t} \overline{b_k} & \text{for } 0 \leq t \leq n, \\ \overline{R_X(-t)} & \text{for } -n \leq t \leq 0, \\ 0 & \text{for } |t| > n, \end{cases}$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{n} b_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

MA(∞): *The causal linear process* is a random sequence defined by

$$X_t = \sum_{j=0}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}, \tag{17}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $c_0, c_1, \dots$ is a sequence of constants such that $\sum_{j=0}^{\infty} |c_j| < \infty$ (this condition implies the sum converges absolutely almost surely). $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 \sum_{k=0}^{\infty} c_{k+t} \overline{c_k} & \text{for } t \geq 0, \\ \overline{R_X(-t)} & \text{for } t \leq 0, \end{cases} \tag{18}$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

AR(m): *The autoregressive sequence of order m* is defined by

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \dots, a_m$ are real-valued constants, $a_m \neq 0$. If all the roots of the polynomial $1 + a_1 z + \cdots + a_m z^m$ lie outside the unit circle in $\mathbb{C}$ (which is equivalent to all the roots of $z^m + a_1 z^{m-1} + \cdots + a_m$ lying inside the unit circle) then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process (17) with coefficients c_j determined by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1}{1 + a_1 z + \cdots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in (17) into the defining relation and by comparing the coefficients by the respective terms Y_{t-j} on both sides. The autocovariance function is given by (18) and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

ARMA(m, n): This model is defined by the equation

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z}, \tag{19}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \ldots, a_m, b_1, \ldots, b_n$ are real-valued constants, $a_m \neq 0$, $b_n \neq 0$. Suppose that the polynomials $1 + a_1 z + \cdots + a_m z^m$ and $1 + b_1 z + \cdots + b_n z^n$ have no common roots and all the roots of the polynomial $1 + a_1 z + \cdots + a_m z^m$ are outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process (17) with coefficients c_j given by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1 + b_1 z + \cdots + b_n z^n}{1 + a_1 z + \cdots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in (17) into the defining relation and by comparing the coefficients by the respective terms Y_{t-j} on both sides. The autocovariance function is given by (18) and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|1 + b_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + b_n \mathrm{e}^{-\mathrm{i}n\lambda}|^2}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

Definition 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m, n) random sequence defined by (19). If there exists a sequence of constants $\{d_j, j \in \mathbb{N}_0\}$ such that $\sum_{j=0}^{\infty} |d_j| < \infty$ and

$$Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}, \quad t \in \mathbb{Z},$$

then $\{X_t, t \in \mathbb{Z}\}$ is called *invertible* (it has an AR(∞) representation).

Theorem 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m, n) random sequence. Let the polynomials $a(z) = 1 + a_1 z + \cdots + a_m z^m$ and $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have no common roots and the polynomial $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have all the roots outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is invertible and the coefficients d_j are given by

$$\sum_{j=0}^{\infty} d_j z^j = \frac{1 + a_1 z + \cdots + a_m z^m}{1 + b_1 z + \cdots + b_n z^n}, \quad |z| \leq 1.$$

Remark: We may obtain the coefficients d_j by solving the equations we get by plugging the equality $Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}$ into the defining formula of the ARMA sequence and comparing the coefficients on both sides.

Exercise 6.1: Determine the autocovariance function and the spectral density of the sequence

$$X_t = Y_t + \theta Y_{t-2}, \quad t \in \mathbb{Z},$$

where $\theta \in \mathbb{C}$ a $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\text{WN}(0, \sigma^2)$.

Solution: The sequence $\{X_t, t \in \mathbb{Z}\}$ is an MA(2) sequence with coefficients $b_0 = 1, b_1 = 0, b_2 = \theta$. Thus it is a centered weakly stationary sequence with only three nonzero values of $R_X(t)$:

$$R_X(0) = \sigma^2(1 + |\theta|^2), \qquad R_X(2) = \sigma^2\theta, \qquad R_X(-2) = \sigma^2\,\overline{\theta}.$$

From the formula for spectral density of an MA(n) model we get

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left|1 + \theta \mathrm{e}^{-\mathrm{i}2\lambda}\right|^2 = \frac{\sigma^2}{2\pi}\left(1 + |\theta|^2 + 2Re(\theta \mathrm{e}^{-\mathrm{i}2\lambda})\right), \quad \lambda \in [-\pi, \pi],$$

and for the special case of θ real $f_X(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + \theta^2 + 2\theta\cos(2\lambda)\right), \lambda \in [-\pi, \pi]$. $\diamondsuit$

Exercise 6.2: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by

$$X_t = \rho X_{t-1} + Y_t, \quad t \in \mathbb{Z}, \tag{20}$$

where $\rho \in \mathbb{R}, 0 < |\rho| < 1$, is a constant and the random sequence $\{Y_t, t \in \mathbb{Z}\}$ is defined by

$$Y_t \;= \begin{cases} Z_t & \text{for } t \text{ even}, \\ \frac{1}{\sqrt{2}}(Z_t^2 - 1) & \text{for } t \text{ odd}. \end{cases}$$

Here $\{Z_t, t \in \mathbb{Z}\}$ is a sequence of independent $N(0,1)$-distributed random variables. Decide whether $\{X_t, t \in \mathbb{Z}\}$ is strictly or weakly stationary and compute its autocovariance function.

Solution: First we must investigate the random sequence $\{Y_t, t \in \mathbb{Z}\}$. Obviously it is a sequence of independent random variables, $\mathbb{E}Y_t = 0, \operatorname{var} Y_t = 1$ for t even, and for t odd

$$\mathbb{E}Y_t = \frac{1}{\sqrt{2}}(1-1) = 0, \quad \operatorname{var} Y_t = \mathbb{E}Y_t^2 = \frac{1}{2}\operatorname{var}(Z_t^2) = 1,$$

since $Z_t \sim N(0,1)$ and Z_t^2 has the χ^2-distribution with one degree of freedom. Thus $\{Y_t, t \in \mathbb{Z}\}$ is a white noise.

$\{X_t, t \in \mathbb{Z}\}$ is then AR(1) sequence with the coefficient $a_1 = -\rho$. The polynomial $1 - \rho z$ has the root $\frac{1}{\rho}$ which lies outside the unit circle (since $|\rho| < 1$ from the assignment). Thus $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process, it is centered and weakly stationary.

From the expansion of the geometric series $\frac{1}{1-\rho z} = \sum_{j=0}^{\infty}(\rho z)^j = \sum_{j=0}^{\infty}\rho^j z^j$ we get the coefficients of the causal representation $c_j = \rho^j, j = 0, 1, \dots$ Thus

$$X_t = \sum_{j=0}^{\infty}\rho^j Y_{t-j}, \quad t \in \mathbb{Z}.$$

Now we see that $\{X_t, t \in \mathbb{Z}\}$ is not strictly stationary since e.g. $\mathbb{E}X_0^3 = \frac{\sqrt{8}\rho^3}{1-\rho^6} \neq \frac{\sqrt{8}}{1-\rho^6} = \mathbb{E}X_1^3$. Moreover,

$$R_X(t) = \sum_{j=0}^{\infty}\rho^{j+t}\rho^j = \rho^t\frac{1}{1-\rho^2}, \quad t \in \mathbb{Z}, t \geq 0,$$

and $R_X(t) = R_X(-t)$ for $t \in \mathbb{Z}, t \leq 0$. $\diamondsuit$

Remark: Note that we could get the causal representation for the AR(1) model also directly by iteratively plugging the equation (20) into itself:

$$X_t = Y_t + \rho X_{t-1} = Y_t + \rho(Y_{t-1} + \rho X_{t-2}) = \cdots = \sum_{j=0}^{k} \rho^j Y_{t-j} + \rho^{k+1} X_{t-k-1} = \sum_{j=0}^{\infty} \rho^j Y_{t-j}.$$

The last equality is correct since the sequence $\rho^{k+1} X_{t-k-1}$ goes to 0 in L_2 and $X_t^k = \sum_{j=0}^{k} \rho^j Y_{t-j}$, $k \in \mathbb{N}_0$, is a Cauchy sequence in L_2 and thus it must have a limit which is exactly $\sum_{j=0}^{\infty} \rho^j Y_{t-j}$.

Exercise 6.3: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by

$$X_t - 0.7X_{t-1} + 0.1X_{t-2} = Y_t, \quad t \in \mathbb{Z}, \tag{21}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Express the random sequence $\{X_t, t \in \mathbb{Z}\}$ as a causal linear process and compute its autocovariance function and spectral density.

Solution: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an AR(2) process with coefficients $a_1 = -0.7, a_2 = 0.1$. The roots of the equation $1 - 0.7z + 0.1z^2 = 0$ are 2 and 5. They lie outside the unit circle and thus $\{X_t, t \in \mathbb{Z}\}$ is really a causal linear process with the representation $X_t = \sum_{k=0}^{\infty} c_k Y_{t-k}, t \in \mathbb{Z}$. We can find the coefficients c_k by plugging the causal representation into the model equation (21) and comparing the coefficients in front of every Y_{t-k}. We obtain the set of equations

$$\begin{aligned} c_0 &= 1, \\ c_1 - 0.7c_0 &= 0, \\ c_k - 0.7c_{k-1} + 0.1c_{k-2} &= 0, \quad k \geq 2. \end{aligned}$$

The characteristic polynomial of the difference equation in the third line is $\lambda^2 - 0.7\lambda + 0.1$ which has roots 0.5 and 0.2. Thus the solution is $c_k = a_1 0.5^k + a_2\, 0.2^k$ for some real values a_1, a_2. We use the first two lines to obtain the boundary conditions:

$$\begin{aligned} c_0 &= a_1 + a_2 = 1, \\ c_1 - 0.7c_0 &= a_1 0.5 + a_2\, 0.2 - 0.7(a_1 + a_2) = 0, \end{aligned}$$

and we obtain $a_1 = \frac{5}{3}, a_2 = -\frac{2}{3}$. The causal representaion is then

$$X_t = \sum_{k=0}^{\infty} \left(\frac{5}{3} 0.5^k - \frac{2}{3} 0.2^k \right) Y_{t-k}, \quad t \in \mathbb{Z}.$$

We may compute the autocovariance function $R_X(t)$ from the formula for the autocovariance function of the causal linear process. For $t \in \mathbb{Z}, t \geq 0$, we get

$$R_X(t) = \sigma^2 \sum_{k=0}^{\infty} c_{k+t} \overline{c_k} = \sigma^2 \sum_{k=0}^{\infty} \left(\frac{5}{3} 0.5^{(k+t)} - \frac{2}{3} 0.2^{(k+t)} \right) \left(\frac{5}{3} 0.5^k - \frac{2}{3} 0.2^k \right) = \sigma^2 \left(\frac{200}{81} 0.5^t - \frac{125}{162} 0.2^t \right) \tag{22}$$

and $R_X(t) = R_X(-t)$ for $t \in \mathbb{Z}, t \leq 0$.

However, we may also compute $R_X(t)$ without the causal representation by using the Yule-Walker equations. For this we need to assume that the white noise sequence $\{Y_t, t \in \mathbb{Z}\}$ is real-valued and the sequence $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process (we have shown that above). The Yule-Walker equations are obtained by multiplying the model equation (21) by X_{t-k} for $k \in \mathbb{N}_0$ and taking expectations:

$$\begin{aligned}
\mathbb{E}X_tX_t - 0.7\mathbb{E}X_{t-1}X_t + 0.1\mathbb{E}X_{t-2}X_t &= \mathbb{E}Y_tX_t, \\
\mathbb{E}X_tX_{t-1} - 0.7\mathbb{E}X_{t-1}X_{t-1} + 0.1\mathbb{E}X_{t-2}X_{t-1} &= \mathbb{E}Y_tX_{t-1}, \\
\mathbb{E}X_tX_{t-2} - 0.7\mathbb{E}X_{t-1}X_{t-2} + 0.1\mathbb{E}X_{t-2}X_{t-2} &= \mathbb{E}Y_tX_{t-2}, \\
\mathbb{E}X_tX_{t-k} - 0.7\mathbb{E}X_{t-1}X_{t-k} + 0.1\mathbb{E}X_{t-2}X_{t-k} &= \mathbb{E}Y_tX_{t-k}, \quad k \geq 2.
\end{aligned}$$

Since we know that $\{X_t, t \in \mathbb{Z}\}$ is centered and weakly stationary we may rewrite the left hand sides as follows:

$$\begin{aligned}
R_X(0) - 0.7R_X(-1) + 0.1R_X(-2) &= \mathbb{E}Y_tX_t, \\
R_X(1) - 0.7R_X(0) + 0.1R_X(-1) &= \mathbb{E}Y_tX_{t-1}, \\
R_X(2) - 0.7R_X(1) + 0.1R_X(0) &= \mathbb{E}Y_tX_{t-2}, \\
R_X(k) - 0.7R_X(k-1) + 0.1R_X(k-2) &= \mathbb{E}Y_tX_{t-k}, \quad k \geq 2.
\end{aligned}$$

On the right hand sides we know that $\mathbb{E}Y_tX_{t-k} = \mathbb{E}(Y_t \sum_{j=0}^{\infty} c_j Y_{t-k-j}) = 0$ for any $k \in \mathbb{N}$ since from the causality $X_{t-k} \in \mathcal{H}(Y_{t-k}, Y_{t-k-1}, \dots)$ and Y_t is uncorrelated with any random variable from $\mathcal{H}(Y_{t-k}, Y_{t-k-1}, \dots)$, and hence also with X_{t-k}. To find the value of $\mathbb{E}Y_tX_t$ we multiply the model equation by Y_t and take expectation:

$$\mathbb{E}X_tY_t - 0.7\mathbb{E}X_{t-1}Y_t + 0.1\mathbb{E}X_{t-2}Y_t = \mathbb{E}Y_t^2,$$

and we obtain

$$\mathbb{E}X_tY_t - 0.7 \cdot 0 + 0.1 \cdot 0 = \sigma^2.$$

Thus, $\mathbb{E}X_tY_t = \sigma^2$. Since $R_X(k)$ is a real-valued autocovariance function we may put $R_X(k) = R_X(-k), k \in \mathbb{Z}$, and the Yule-Walker equations finaly look like

$$\begin{aligned}
R_X(0) - 0.7R_X(1) + 0.1R_X(2) &= \sigma^2, \\
R_X(1) - 0.7R_X(0) + 0.1R_X(1) &= 0, \\
R_X(k) - 0.7R_X(k-1) + 0.1R_X(k-2) &= 0, \quad k \geq 2.
\end{aligned}$$

The characteristic polynomial of the difference equation in the third line is $\lambda^2 - 0.7\lambda + 0.1$ which has roots 0.5 and 0.2. Thus $R_X(k) = a_1 0.5^k + a_2\, 0.2^k, k \geq 0$. Using the first two equations to get the boundary conditions we would obtain the solution (22).

Note, however, that from the second equation we easily obtain $r_X(1) = \frac{7}{11}$. We also know that $r_X(0) = 1$ since that always holds for the autocorrelation function. By dividing the third equation by $R_X(0)$ we get the same difference equation

$$r_X(k) - 0.7r_X(k-1) + 0.1r_X(k-2) = 0, \quad k \geq 2,$$

also for the autocorrelation function. Thus $r_X(k) = b_1 0.5^k + b_2 0.2^k, k \geq 0$, and the boundary conditions are simpler than those for the autocovariance function. We easily compute $b_1 = \frac{16}{11}, b_2 = -\frac{5}{11}$. Since $r_X(2) = \frac{19}{55}$ we may express $R_X(0)$ from the first Yule-Walker equation as

$$R_X(0) = \frac{\sigma^2}{1 - 0.7\frac{7}{11} + 0.1\frac{19}{55}} = \frac{275}{162}\sigma^2.$$

Together, we get for the autocovariance function

$$R_X(k) = r_X(k)R_X(0) = \sigma^2 \left(\frac{200}{81} 0.5^{|k|} - \frac{125}{162} 0.2^{|k|} \right), \quad k \in \mathbb{Z}.$$

The spectral density $f_X(\lambda)$ is obtained from the formula for AR model:

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - 0.7\mathrm{e}^{-\mathrm{i}\lambda} + 0.1\mathrm{e}^{-\mathrm{i}2\lambda}|^2} = \frac{\sigma^2}{2\pi} \frac{50}{75 - 77\cos(\lambda) + 10\cos(2\lambda)}, \quad \lambda \in [-\pi, \pi].$$

$\diamondsuit$

Remark: Note that if we substitute $\lambda = \frac{1}{z}$ into $1 - 0.7z + 0.1z^2 = 0$ we get

$$\begin{aligned} 1 - \frac{0.7}{\lambda} + \frac{0.1}{\lambda} &= 0 \qquad / \cdot \lambda^2 \\ \lambda^2 - 0.7\lambda + 0.1 &= 0. \end{aligned}$$

Thus, if the roots $z_{1,2}$ of the first equation lie outside the unit circle, the roots $\lambda_{1,2}$ of the other equation are their reciprocal values and as such lie inside the unit circle. Since the other equation is the characteristic equation for the system of difference equations for the causal coefficients c_k and for the system of difference equations derived from the Yule-Walker equations for $R_X(t)$, it follows that $\{c_k\}_{k=0}^{\infty}$ and $\{R_X(t)\}_{t=0}^{\infty}$ are linear combinations of geometric series with quotients in absolute value smaller than 1. Thus, we see again that for the causal AR(2) model $R_X(t) \to 0, t \to \infty$, and $\sum_{k=0}^{\infty} |c_k| < \infty$.

Exercise 6.4: Solve the Yule-Walker equations and determine the autocovariance function of the random sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t - 0.4X_{t-1} + 0.04X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$.

Solution: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an AR(2) process with coefficients $a_1 = -0.4, a_2 = 0.04$. The equation $1 - 0.4z + 0.04z^2 = 0$ has a double root 5 which lies outside the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary causal linear process.

Analogously to the previous exercise we derive the Yule-Walker equations:

$$\begin{aligned} R_X(0) - 0.4R_X(1) + 0.04R_X(2) &= \sigma^2, \\ R_X(1) - 0.4R_X(0) + 0.04R_X(1) &= 0, \\ R_X(k) - 0.4R_X(k-1) + 0.04R_X(k-2) &= 0, \quad k \geq 2. \end{aligned}$$

From the second equation we get $r_X(1) = \frac{5}{13}$. The homogeneous difference equation for the autocorrelation function,

$$r_X(k) - 0.4r_X(k-1) + 0.04r_X(k-2) = 0, \quad k \geq 2,$$

has the characteristic polynomial $\lambda^2 - 0.4\lambda + 0.04$ with double root 0.2. Thus the autocorrelation function is $r_X(k) = (a + bk)0.2^k, k \geq 0$. From the boundary conditions $r_X(0) = 1$ and $r_X(1) = \frac{5}{13}$ we get $a = 1$, $b = \frac{12}{13}$ and hence

$$r_X(k) = \left(1 + \frac{12}{13}|k|\right) 0.2^{|k|}, \quad k \in \mathbb{Z}.$$

We get the variance $R_X(0)$ from the first Yule-Walker equation

$$R_X(0) = \frac{\sigma^2}{1 - 0.4r_X(1) + 0.04r_X(2)} = \frac{\sigma^2}{1 - \frac{2}{5}\frac{5}{13} + \frac{1}{25}\frac{37}{25 \cdot 13}} = \frac{8\,125}{6\,912}\sigma^2.$$

$\diamondsuit$

Exercise 6.5: Solve the Yule-Walker equations and determine the autocovariance function of the random sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t - 1.4X_{t-1} + 0.98X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$.

Solution: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an AR(2) process with coefficients $a_1 = -1.4, a_2 = 0.98$. The equation $1 - 1.4z + 0.98z^2 = 0$ has complex roots $\frac{5}{7}(1 \pm i)$ which lie outside the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary causal linear process.

Analogously to the previous exercise we derive the Yule-Walker equations:

$$\begin{aligned} R_X(0) - 1.4R_X(1) + 0.98R_X(2) &= \sigma^2, \\ R_X(1) - 1.4R_X(0) + 0.98R_X(1) &= 0, \\ R_X(k) - 1.4R_X(k-1) + 0.98R_X(k-2) &= 0, \quad k \geq 2. \end{aligned}$$

The homogeneous difference equation for the autocorrelation function,

$$r_X(k) - 1.4r_X(k-1) + 0.98r_X(k-2) = 0, \quad k \geq 2,$$

has the characteristic polynomial $\lambda^2 - 1.4\lambda + 0.98$ with roots $\lambda_{1,2} = \frac{7}{10}(1 \pm i) = \frac{7\sqrt{2}}{10}\mathrm{e}^{\pm i\frac{\pi}{4}}$. We could look for $r_X(k)$ in the form $r_X(k) = a\lambda_1^k + b\lambda_2^k$ but we know that the correlation function should be real-valued and thus it must be possible to write it as $r_X(k) = c_1\rho^k \cos(k\omega) + c_2\rho^k \sin(k\omega)$, where $\rho = |\lambda_1| = \frac{7\sqrt{2}}{10}$ and $\omega = \frac{\pi}{4}$.

From the second Yule-Walker equation we get $r_X(1) = \frac{70}{99}$ and we compute $c_1 = 1$ and $c_2 = \frac{1}{99}$. It follows that $r_X(2) = 49/4950$ and $R_X(0) = \frac{247500}{4901}\sigma^2$. ♢

Exercise 6.6: Let $\{X_t, t \in \mathbb{Z}\}$ be an ARMA(2,1) random sequence defined by

$$X_t - X_{t-1} + \frac{1}{4}X_{t-2} = Y_t + Y_{t-1}, \quad t \in \mathbb{Z}, \tag{23}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the MA(∞) representation of X_t and compute its autocovariance function and spectral density. Is the process invertible?

Solution: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an ARMA(2,1) process with coefficients $a_1 = -1$, $a_2 = \frac{1}{4}$, $b_1 = 1$. The polynomials $1 - z + \frac{1}{4}z^2$ and $1 + z$ have no common roots and the first one has a double root 2 which lies outside the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary causal linear process. The root of the second polynomial $z = -1$ does not lie outside of the unit circle and thus $\{X_t, t \in \mathbb{Z}\}$ is not invertible (but see the remark below this exercise).

We can find the coefficients c_k of the MA(∞) representation by plugging the causal representation into the model equation and comparing the coefficients in front of every Y_{t-k}. We obtain the following set of equations:

$$\begin{aligned} c_0 &= 1, \\ c_1 - c_0 &= 1, \\ c_k - c_{k-1} + \frac{1}{4}c_{k-2} &= 0, \quad k \geq 2. \end{aligned}$$

The characteristic polynomial of the difference equation in the third line is $\lambda^2 - \lambda + \frac{1}{4}$ which has a double root 0.5. Thus the solution is $c_k = (a + bk)0.5^k, k \in \mathbb{N}_0$. We use the first two lines to get the boundary conditions:

$$\begin{aligned} c_0 = a &= 1, \\ c_1 - c_0 = (a + b)/2 &= 0. \end{aligned}$$

We obtain $a = 1, b = 3$. The causal representaion is

$$X_t = \sum_{k=0}^{\infty}(1+3k)\frac{1}{2^k}Y_{t-k}, \quad k \in \mathbb{Z}.$$

We could compute the autocovariance function from the formula for the MA(∞) process, but since we have the double root 0.5 in the causal representation we prefer to use the Yule-Walker equations. By multiplying the model equation by $X_{t-k}, k \geq 0$ and taking expectations we obtain

$$\begin{aligned}
R_X(0) - R_X(1) + \frac{1}{4}R_X(2) &= \mathbb{E}X_tY_t + \mathbb{E}X_tY_{t-1},\\
R_X(1) - R_X(0) + \frac{1}{4}R_X(1) &= \mathbb{E}X_{t-1}Y_t + \mathbb{E}X_{t-1}Y_{t-1},\\
R_X(k) - R_X(k-1) + \frac{1}{4}R_X(k-2) &= \mathbb{E}X_{t-k}Y_t + \mathbb{E}X_{t-k}Y_{t-1}, \quad k \geq 2.
\end{aligned}$$

From causality we get $\mathbb{E}X_{t-k}Y_t = 0$ for $k > 0$ and $\mathbb{E}X_{t-k}Y_{t-1} = 0$ for $k > 1$ using the same arguments as in Exercise 6.3. Thus it remains to find the values $\mathbb{E}X_tY_t$ and $\mathbb{E}X_tY_{t-1}$ to be able to determine the values of all the right-hand sides. Let us multiply the model equation with Y_t and Y_{t-1} and take expectations. We get

$$\begin{aligned}
\mathbb{E}X_tY_t - \mathbb{E}X_{t-1}Y_t + \frac{1}{4}\mathbb{E}X_{t-2}Y_t &= \mathbb{E}Y_t^2 + \mathbb{E}Y_{t-1}Y_t,\\
\mathbb{E}X_tY_{t-1} - \mathbb{E}X_{t-1}Y_{t-1} + \frac{1}{4}\mathbb{E}X_{t-2}Y_{t-1} &= \mathbb{E}Y_tY_{t-1} + \mathbb{E}Y_{t-1}^2,
\end{aligned}$$

and by plugging-in the 0's and σ^2 we get

$$\begin{aligned}
\mathbb{E}X_tY_t &= \mathbb{E}Y_t^2 = \sigma^2,\\
\mathbb{E}X_tY_{t-1} - \mathbb{E}X_{t-1}Y_{t-1} &= \mathbb{E}Y_{t-1}^2 = \sigma^2.
\end{aligned}$$

Thus $\mathbb{E}X_tY_t = \sigma^2$ and $\mathbb{E}X_tY_{t-1} = 2\sigma^2$. The Yule-Walker equations are thus the following:

$$\begin{aligned}
R_X(0) - R_X(1) + \frac{1}{4}R_X(2) &= 3\sigma^3,\\
R_X(1) - R_X(0) + \frac{1}{4}R_X(1) &= \sigma^2,\\
R_X(k) - R_X(k-1) + \frac{1}{4}R_X(k-2) &= 0, \quad k \geq 2.
\end{aligned}$$

The homogeneous difference equation from the third line has the characteristic polynomial $\lambda^2 - \lambda + \frac{1}{4}$ with the double root $\frac{1}{2}$. Thus $R_X(k) = (a+bk)\frac{1}{2^k}, k \in \mathbb{N}_0$, and the first two equations give

$$\begin{aligned}
a - \frac{a+b}{2} + \frac{1}{4}\frac{a+2b}{4} &= 3\sigma^2,\\
\frac{5}{4}\frac{a+b}{2} - a &= \sigma^2.
\end{aligned}$$

The solution is $a = \frac{32}{3}\sigma^2$, $b = 8\sigma^2$. The autocovariance function is then

$$R_X(k) = \sigma^2\left(\frac{32}{3} + 8|k|\right)\left(\frac{1}{2}\right)^{|k|}, \quad k \in \mathbb{Z}.$$

We compute the spectral density of $\{X_t, t \in \mathbb{Z}\}$ using the formula for the spectral density of the ARMA model:

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|1 + e^{-i\lambda}|^2}{\left|1 - e^{-i\lambda} + \frac{1}{4}e^{-i2\lambda}\right|^2} = \frac{\sigma^2}{2\pi} \frac{32(1 + \cos\lambda)}{(5 - 4\cos\lambda)^2}, \quad \lambda \in [-\pi, \pi].$$

$\Diamond$

Remark: Note that it would not be effective to use the trick with the autocorrelation function for the ARMA model since we have more than one non-zero right-hand side in the Yule-Walker equations and we cannot switch from the autocovariance function to the autocorrelation function without knowing the variance $R_X(0)$.

Remark: Concerning the (non)invertibility of the ARMA model: Theorem 6.1 as it is formulated gives sufficient conditions for invertibility. However, these conditions are also necessary conditions for invertibility. We show this for the special case of the ARMA model from Exercise 6.6 but the same argument is valid in general.

Suppose that $\{X_t, t \in \mathbb{Z}\}$ is invertible. Then there must exist inverted representation

$$Y_t = \sum_{k=0}^{\infty} d_k X_{t-k}, \qquad t \in \mathbb{Z},$$

with $\{d_k\}_{k=0}^{\infty}$ satisfying $\sum_{k=0}^{\infty} |d_k| < \infty$. But then we can plug this representation into the model equation (23) and compare the coefficients in front of each X_{t-k}. We get a set of equations for the sequence $\{d_k\}_{k=0}^{\infty}$:

$$\begin{aligned} d_0 &= 1, \\ d_1 + d_0 &= -1, \\ d_2 + d_1 &= \frac{1}{4}, \\ d_k + d_{k-1} &= 0, \qquad k \geq 3. \end{aligned}$$

We solve it to see that $d_0 = 1$, $d_1 = -2$, $d_2 = 2.25$ and the homogeneous difference equation on the fourth line has the general solution $d_k = a(-1)^k, k \geq 2$. Using the boundary condition $d_2 = 2.25$ we get the unique solution

$$d_0 = 1, \quad d_1 = -2, \quad d_k = 2.25(-1)^k, k \geq 2.$$

But these coefficients are not summable ($\sum_{k=0}^{\infty} |d_k| = \infty$) and our computed "inverted representation" $Y_t = \sum_{k=0}^{\infty} d_k X_{t-k}, t \in \mathbb{Z}$, does not converge in L_2 (it is not Cauchy, it cannot converge). Thus we have obtained a contradiction with our assumption of existence of inverted representation and we have proved that the ARMA model in question is not invertible.

Exercise 6.7: Consider the ARMA(2,1) model defined by

$$X_t - 0.1X_{t-1} - 0.12X_{t-2} = Y_t - 0.7Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Decide whether it is causal. Decide whether it is invertible. If it is, determine the coefficients of the AR(∞) representation.

Solution: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an ARMA(2,1) model with coefficients $a_1 = -0.1, a_2 = -0.12, b_1 = -0.7$. The polynomials $1 - 0.1z - 0.12z^2$ and $1 - 0.7z$ have no common roots and the roots of the first one $\frac{5}{2}, -\frac{10}{3}$ lie outside of the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary causal linear process. The root of the second polynomial $z = \frac{10}{7}$ lies outside of the unit circle as well and thus $\{X_t, t \in \mathbb{Z}\}$ is invertible.

We can find the coefficients d_k of the AR(∞) representation by plugging the inverted representation $Y_t = \sum_{k=0}^{\infty} d_k X_{t-k}$ into the model equation and comparing the coefficients in front of every X_{t-k}. We obtain the following set of equations:

$$\begin{aligned} d_0 &= 1, \\ d_1 - 0.7d_0 &= -0.1, \\ d_2 - 0.7d_1 &= -0.12, \\ d_k - 0.7d_{k-1} &= 0, \qquad\qquad k \geq 3. \end{aligned}$$

From the first three equations with nonzero right-hand side we get $d_0 = 1$, $d_1 = 0.6$, $d_2 = 0.3$. The characteristic polynomial of the homogeneous difference equation in the fourth line is $\lambda - 0.7$. It has the root 0.7 and thus the general solution is $d_k = a\,0.7^k, k \geq 2$. Note that the homogeneous difference equation should be satisfied only from $k = 3$ on, thus only $d_k, k \geq 2$, are obliged to be in the shape of the general solution. The boundary condition is thus $d_2 = 0.3 = a\,0.7^k$ and $a = 0.3/0.7^2$. It follows that $d_k = 0.3(0.7)^{k-2}, k \geq 2$. ♢

Further exercises

Exercise 6.8: Let

$$Z_t = Y_t + W_t \qquad \text{and} \qquad Y_t + \beta Y_{t-1} = U_t, \quad t \in \mathbb{Z},$$

where $|\beta| < 1$ and $\{U_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$ are independent centered sequences of independent random variables with $\operatorname{var} U_t = \sigma^2$, $\operatorname{var} W_t = \tau^2$ for all $t \in \mathbb{Z}$. Determine the spectral density of the random sequence $\{Z_t, t \in \mathbb{Z}\}$.

Exercise 6.9: Determine the autocovariance function and the spectral density of the sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t = Y_t + c_1 Y_{t-1} + c_2 Y_{t-2}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and c_1, c_2 are the coefficients from the equation $z^2 + c_1 z + c_2 = 0$ with roots $z_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $z_2 = \rho \mathrm{e}^{-\mathrm{i}\theta}$, where $\rho > 0$, $\theta \in (0, \pi)$.

Exercise 6.10: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by

$$X_t - \frac{2}{15} X_{t-1} - \frac{1}{15} X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Express the random sequence $\{X_t, t \in \mathbb{Z}\}$ as a causal linear process and compute its autocovariance function and spectral density.

Exercise 6.11: Let $\{X_t, t \in \mathbb{Z}\}$ be an AR(2) random sequence defined by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} = Y_t, \quad t \in \mathbb{Z}.$$

Determine for which values of a_1 and a_2 is $\{X_t, t \in \mathbb{Z}\}$ a causal linear process. Express the variance of $\{X_t, t \in \mathbb{Z}\}$ by means of a_1 and a_2 and the white noise variance σ^2.

Exercise 6.12: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by the equation

$$X_t - \frac{5}{4} X_{t-1} + \frac{1}{2} X_{t-2} - \frac{1}{16} X_{t-3} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$.

Exercise 6.13: Consider the ARMA(1,1) model defined by

$$X_t + 0.7X_{t-1} = Y_t + 0.3Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the $\mathrm{AR}(\infty)$ representation.

Exercise 6.14: Consider the ARMA(1,1) model defined by

$$X_t + 0.6X_{t-1} = Y_t + 1.2Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$.

Exercise 6.15: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by the equation

$$X_t - (a+b)X_{t-1} + abX_{t-2} = Y_t - aY_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and $a \neq 0, b \neq 0$ are real constants. For which values of a, b is the process causal? For which values of a, b is the process invertible? Derive the causal ($\mathrm{MA}(\infty)$) and inverted ($\mathrm{AR}(\infty)$) representation. Compute the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$.

Exercise 6.16: Consider the ARMA(2,1) model defined by

$$X_t - 0.5X_{t-1} + 0.04X_{t-2} = Y_t + 0.25Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the $\mathrm{AR}(\infty)$ representation.

6.2 Linear filters

Definition 6.2: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence. Let $\{c_j, j \in \mathbb{Z}\}$ be a sequence of (complex-valued) numbers such that $\sum_{j=-\infty}^{\infty} |c_j| < \infty$.
We say that a random sequence $\{X_t, t \in \mathbb{Z}\}$ is obtained by filtration of the sequence $\{Y_t, t \in \mathbb{Z}\}$ if

$$X_t = \sum_{j=-\infty}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}.$$

The sequence $\{c_j, j \in \mathbb{Z}\}$ is called *time-invariant linear filter*.
Provided that $c_j = 0$ for all $j < 0$, we say that the filter $\{c_j, j \in \mathbb{Z}\}$ is *causal*.

Theorem 6.2: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with an autocovariance function R_Y and spectral density f_Y and let $\{c_k, k \in \mathbb{Z}\}$ be a linear filter such that $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Then $\{X_t, t \in \mathbb{Z}\}$, where $X_t = \sum_{k=-\infty}^{\infty} c_k Y_{t-k}$, is a centered weakly stationary sequence with the autocovariance function

$$R_X(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j \overline{c_k} R_Y(t - j + k), \quad t \in \mathbb{Z},$$

and spectral density

$$f_X(\lambda) = |\Psi(\lambda)|^2 f_Y(\lambda), \quad \lambda \in [-\pi, \pi],$$

where

$$\Psi(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\lambda}, \quad \lambda \in [-\pi, \pi],$$

is called *the transfer function of the filter.*

Exercise 6.17: Let $\{Z_t, t \in \mathbb{Z}\}$ be a white noise WN$(0,1)$ and let $\{X_t, t \in \mathbb{Z}\}$ be a causal linear process defined by

$$X_t - \frac{1}{3}X_{t-1} = Z_t, \quad t \in \mathbb{Z}. \tag{24}$$

Let $\{Y_t, t \in \mathbb{Z}\}$ be a process obtained by the filtration $Y_t = X_{t+2} - \frac{1}{2}X_{t+1} + \frac{1}{4}X_t, t \in \mathbb{Z}$. Derive the transfer function of the filter and compute the spectral density of $\{Y_t, t \in \mathbb{Z}\}$.

Solution: $\{X_t, t \in \mathbb{Z}\}$ is a causal AR(1) model since the root of the polynomial $1 - \frac{1}{3}z$ is 3 and it lies outside the unit circle. We compute the spectral density according to the formula for the AR model

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\frac{1}{|1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\lambda}|^2} = \frac{\sigma^2}{2\pi}\frac{9}{10 - 6\cos\lambda}, \quad \lambda \in [-\pi, \pi].$$

The coefficients of the filter are $c_0 = \frac{1}{4}, c_{-1} = -\frac{1}{2}, c_{-2} = 1$. Then according to the Theorem 6.2 the transfer function of the filter is

$$\Psi_Y(\lambda) = \frac{1}{4} - \frac{1}{2}\mathrm{e}^{\mathrm{i}\lambda} + \mathrm{e}^{\mathrm{i}2\lambda}, \quad \lambda \in [-\pi, \pi],$$

and the spectral density of $\{Y_t, t \in \mathbb{Z}\}$ is

$$\begin{aligned} f_Y(\lambda) = |\Psi_Y(\lambda)|^2 f_X(\lambda) &= \frac{\sigma^2}{2\pi}\frac{\left|\frac{1}{4} - \frac{1}{2}\mathrm{e}^{\mathrm{i}\lambda} + \mathrm{e}^{\mathrm{i}2\lambda}\right|^2}{|1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\lambda}|^2} = \frac{\sigma^2}{2\pi}\frac{21 - 20\cos\lambda + 8\cos 2\lambda}{16}\frac{9}{10 - 6\cos\lambda} \\ &= \frac{\sigma^2}{2\pi}\frac{9(21 - 20\cos\lambda + 8\cos 2\lambda)}{32(5 - 3\cos\lambda)}, \qquad \lambda \in [-\pi, \pi]. \end{aligned}$$

$\Diamond$

Remark: We could ask what does the process $\{Y_t, t \in \mathbb{Z}\}$ look like. Obviously it is centered and weakly stationary. If we have a look on the spectral density of $\{Y_t, t \in \mathbb{Z}\}$ we see that

$$\begin{aligned} f_Y(\lambda) &= \frac{\sigma^2}{2\pi}\frac{\left|\frac{1}{4} - \frac{1}{2}\mathrm{e}^{\mathrm{i}\lambda} + \mathrm{e}^{\mathrm{i}2\lambda}\right|^2}{|1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\lambda}|^2} = \frac{\sigma^2}{2\pi}\frac{\left|\mathrm{e}^{\mathrm{i}2\lambda}\left(\frac{1}{4}\mathrm{e}^{-\mathrm{i}2\lambda} - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\lambda} + 1\right)\right|^2}{|1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\lambda}|^2} \\ &= \frac{\sigma^2}{2\pi}\frac{\left|1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\lambda} + \frac{1}{4}\mathrm{e}^{-\mathrm{i}2\lambda}\right|^2}{|1 - \frac{1}{3}\mathrm{e}^{-\mathrm{i}\lambda}|^2}, \qquad \lambda \in [-\pi, \pi], \end{aligned}$$

which is the spectral density of an ARMA(1,2) model given by the equation

$$V_t - \frac{1}{3}V_{t-1} = W_t - \frac{1}{2}W_{t-1} + \frac{1}{4}W_{t-2}, \quad t \in \mathbb{Z},$$

for some white noise $\{W_t, t \in \mathbb{Z}\}$ with variance σ^2. Thus $\{Y_t, t \in \mathbb{Z}\}$ is equivalent to an ARMA(1,2) model. This can be verified by expressing $Y_t - \frac{1}{3}Y_{t-1}$ by plugging (24) into the defining formula for $Y_t, t \in \mathbb{Z}$, obtaining $Y_t - \frac{1}{3}Y_{t-1} = Z_{t+2} - \frac{1}{2}Z_{t+1} + \frac{1}{4}Z_t, t \in \mathbb{Z}$.

Exercise 6.18: Let $\{Y_t, t \in \mathbb{Z}\}$ be a white noise $\mathrm{WN}(0, \sigma^2)$. Let it be transformed by a linear filter to $\{X_t, t \in \mathbb{Z}\}$ so that

$$X_t - 2X_{t-1} = Y_t, \quad t \in \mathbb{Z}, \tag{25}$$

holds. Determine the coefficients of the linear filter, the transfer function of the filter and compute the autocovariance function and the spectral density of $\{X_t, t \in \mathbb{Z}\}$.

Solution: The equation (25) corresponds to an AR(1) model. However, the root of $1 - 2z$ is $\frac{1}{2}$ and it lies inside the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is not a well-defined causal autoregressive model. Nevertheless, it is still possible to express $\{X_t, t \in \mathbb{Z}\}$ as a filtered white noise. We try to expand X_t into the future using (25):

$$-X_t = \frac{Y_{t+1} - X_{t+1}}{2} = \frac{Y_{t+1}}{2} + \frac{1}{2}\frac{Y_{t+2} - X_{t+2}}{2} = \cdots = \sum_{k=1}^{m} \frac{Y_{t+k}}{2^k} - \frac{X_{t+m}}{2^{t+m}} = \cdots = \sum_{k=1}^{\infty} \frac{Y_{t+k}}{2^k}.$$

Thus

$$X_t = \sum_{k=1}^{\infty} -\frac{1}{2^k} Y_{t+k} = \sum_{l=-\infty}^{-1} -2^l Y_{t-l}, \quad t \in \mathbb{Z},$$

is filtered from the white noise by a non-causal filter with coefficients $c_l = -2^l$ for $l < 0$ and $c_l = 0$ for $l \geq 0$. This filter satisfies the condition $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, thus according to the Theorem 6.2 $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary sequence with the transfer function

$$\Psi_X(\lambda) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{-\mathrm{i}k\lambda} = \sum_{k=-\infty}^{-1} -2^k \mathrm{e}^{-\mathrm{i}k\lambda} = -\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \mathrm{e}^{\mathrm{i}k\lambda} = -\frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\lambda}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\lambda}} = \frac{1}{1 - 2\mathrm{e}^{-\mathrm{i}\lambda}}, \lambda \in [-\pi, \pi],$$

and the spectral density

$$f_X(\lambda) = |\Psi_X(\lambda)|^2 f_Y(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - 2\mathrm{e}^{-\mathrm{i}\lambda}|^2} = \frac{\sigma^2}{2\pi} \frac{1}{5 - 4\cos\lambda}, \quad \lambda \in [-\pi, \pi],$$

since $\{Y_t, t \in \mathbb{Z}\}$ is a white noise.

We compute the autocovariance function $R_X(t)$ according to the Theorem 6.2:

$$R_X(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j c_k R_Y(t - j + k) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j c_k \sigma^2 \mathbf{1}(t - j + k = 0) = \sum_{j=-\infty}^{-1} c_j c_{j-t} \sigma^2,$$

where we used the fact that the filter is real-valued and that $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. For $t \geq 0$ we get

$$R_X(t) = \sum_{j=-\infty}^{-1} 2^{2j-t} \sigma^2 = \sigma^2 2^{-t} \sum_{l=1}^{\infty} 4^{-l} = \frac{\sigma^2}{3} \left(\frac{1}{2}\right)^t.$$

Thus $R_X(t) = \frac{\sigma^2}{3} 2^{-|t|}, t \in \mathbb{Z}$. ◊

Remark: In the previous exercise we found a centered weakly stationary random sequence that satisfies equation (25). Analogously we would be able to find a centered weakly stationary sequence satisfying the equation

$$X_t - \rho X_{t-1} = Y_t, \quad t \in \mathbb{Z}$$

for any $\rho \in \mathbb{R}, |\rho| > 1$. For $\rho \in \mathbb{R}, |\rho| < 1$, we found a causal linear process satisfying the same equation in Exercise 6.2.

To complete the picture, we could ask about the case $\rho = \pm 1$. In this case we are not able to find a centered weakly stationary solution of equation (20). But if we change the assignment to

$$X_t - X_{t-1} = Y_t, \quad t \in \mathbb{N},$$

where $\{Y_t, t \in \mathbb{N}_0\}$ is an i.i.d. white noise, we see that it is satisfied by the random walk

$$X_t = \sum_{k=0}^{t} Y_k, \quad t \in \mathbb{N}, \quad X_0 = Y_0,$$

with the same incremental and starting distribution, equal to the distribution of Y_0.

Further exercises

Exercise 6.19: Let $\{Z_t, t \in \mathbb{Z}\}$ be a white noise WN$(0,1)$ and let $\{X_t, t \in \mathbb{Z}\}$ be a causal linear process defined by

$$X_t - 0.99X_{t-3} = Z_t, \quad t \in \mathbb{Z}.$$

Let $\{Y_t, t \in \mathbb{Z}\}$ be a process obtained by the filtration $Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$. Derive the transfer function of the filter and compute the spectral density of $\{Y_t, t \in \mathbb{Z}\}$.

Exercise 6.20: Consider a random sequence given by the formula

$$X_t - \frac{1}{3}X_{t-1} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a centered real-valued white noise with positive finite variance σ^2. Let $\{Z_t, t \in \mathbb{Z}\}$ be a process obtained by the filtration

$$Z_t = X_t - \frac{1}{2}X_{t-1}, \quad t \in \mathbb{Z}.$$

Derive the transfer function of the filter and compute the spectral density of $\{Z_t, t \in \mathbb{Z}\}$. Compute the autocovariance function of $\{Z_t, t \in \mathbb{Z}\}$.

7 Prediction

7.1 Prediction based on infinite history

Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence. $H_{-\infty}^n = \mathcal{H}\{\dots, X_{n-1}, X_n\}$ denotes the Hilbert space generated by the random variables $\{X_t, t \le n\}$, i.e. by the history of the process $\{X_t, t \in \mathbb{Z}\}$ up to time n.

Prediction $\widehat{X}_{n+h}(n)$ of X_{n+h} (where $h \in \mathbb{N}$) based on the infinite history $X_n, X_{n-1}, \dots$ is the orthogonal projection of the random variable X_{n+h} into the space $H_{-\infty}^n$. We denote $\widehat{X}_{n+h}(n) = P_{H_{-\infty}^n}(X_{n+h})$ and use a shorter notation $\widehat{X}_{n+1}$ for $\widehat{X}_{n+1}(n)$.

Prediction error (*residual variance*) is defined as $\mathbb{E}|X_{n+h} - \widehat{X}_{n+h}(n)|^2$.

Consider a causal and invertible ARMA(m, n) sequence. Invertibility implies that

$$X_{n+1} = -\sum_{j=1}^{\infty} d_j X_{n+1-j} + Y_{n+1}, \quad n \in \mathbb{Z}.$$

Causality implies that $X_n \in \mathcal{H}\{\dots, Y_{n-1}, Y_n\} \perp Y_{n+1}$. Thus the random variable Y_{n+1} is uncorrelated with X_n, and similarly with any other $X_{n-k}, k \in \mathbb{N}$. From linearity and continuity of the inner product we get $Y_{n+1} \perp H_{-\infty}^n$. Furthermore, $-\sum_{j=1}^{\infty} d_j X_{n+1-j} \in H_{-\infty}^n$. Thus the best linear prediction of X_{n+1} based on the whole history X_n, X_{n-1}, ... is the projection

$$\widehat{X}_{n+1} = -\sum_{j=1}^{\infty} d_j X_{n+1-j}, \tag{26}$$

and the prediction error is

$$\mathbb{E}|X_{n+1} - \widehat{X}_{n+1}|^2 = \mathbb{E}|Y_{n+1}|^2 = \sigma^2.$$

Exercise 7.1: Consider the ARMA(2,1) model defined by

$$X_t - 0.1X_{t-1} - 0.12X_{t-2} = Y_t - 0.7Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. Assume the whole history up to time n is known. Find the prediction of X_{n+1} and X_{n+2} based on X_n, X_{n-1}, ... and their prediction error.

Solution: Recall that in Exercise 6.7 we found that this model is causal and invertible with the inverted representation $Y_t = \sum_{k=0}^{\infty} d_k X_{t-k}$ and $d_0 = 1$, $d_1 = 0.6$, $d_k = 0.3(0.7)^{k-2}, k \ge 2$. Thus, according to the observation above (equation (26)),

$$\widehat{X}_{n+1} = -0.6X_n - \sum_{j=2}^{\infty} 0.3(0.7)^{j-2} X_{n+1-j},$$

and the prediction error is

$$\mathbb{E}|X_{n+1} - \widehat{X}_{n+1}|^2 = \mathbb{E}|Y_{n+1}|^2 = \sigma^2.$$

For the prediction of X_{n+2} based on X_n, X_{n-1}, ..., i.e. prediction two steps ahead, we express from the model equation

$$X_{n+2} = 0.1X_{n+1} + 0.12X_n + Y_{n+2} - 0.7Y_{n+1}.$$

Further, $Y_{n+2}, Y_{n+1} \perp H_{-\infty}^{n}$ from the causality of the model and $X_n \in H_{-\infty}^{n}$ by definition of $H_{-\infty}^{n}$. Thus from the linearity of projection we get

$$\begin{aligned}\widehat{X}_{n+2}(n) &= 0.1\widehat{X}_{n+1} + 0.12X_n + 0 - 0 = -0.06X_n - \sum_{j=2}^{\infty} 0.03(0.7)^{k-2}X_{n+1-j} + 0.12X_n \\ &= 0.06X_n - \sum_{j=2}^{\infty} 0.03(0.7)^{k-2}X_{n+1-j}.\end{aligned}$$

The prediction error is computed by finding all the terms which were projected to 0:

$$\mathbb{E}|X_{n+2} - \widehat{X}_{n+2}(n)|^2 = \mathbb{E}|Y_{n+2} - 0.7Y_{n+1} + 0.1Y_{n+1}|^2 = \mathbb{E}|Y_{n+2} - 0.6Y_{n+1}|^2 = 1.36\sigma^2.$$

Alternatively, we can obtain the same result by recalling $X_{n+1} - \widehat{X}_{n+1} = Y_{n+1}$ and computing

$$\mathbb{E}|X_{n+2} - \widehat{X}_{n+2}(n)|^2 = \mathbb{E}|0.1X_{n+1} + 0.12X_n + Y_{n+2} - 0.7Y_{n+1} - (0.1\widehat{X}_{n+1} + 0.12X_n)|^2.$$

$\Diamond$

Remark: If we do not remember the trick with equation (26) we could also find the prediction directly from the model equation. We write

$$X_{n+1} = 0.1X_n + 0.12X_{n-1} + Y_{n+1} - 0.7Y_n, \quad t \in \mathbb{Z}.$$

By the same argumentation as above we get

$$\widehat{X}_{n+1} = 0.1\widehat{X}_n + 0.12\widehat{X}_{n-1} - 0.7\widehat{Y}_n, \quad t \in \mathbb{Z}. \tag{27}$$

We know from invertibility of the model that $Y_n \in H_{-\infty}^{n}$ but we must express it using $X_n, X_{n-1}, \ldots$ since these are the only observed random variables. The white noise $\{Y_t, t \in \mathbb{Z}\}$ is not observed. However, if we do that – express Y_n by the inverted representation and plug it into (27) – we obtain the equation (26).

Exercise 7.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t - \frac{1}{2}X_{t-1} + \frac{1}{16}X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Suppose we know the history of the process up to time $t = 100$. Compute the predictions $\widehat{X}_{101}(100), \widehat{X}_{102}(100)$ and $\widehat{X}_{103}(100)$ and the respective prediction errors for $\widehat{X}_{101}(100)$ and $\widehat{X}_{102}(100)$.

Solution: First we check the causality of the model. The random sequence $\{X_t, t \in \mathbb{Z}\}$ is an AR(2) process with coefficients $a_1 = -\frac{1}{2}, a_2 = \frac{1}{16}$. The equation $1 - \frac{1}{2}z + \frac{1}{16}z^2 = 0$ has a double root 4 which lies outside the unit circle. Thus $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary causal linear process. Trivially the model is also invertible. We could use the general formula (26) for the ARMA model but observe that the inverted representation is actually given by the model equation. Thus we can write directly from the model equation:

$$X_{n+1} = \frac{1}{2}X_n - \frac{1}{16}X_{n-1} + Y_{n+1}.$$

Since $Y_{n+1} \perp H_{-\infty}^{n}$ from the causality of the model (and linearity and continuity of the inner product) and $X_n, X_{n-1} \in H_{-\infty}^{n}$ by definition of $H_{-\infty}^{n}$ we get

$$\widehat{X}_{n+1} = \frac{1}{2}\widehat{X}_n - \frac{1}{16}\widehat{X}_{n-1} + \widehat{Y}_{n+1} = \frac{1}{2}X_n - \frac{1}{16}X_{n-1},$$

and the prediction error is

$$\mathbb{E}|X_{n+1} - \widehat{X}_{n+1}|^2 = \mathbb{E}|Y_{n+1}|^2 = \sigma^2.$$

Further,

$$X_{n+2} = \frac{1}{2}X_{n+1} - \frac{1}{16}X_n + Y_{n+2},$$

thus

$$\widehat{X}_{n+2}(n) = \frac{1}{2}\widehat{X}_{n+1} - \frac{1}{16}X_n = \frac{1}{2}\left(\frac{1}{2}X_n - \frac{1}{16}X_{n-1}\right) - \frac{1}{16}X_n = \frac{3}{16}X_n - \frac{1}{32}X_{n-1},$$

and

$$\mathbb{E}|X_{n+2} - \widehat{X}_{n+2}(n)|^2 = \mathbb{E}\left|Y_{n+2} + \frac{1}{2}Y_{n+1}\right|^2 = \frac{5}{4}\sigma^2.$$

We continue analogically:

$$\begin{aligned}\widehat{X}_{n+3}(n) &= \frac{1}{2}\widehat{X}_{n+2}(n) - \frac{1}{16}\widehat{X}_{n+1} = \frac{1}{2}\left(\frac{3}{16}X_n - \frac{1}{32}X_{n-1}\right) - \frac{1}{16}\left(\frac{1}{2}X_n - \frac{1}{16}X_{n-1}\right)\\ &= \frac{1}{16}X_n - \frac{3}{256}X_{n-1}.\end{aligned}$$

Thus the answer to our assignment is

$$\begin{aligned}\widehat{X}_{101}(100) &= \frac{1}{2}X_{100} - \frac{1}{16}X_{99}, \qquad & \mathbb{E}|X_{101} - \widehat{X}_{101}(100)|^2 = \sigma^2,\\ \widehat{X}_{102}(100) &= \frac{3}{16}X_{100} - \frac{1}{32}X_{99}, \qquad & \mathbb{E}|X_{102} - \widehat{X}_{102}(100)|^2 = \frac{5}{4}\sigma^2,\\ \widehat{X}_{103}(100) &= \frac{1}{16}X_{100} - \frac{3}{256}X_{99}.\end{aligned}$$

$\Diamond$

Exercise 7.3: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is WN$(0, \sigma^2)$. Determine $\widehat{X}_{n+h}(n)$ for $h \in \mathbb{N}$ and compute the prediction error.

Solution: Any MA model is trivially causal and this particular MA(1) model is also invertible, since $1 - 0.5z$ has the root 2 which lies outside the unit circle.

The inverted representation is $Y_t = \sum_{j=0}^{\infty} \frac{1}{2^j} X_{t-j}$. Using formula (26) we get

$$\widehat{X}_{n+1} = -\sum_{j=1}^{\infty} \frac{1}{2^j} X_{n+1-j},$$

and the prediction error is

$$\mathbb{E}|X_{n+1} - \widehat{X}_{n+1}|^2 = \mathbb{E}|Y_{n+1}|^2 = \sigma^2.$$

We would get the same result from the model equation, the fact that $Y_{n+1} \perp H_{-\infty}^n$ (which follows from the model equation and the linearity and continuity of the inner product) and from the inverted representation:

$$\widehat{X}_{n+1} = \widehat{Y}_{n+1} - 0.5\widehat{Y}_n = 0 - 0.5Y_n = -0.5\sum_{j=0}^{\infty} \frac{1}{2^j} X_{n-j} = -\sum_{j=1}^{\infty} \frac{1}{2^j} X_{n+1-j}.$$

For $\widehat{X}_{n+k}(n), k \geq 2$, we get from the model equation and the fact $Y_{n+k}, Y_{n+k-1}, \dots Y_{n+1} \perp H^n_{-\infty}$

$$\widehat{X}_{n+k}(n) = \widehat{Y}_{n+k}(n) - 0.5\widehat{Y}_{n+k-1}(n) = 0,$$

and

$$\mathbb{E}|X_{n+k} - \widehat{X}_{n+k}(n)|^2 = \mathbb{E}|X_{n+k}|^2 = \mathbb{E}|Y_{n+k} - 0.5Y_{n+k-1}|^2 = \frac{5}{4}\sigma^2.$$

Note that for these predictions $\widehat{X}_{n+k}, k \geq 2$, we do not need invertibility of the model in question. $\diamondsuit$

Further exercises

Exercise 7.4: Consider the AR(2) model defined by

$$X_t - 0.4X_{t-1} + 0.04X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Find the prediction of X_{n+1}, X_{n+2} and X_{n+3} based on the history X_n, X_{n-1}, ... Compute the prediction errors.

Exercise 7.5: Consider the ARMA(1,1) model defined by

$$X_t + 0.7X_{t-1} = Y_t + 0.3Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the AR(∞) representation. Find the prediction of X_{n+1} and X_{n+2} based on the history X_n, X_{n-1}, ... Compute the prediction errors.

Exercise 7.6: Consider the ARMA(2,1) model defined by

$$X_t - 0.5X_{t-1} + 0.04X_{t-2} = Y_t + 0.25Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Find the prediction of X_{n+1} and X_{n+2} based on the history X_n, X_{n-1}, ... Compute the prediction errors.

7.2 Prediction based on finite history

Concerning the prediction based on the finite history we denote $H_1^n = \mathcal{H}\{X_1, \dots, X_n\}$ the Hilbert space generated by the random variables $X_1, \dots, X_n$.

The best linear prediction of X_{n+h} (for $h \in \mathbb{N}$) is the orthogonal projection into the space H_1^n, i.e. $\widehat{X}_{n+h}(n) = \sum_{j=1}^n c_j X_j \in H_1^n$ such that $X_{n+h} - \widehat{X}_{n+h}(n) \perp H_1^n$. We use a shorter notation $\widehat{X}_{n+1}$ for $\widehat{X}_{n+1}(n)$.

Prediction error (*residual variance*) is again defined as $\mathbb{E}|X_{n+h} - \widehat{X}_{n+h}(n)|^2$.

Exercise 7.7: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a real-valued $\mathrm{WN}(0, \sigma^2)$. Determine $\widehat{X}_4, \widehat{X}_5$ based on the observations X_1, X_2, X_3 and compute the prediction errors.

Solution: Note that for this sequence we have already determined the predictions based on infinite history in Exercise 7.3. Now we are looking for the projection of X_4 into $H_1^3 = \mathcal{H}\{X_1, X_2, X_3\}$, i.e. $\widehat{X}_4 = c_1X_1 + c_2X_2 + c_3X_3$ and $(X_4 - \widehat{X}_4) \perp X_j, j = 1,2,3$. Thus

$$\mathbb{E}\left(X_4 - c_1X_1 - c_2X_2 - c_3X_3\right)X_j = 0, \quad j = 1,2,3,$$

must be fulfilled. This gives the following set of equations:

$$\begin{aligned}
R_X(3) &= c_1R_X(0) + c_2R_X(1) + c_3R_X(2),\\
R_X(2) &= c_1R_X(1) + c_2R_X(0) + c_3R_X(1),\\
R_X(1) &= c_1R_X(2) + c_2R_X(1) + c_3R_X(0).
\end{aligned}$$

In this exercise the sequence $\{X_t, t \in \mathbb{Z}\}$ is an MA(1) model and we can easily determine the required values of the autocovariance function: $R_X(0) = 1.25\sigma^2, R_X(1) = -0.5\sigma^2, R_X(k) = 0, k = 2,3,\dots$ It follows that the solution of the set of equations above is $c_1 = -\frac{8}{85}, c_2 = -\frac{4}{17}, c_3 = -\frac{42}{85}$ and

$$\widehat{X}_4 = -\frac{8}{85}X_1 - \frac{4}{17}X_2 - \frac{42}{85}X_3. \tag{28}$$

When looking for the projection of X_5 into H_1^3 we obtain a similar set of equations as above. The right-hand sides will stay unchanged while the left-hand sides will be changed to $R_X(4), R_X(3)$ and $R_X(2)$, respectively. It follows that all the left-hand sides will be 0 and hence $c_1 = c_2 = c_3 = 0$. As a result, $\widehat{X}_5(3) = 0$, i.e. the value of X_5 is predicted by $\mathbb{E}X_5 = 0$ with no influence of the observed values X_1, X_2, X_3. This is consistent with the fact that X_5 depends on the values of the white noise only through Y_4 and Y_5 and the observed values X_1, X_2 and X_3 do not contain any information about them.

Concerning the prediction errors, it is straightforward to calculate $\mathbb{E}|X_5 - \widehat{X}_5(3)|^2 = \mathbb{E}|X_5|^2 = R_X(0) = 1.25\sigma^2$ since $\widehat{X}_5(3) = 0$. The prediction error of $\widehat{X}_4$ can be calculated as follows:

$$\begin{aligned}
\mathbb{E}|X_4 - \widehat{X}_4|^2 = \mathbb{E}|X_4|^2 - \mathbb{E}|\widehat{X}_4|^2 &= R_X(0) - \mathbb{E}\left|-\frac{8}{85}X_1 - \frac{4}{17}X_2 - \frac{42}{85}X_3\right|^2\\
&= R_X(0) - \left[\left(\frac{8}{85}\right)^2 R_X(0) + 2\cdot\frac{8}{85}\cdot\frac{4}{17}R_X(1) + 2\cdot\frac{8}{85}\cdot\frac{42}{85}R_X(2)\right.\\
&\quad \left. + \left(\frac{4}{17}\right)^2 R_X(0) + 2\cdot\frac{4}{17}\cdot\frac{42}{85}R_X(1) + \left(\frac{42}{85}\right)^2 R_X(0)\right] = \frac{5797}{5780}\sigma^2.
\end{aligned}$$

We remark that this procedure corresponds exactly to the formula one line above (76) on p.83 of the lecture notes [4]. Another possibility is to take advantage of the properties of the MA(1) model and express the values of X_k using the values of the white noise Y_k, Y_{k-1} for $k = 1,2,3,4$. In this way we only have to calculate the second moment of a linear combination of uncorrelated random variables and we obtain the same result as above. $\diamondsuit$

Remark: Note that when computing the prediction error for the prediction based on infinite past we usually determine the difference $X_{n+k} - \widehat{X}_{n+k}(n)$ and compute its variance $\mathbb{E}|X_{n+k} - \widehat{X}_{n+k}(n)|^2$. Since $X_{n+k} - \widehat{X}_{n+k}(n)$ is expressible as a linear combination of marginals of the white noise it is easy to compute its variance. On the other hand, when computing the prediction error for the prediction based on finite past we usually do not know any formula for the difference $X_{n+k} - \widehat{X}_{n+k}(n)$. Thus we have to use the formula

$$\mathbb{E}|X_{n+k} - \widehat{X}_{n+k}(n)|^2 = \mathbb{E}|X_{n+k}|^2 - \mathbb{E}|\widehat{X}_{n+k}(n)|^2,$$

even though it may be a bit laborious to compute $\mathbb{E}|\widehat{X}_{n+k}(n)|^2$.

Remark: It is good to recall one more time the assumptions which need to be satified for our predictions to be correct. For predictions in ARMA models based on infinite past we need both invertibility and causality of the model. For predictions based on finite past we only need the sequence to be weakly stationary and centered. Also, we must be able to compute the required values of the autocovariance function. Thus no invertibility or causality is needed for the prediction. Of course, we may need the causality for computing the autocovariance function.

Exercise 7.8: Consider a stationary AR(1) process $\{X_t, t \in \mathbb{Z}\}$ defined by the equation

$$X_t + \frac{1}{3}X_{t-1} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Predict the values of X_{k+1} for $k \in \mathbb{N}$, supposing we have observed the values $X_0 = X_1 = 1$. Compute the prediction error.

Solution: We are looking for the projection of X_{k+1} into $\mathcal{H}_0^1 = H\{X_1, X_0\}$, i.e. $\widehat{X}_{k+1}(1) = c_0X_0 + c_1X_1$ and $(X_{k+1} - \widehat{X}_{k+1}(1)) \perp X_j, j = 0, 1$. Thus

$$\mathbb{E}\left(X_{k+1} - c_0X_0 + c_1X_1\right)X_j = 0, \quad j = 0, 1,$$

must be fulfilled. This gives a set of two equations:

$$\begin{aligned} R_X(k+1) &= c_0R_X(0) + c_1R_X(1), \\ R_X(k) &= c_0R_X(1) + c_1R_X(0), \end{aligned}$$

with the solution

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} R_X(0) & R_X(1) \\ R_X(1) & R_X(0) \end{pmatrix}^{-1} \begin{pmatrix} R_X(k+1) \\ R_X(k) \end{pmatrix} = \frac{1}{1 - r^2(1)} \begin{pmatrix} r_X(k+1) - r_X(k)r_X(1) \\ -r_X(k+1)r_X(1) + r_X(k) \end{pmatrix}.$$

For our particular model with autocorrelation function $r_X(k) = \left(-\frac{1}{3}\right)^k, k \geq 1$, we get $c_0 = 0, c_1 = \left(-\frac{1}{3}\right)^k, k \geq 1$. Since $X_{k+1} - \widehat{X}_{k+1}(1) \perp \widehat{X}_{k+1}(1) \in H_1^n$ we have

$$\begin{aligned} \mathbb{E}|X_{k+1} - \widehat{X}_{k+1}(1)|^2 &= \mathbb{E}|X_{k+1}|^2 - \mathbb{E}|\widehat{X}_{k+1}(1)|^2 = R_X(0) - \left(-\frac{1}{3}\right)^{2k} R_X(0) \\ &= \frac{1 - \left(-\frac{1}{3}\right)^{2k}}{1 - \frac{1}{9}}\sigma^2 = \frac{9}{8}\left(1 - \left(-\frac{1}{3}\right)^{2k}\right)\sigma^2. \end{aligned}$$

Another method how to obtain the prediction is realizing that the prediction of X_{k+1} based on $H_{-\infty}^1$ is $\left(-\frac{1}{3}\right)^k X_1$ (by using the methods shown in the preceeding chapter 7.1) which is contained in H_0^1. Thus it must be equal to the prediction based on the finite past. Note also that from plugging the model equation iteratively into itself we get

$$X_{k+1} = Y_{k+1} - \frac{1}{3}X_k = \cdots = \sum_{j=0}^{k-1}\left(-\frac{1}{3}\right)^j Y_{k+1-j} + \left(-\frac{1}{3}\right)^k X_1,$$

which is another way how to get the prediction based on the finite past and also how to get the corresponding prediction error since

$$\mathbb{E}|X_{k+1} - \widehat{X}_{k+1}(1)|^2 = \mathbb{E}\left|\sum_{j=0}^{k-1}\left(-\frac{1}{3}\right)^j Y_{k+1-j}\right|^2 = \frac{1 - \left(-\frac{1}{3}\right)^{2k}}{1 - \frac{1}{9}}\sigma^2 = \frac{9}{8}\left(1 - \left(-\frac{1}{3}\right)^{2k}\right)\sigma^2.$$

$\diamondsuit$

Exercise 7.9: Consider a stationary AR(2) process $\{X_t, t \in \mathbb{Z}\}$ defined by the equation

$$X_t + \frac{1}{3}X_{t-1} + \frac{1}{3}X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Assume that you have observed the values of the process
a) $X_0 = X_1 = 1$,
b) $X_0 = 1$,
c) $X_1 = 1$,
d) $X_0 = X_1 = X_{-1} = 1$.
Predict the value of X_2 and compute the prediction error.

Solution: The considered model is a causal AR(2) model and using the methods from Chapter 7.1 we obtain the prediction based on the infinite history, i.e. the projection into $H^1_{-\infty}$,

$$\tilde{X}_2 = -\frac{1}{3}X_1 - \frac{1}{3}X_0 = -\frac{2}{3},$$

with the prediction error

$$\mathbb{E}|X_2 - \tilde{X}_2|^2 = \sigma^2.$$

Since $\tilde{X}_2 \in H^1_0 \subset H^1_{-1}$, $\tilde{X}_2$ is the correct answer in the cases a) and d).

For the other two cases $\tilde{X}_2 \notin H^1_1, \tilde{X}_2 \notin H^0_0$. Thus we must project $\tilde{X}_2$ further. Let us start with the case b), i.e. the projection into H^0_0. From the linearity of projection we have

$$\widehat{X}_2(0) = \widehat{\tilde{X}_2} = \widehat{\left(-\frac{1}{3}X_1 - \frac{1}{3}X_0\right)} = -\frac{1}{3}\widehat{X_1} - \frac{1}{3}X_0,$$

since the second term is in H^0_0. The best linear prediction $\widehat{X}_1 = cX_0$ of X_1 based on X_0 must fulfill $\mathbb{E}(X_1 - \widehat{X}_1)X_0 = 0$, i.e. $R_X(1) = cR_X(0)$. This is solved by $c = r_X(1) = -\frac{1}{4}$ (we get this very quickly from the second Yule-Walker equation). Thus together

$$\widehat{X}_2(0) = -\frac{1}{3}\left(-\frac{1}{4}X_0\right) - \frac{1}{3}X_0 = -\frac{1}{4}X_0 = -\frac{1}{4},$$

and

$$\begin{aligned}\mathbb{E}|X_2 - \widehat{X}_2(0)|^2 &= \mathbb{E}|X_2|^2 - \mathbb{E}|\widehat{X}_2(0)|^2 = \mathbb{E}|X_2|^2 - \mathbb{E}\left|\frac{1}{4}X_0\right|^2 = R_X(0) - \frac{1}{16}R_X(0) \\ &= \frac{6}{5}\sigma^2\left(1 - \frac{1}{16}\right) = \frac{9}{8}\sigma^2.\end{aligned}$$

Here we used $R_X(0) = \frac{6}{5}\sigma^2$. For determining the variance $R_X(0)$ we also needed $r_X(2) = -\frac{1}{4}$. These two values are easily obtained from the first three Yule-Walker equations for $\{X_t, t \in \mathbb{Z}\}$.

For the question c) we get analogically

$$\widehat{X}_2 = -\frac{1}{4}, \qquad |X_2 - \widehat{X}_2|^2 = \frac{9}{8}\sigma^2.$$

◊

Remark: In the previous exercise we could of course also directly solve the projection equations. For cases b) and c) it would be more effective since for b) we have

$$\widehat{X}_2(0) = c_0X_0 \qquad \text{such that} \qquad \mathbb{E}(X_2 - c_0X_0)X_0 = 0,$$

which gives

$$R_X(2) = c_0 R_X(0) \qquad \text{and thus} \qquad c_0 = r_X(2).$$

For c) we have analogically

$$\widehat{X}_2 = c_1 X_1 \qquad \text{such that} \qquad \mathbb{E}(X_2 - c_1 X_1)X_1 = 0,$$

which gives

$$R_X(1) = c_1 R_X(0) \qquad \text{and thus} \qquad c_1 = r_X(1).$$

On the other hand, for the case d) when the observed history is longer it is more effective to use the knowledge about the prediction based on infinite history since the projection equations would lead to

$$\begin{aligned}
R_X(3) &= c_{-1}R_X(0) + c_0 R_X(1) + c_1 R_X(2),\\
R_X(2) &= c_{-1}R_X(1) + c_0 R_X(0) + c_1 R_X(1),\\
R_X(1) &= c_{-1}1R_X(2) + c_0 R_X(1) + c_1 R_X(0),
\end{aligned}$$

and it takes some time to solve this set of equations.

Further exercises

Exercise 7.10: Consider a stationary ARMA(1,1) process $\{X_t, t \in \mathbb{Z}\}$ given by the equation

$$X_t + \frac{1}{3}X_{t-1} = Y_t - Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Predict the values of X_{k+1} for $k \in \mathbb{N}$, supposing we have observed the values $X_0 = -1$, $X_1 = 2$.

Exercise 7.11: We know the values $X_1 = 5.9$, $X_2 = 4.9$, $X_3 = 2.2$, $X_4 = 2.0$, $X_5 = 4.9$ of the process

$$(X_t - 4) - 0.8(X_{t-1} - 4) = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a centered white noise with the variance $\sigma^2 = 0.7$. Find the prediction of X_6 and X_7. Compute the respective prediction errors.
Hint: One has to be careful here – our methodology for predictions based on finite past is developed for *centered* weakly stationary sequences. Thus we have to determine what is the centered ARMA sequence in this exercise and compute the predictions for it. In the second step the prediction is recomputed for the non-centered sequence $\{X_t, t \in \mathbb{Z}\}$. Note (try it out) that applying the methodology directly to $\{X_t, t \in \mathbb{Z}\}$ would lead to a wrong answer.

8 Ergodicity

Definition 8.1: We say that a stationary sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ is *mean square ergodic* or it follows the law of large numbers in $L_2(\Omega, \mathcal{A}, P)$ if, as $n \to \infty$,

$$\overline{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t \to \mu \qquad \text{in the mean square.} \tag{29}$$

If $\{X_t, t \in \mathbb{Z}\}$ is a sequence that is mean square ergodic then

$$\frac{1}{n} \sum_{t=1}^{n} X_t \xrightarrow{P} \mu,$$

i.e. $\{X_t, t \in \mathbb{Z}\}$ satisfies the weak law of large numbers for stationary sequences.

Definition 8.2: A stationary mean square continuous process $\{X_t, t \in \mathbb{R}\}$ with mean μ is *mean square ergodic* if, as $\tau \to \infty$,

$$\overline{X}_\tau = \frac{1}{\tau} \int_0^\tau X_t \,\mathrm{d}t \to \mu \qquad \text{in the mean square.} \tag{30}$$

Remark: The above described convergences imply that the empirical average (29) or the integral (30) are weakly consistent estimates of the mean value μ of the random sequence or process $\{X_t\}$, respectively.

Theorem 8.1: A stationary random sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ and autocovariance function R is mean square ergodic if and only if

$$\frac{1}{n} \sum_{t=1}^{n} R(t) \to 0 \text{ as } n \to \infty.$$

If the sequence is real-valued and moreover satisfies $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$ then

$$n \operatorname{var}(\overline{X}_n) \to \sum_{t=-\infty}^{\infty} R(t) \text{ as } n \to \infty.$$

Theorem 8.2: A stationary, mean square continuous process $\{X_t, t \in \mathbb{R}\}$ is mean square ergodic if and only if its autocovariance function satisfies the condition

$$\frac{1}{\tau} \int_0^\tau R(t) \,\mathrm{d}t \to 0 \quad \text{as } \tau \to \infty.$$

If the process is real-valued and moreover satisfies $\int_{-\infty}^{\infty} |R(t)| \,\mathrm{d}t < \infty$ then $\tau \operatorname{var}(\overline{X}_\tau) \to \int_{-\infty}^{\infty} R(t) \,\mathrm{d}t$.

Exercise 8.1: Are the AR models from Exercises 6.3–6.5 mean square ergodic? And what about the ARMA(2,1) model from Exercise 6.6?

Solution: We will check the condition $\sum_{t=-\infty}^{\infty}|R(t)| < \infty$. The autocovariance functions of the considered AR and ARMA models are

$$R_{X_{6.3}}(k) = \sigma^2\left(\frac{200}{81}0.5^{|k|} - \frac{125}{162}0.2^{|k|}\right), \quad k \in \mathbb{Z},$$

$$R_{X_{6.4}}(k) = \frac{8\,125}{6\,912}\sigma^2\left(1 + \frac{12}{13}|k|\right)0.2^{|k|}, \quad k \in \mathbb{Z},$$

$$R_{X_{6.5}}(k) = const_1\left(\frac{7\sqrt{2}}{10}\mathrm{e}^{i\frac{\pi}{4}}\right)^{|k|} + const_2\left(\frac{7\sqrt{2}}{10}\mathrm{e}^{-i\frac{\pi}{4}}\right)^{|k|}, \quad k \in \mathbb{Z},$$

$$R_{X_{6.6}}(k) = \sigma^2\left(\frac{32}{3} + 8|k|\right)\left(\frac{1}{2}\right)^{|k|}, \quad k \in \mathbb{Z},$$

as we have computed in the respective Exercises. All of these sequences are linear combinations of a finite number of (possibly differentiated) geometric sequences with quotients inside the unit circle in $\mathbb{C}$. Thus all of the sequences are summable and according to the Theorem 8.1 all the considered models are ergodic. $\Diamond$

Remark: Actually any causal ARMA model is ergodic and even satisfies $\sum_{t=-\infty}^{\infty}|R(t)| < \infty$. Thus according to the Theorem 8.1 we can even estimate the asymptotic variance $n\,\mathrm{var}(\overline{X}_n) \to \sum_{k=-\infty}^{\infty} R(k)$. The argument is the same as in the previous exercise, since any causal ARMA model has the autocovariance function expressible as a linear combination of finitely many (possibly differentiated) geometric sequences with quotients in absolute value smaller than 1.

Exercise 8.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a real-valued $\mathrm{WN}(0, \sigma^2)$. Discuss the mean square ergodicity of the sequence $\{X_t, t \in \mathbb{Z}\}$ and determine the limit of $n\,\mathrm{var}(\overline{X}_n), n \to \infty$.

Solution:

Note that this is the same sequence as discussed in Exercises 7.3 and 7.6. It is an MA(1) model with the autocovariance function $R_X(0) = 1.25\sigma^2, R_X(1) = -0.5\sigma^2, R_X(k) = 0, k = 2, 3, \ldots$ Looking at the Theorem 8.1 we see that the condition $\frac{1}{n}\sum_{t=1}^{n} R_X(t) \to 0$ as $n \to \infty$ is fulfilled since the autocovariance function has only finitely many non-zero values. It follows that the sequence is mean square ergodic.

We also easily check the additional condition of Theorem 8.1, $\sum_{t=-\infty}^{\infty}|R_X(t)| < \infty$, and hence $n\,\mathrm{var}(\overline{X}_n) \to \sum_{t=-\infty}^{\infty} R_X(t) = 1.25\sigma^2 - 2\cdot 0.5\sigma^2 = \frac{1}{4}\sigma^2, n \to \infty$.

The limit of $n\,\mathrm{var}(\overline{X}_n)$ can be also obtained directly in this case. We write explicitly the values of X_i as $X_1 = -\frac{Y_0}{2} + Y_1, X_2 = -\frac{Y_1}{2} + Y_2, X_3 = -\frac{Y_2}{2} + Y_3, \ldots$ It follows easily that

$$\sum_{k=1}^{n} X_k = -\frac{Y_0}{2} + \frac{1}{2}\sum_{k=1}^{n-1} Y_k + Y_n,$$

$$\mathrm{var}\left(\sum_{k=1}^{n} X_k\right) = \frac{\sigma^2}{4} + (n-1)\frac{\sigma^2}{4} + \sigma^2,$$

$$\mathrm{var}(\overline{X}_n) = \frac{\sigma^2}{4n} + \frac{\sigma^2}{n^2},$$

$$n\,\mathrm{var}(\overline{X}_n) = \frac{\sigma^2}{4} + \frac{\sigma^2}{n} \to \frac{\sigma^2}{4}, n \to \infty.$$

$\Diamond$

Exercise 8.3: Is the mean square continuous process with spectral density $f(\lambda) = |\lambda| I(|\lambda| \leq 1)$, $\lambda \in \mathbb{R}$, mean square ergodic?

Solution: The autocovariance function of an L_2-continuous process with spectral density $f(\lambda)$ may be written as $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda)\,\mathrm{d}\lambda$. Thus we may compute $R(t), t \in \mathbb{R}$, as follows:

$$R(t) = \int_{-1}^{1} |\lambda| \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\lambda = 2\int_{0}^{1} \lambda \cos(t\lambda)\,\mathrm{d}\lambda = 2\left[\frac{\cos(t\lambda)}{t^2} + \frac{\lambda \sin(t\lambda)}{t}\right]_0^1 = 2\left(\frac{\cos t}{t^2} - \frac{1}{t^2} + \frac{\sin t}{t}\right).$$

Now we could compute $\frac{1}{\tau}\int_0^\tau R(t)\,\mathrm{d}t$ directly and compute its limit for $\tau \to \infty$, which is a bit laborious.

Or, we could observe that $R(t)$ is a continuous function on $(-\infty, \infty)$ – it is obviously continuous in every $t \neq 0$ and in 0 it is continuous since the corresponding process is mean square continuous (Corollary 3.1). Thus $\int_0^\tau R(t)\,\mathrm{d}t$ exists for any $\tau > 0$ and

$$|R(t)| \leq const \cdot \min\left(\frac{1}{t}, 1\right), \quad t \in \mathbb{R}.$$

Thus

$$\begin{aligned}\lim_{\tau\to\infty}\left|\frac{1}{\tau}\int_0^\tau R(t)\,\mathrm{d}t\right| &\leq \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau |R(t)|\,\mathrm{d}t \leq \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau const \cdot \min\left(\frac{1}{t}, 1\right)\mathrm{d}t \\ &\leq const \cdot \lim_{n\to\infty}\frac{1}{n}\left(1 + \sum_{t=1}^{n-1}\frac{1}{t}\right),\end{aligned}$$

where the last inequality uses the Riemannian upper bound on the integral. But

$$\lim_{n\to\infty}\frac{1}{n}\left(1 + \sum_{t=1}^{n-1}\frac{1}{t}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{t},$$

if the latter limit exists. And it does since $\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{1}{t} = 0$ is the Cesaro limit of the sequence $\frac{1}{n}$, see [2, p.81], which is equal to the ordinary limit $\lim_{n\to\infty}\frac{1}{n} = 0$.

Thus we proved $\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau R(t)\,\mathrm{d}t = 0$ and according to the Theorem 8.2 the corresponding mean square continuous process is also mean square ergodic. ♢

Remark: Note that $\frac{1}{t}$ is not integrable in the neighbourhood of $+\infty$ thus our upper bound function $const \cdot \min(\frac{1}{t}, 1)$ would not be small enough to imply the stronger condition $\int_{-\infty}^{\infty}|R(t)|\,\mathrm{d}t < \infty$. Moreover, it is not just a problem of the chosen upper bound since $\int_{-\infty}^{\infty}\left|\frac{\cos(t)}{t^2} - \frac{1}{t^2} + \frac{\sin(t)}{t}\right|\mathrm{d}t = \infty$.

Further exercises

Exercise 8.4: Let $\{X_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with autocovariance function $R(t) = \cos(\pi t), t \in \mathbb{Z}$. Is $\{X_t, t \in \mathbb{Z}\}$ mean square ergodic?

Exercise 8.5: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary process with autocovariance function $R(t) = \cos t, t \in \mathbb{R}$. Is $\{X_t, t \in \mathbb{R}\}$ mean square ergodic?

Exercise 8.6: Let $\{X_t, t \in \mathbb{R}\}$ be a centered weakly stationary L_2-continuous process with spectral density $f_X(\lambda) = a\mathbf{1}(\lambda \in (-b, b))$ for some constants $a, b > 0$. Is $\{X_t, t \in \mathbb{R}\}$ mean square ergodic?

9 Partial autocorrelation function

Definition 9.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a real-valued centered weakly stationary sequence. The *partial autocorrelation function* of $\{X_t, t \in \mathbb{Z}\}$ is defined to be

$$\alpha(k) = \begin{cases} \operatorname{corr}(X_1, X_{k+1}), & k = 1, \\ \operatorname{corr}(X_1 - \widetilde{X}_1, X_{k+1} - \widetilde{X}_{k+1}), & k > 1, \end{cases}$$

where $\widetilde{X}_1$ is the linear projection of X_1 onto the Hilbert space $H_2^k = \mathcal{H}\{X_2, \dots, X_k\}$ and $\widetilde{X}_{k+1}$ is the linear projection of X_{k+1} onto H_2^k.

Definition 9.2: (Alternative definition of the partial correlation function) Let $\{X_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence, let $P_{H_1^k}(X_{k+1})$ be the best linear prediction of X_{k+1} based on $X_1, \dots, X_k$. If $H_1^k = \mathcal{H}\{X_1, \dots, X_k\}$ and $P_{H_1^k}(X_{k+1}) = \varphi_1 X_k + \dots + \varphi_k X_1$, then the partial autocorrelation function at lag k is defined to be $\alpha(k) = \varphi_k$.

Theorem 9.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a centered real-valued weakly stationary sequence with the autocovariance function R such that $R(0) > 0, R(t) \to 0$ as $t \to \infty$. Then both definitions of the partial autocorrelation function are equivalent and it holds that $\alpha(1) = r(1)$,

$$\alpha(k) = \frac{\begin{vmatrix} 1 & r(1) & \cdots & r(k-2) & r(1) \\ r(1) & 1 & \cdots & r(k-3) & r(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r(k-1) & r(k-2) & \cdots & r(1) & r(k) \end{vmatrix}}{\begin{vmatrix} 1 & r(1) & \cdots & r(k-1) \\ r(1) & 1 & \cdots & r(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ r(k-1) & r(k-2) & \cdots & 1 \end{vmatrix}}, \quad k > 1,$$

where r is the autocorrelation function of the sequence $\{X_t, t \in \mathbb{Z}\}$.

Exercise 9.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5 Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a real-valued $\mathrm{WN}(0, \sigma^2)$. Determine the values of the partial autocorrelation function $\alpha(1), \alpha(2), \alpha(3)$.

Solution: Note that this is the same sequence as discussed in Exercises 7.3, 7.6 and 8.2. It is an MA(1) model with the autocovariance function $R_X(0) = 1.25\sigma^2, R_X(1) = -0.5\sigma^2, R_X(k) = 0, k = 2, 3, \dots$ Hence the autocorrelation function is $r_X(0) = 1, r_X(1) = -\frac{2}{5}, r_X(k) = 0, k = 2, 3, \dots$ Clearly the assumptions of the Theorem 9.1 are fulfilled and we may compute

$$\begin{aligned}
\alpha(1) &= r_X(1) = -\frac{2}{5}, \\
\alpha(2) &= \frac{\begin{vmatrix} 1 & -\frac{2}{5} \\ -\frac{2}{5} & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -\frac{2}{5} \\ -\frac{2}{5} & 1 \end{vmatrix}} = \frac{-\frac{4}{25}}{1 - \frac{4}{25}} = -\frac{4}{21}, \\
\alpha(3) &= \frac{\begin{vmatrix} 1 & -\frac{2}{5} & -\frac{2}{5} \\ -\frac{2}{5} & 1 & 0 \\ 0 & -\frac{2}{5} & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -\frac{2}{5} & 0 \\ -\frac{2}{5} & 1 & -\frac{2}{5} \\ 0 & -\frac{2}{5} & 1 \end{vmatrix}} = \frac{-\frac{8}{125}}{1 - \frac{4}{25} - \frac{4}{25}} = -\frac{\frac{8}{125}}{\frac{17}{25}} = -\frac{8}{85}.
\end{aligned}$$

We remark here that the value of $\alpha(3)$ calculated above is the same as the coefficient c_1 at X_1 in the best linear prediction of X_4 based on the values of X_1, X_2, X_3 we calculated in Exercise 7.2, see Equation (28). This illustrates the fact that the values calculated according to Theorem 9.1 are in fact the values from Definition 9.2. $\diamondsuit$

Exercise 9.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t - \frac{1}{4} X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a real-valued $\mathrm{WN}(0, \sigma^2)$. Determine the values of the partial autocorrelation function $\alpha(1), \alpha(2), \alpha(3)$.

Solution: Since the sequence $\{X_t, t \in \mathbb{Z}\}$ is causal and the values of the autocorrelation function r_X can be easily calculated using the Yule-Walker equations, this exercise can be solved similarly to the previous one. However, we will follow the approach from Definition 9.2.

To determine $\alpha(k), k = 1, 2, 3$, we have to find the prediction $P_{H_1^k}(X_{k+1})$. Using the same arguments as in the solution of Exercise 7.9, part b), we get that $P_{H_1^1}(X_2) = r_X(1) \cdot X_1$. From the Yule-Walker equation $R_X(1) - \frac{1}{4} R_X(1) = 0$ we easily get $r_X(1) = 0$. Hence, $\alpha(1) = r_X(1) = 0$.

Using the model equation and causality we obtain

$$\begin{aligned} P_{H_1^2}(X_3) &= 0 \cdot X_2 + \frac{1}{4} X_1, \\ P_{H_1^3}(X_4) &= 0 \cdot X_3 + \frac{1}{4} X_2 + 0 \cdot X_1. \end{aligned}$$

It follows that $\alpha(2) = \frac{1}{4}$ and $\alpha(3) = 0$. $\diamondsuit$

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List of symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of non-negative integers
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbf{1}$	indicator function
$\|\cdot\|$	norm in a Hilbert space
$\mathcal{B}$	Borel $\sigma-$algebra
$\mathcal{N}(\mu,\sigma^2)$	normal distribution with parameters μ,σ^2
$X\sim\mathcal{N}(\mu,\sigma^2)$	random variable with distribution $\mathcal{N}(\mu,\sigma^2)$
$\{X_t,\, t\in T\}$	stochastic process indexed by set T
$\mathcal{M}\{X_t,\, t\in T\}$	linear span of $\{X_t,\, t\in T\}$
$\mathcal{H}\{X_t,\, t\in T\}$	Hilbert space generated by the stochastic process $\{X_t,\, t\in T\}$
AR(p)	autoregressive sequence of order p
MA(q)	moving average sequence of order q
ARMA(p,q)	mixed ARMA sequence of orders p and q
WN($0,\sigma^2$)	white noise with zero mean and variance σ^2
$X\perp Y$	orthogonal (perpendicular) random variables
$\overline{\lim}$	limes superior
$\xrightarrow{\text{P}}$	convergence in probability
$\xrightarrow{\text{D}}$	convergence in distribution
l. i. m.	convergence in mean square (limit in the mean)