Stochastic processes 2 – topics for oral examination

1. Definition of a stochastic process, Daniell-Kolmogorov theorem. Example of a consistent system of distributions, example of a non-consistent system of distributions. Theorem 2.2 with proof.

2. Strict and weak stationarity, their relationship (Theorem 2.1 with proof). Example of a process which is weakly stationary but not strictly stationary, example of a process which is strictly stationary but not weakly stationary.

3. Definition of the autocovariance function, its properties (Theorems 2.3, 2.5 and 2.7 with proofs, Theorems 2.4 and 2.6 without proofs).

4. Hilbert space, scalar product, its continuity (Theorem 3.2 with proof). Construction of the space $L_2(\Omega, \mathcal{A}, \mathbb{P})$, mean square convergence.

5. Mean square continuity, necessary and sufficient conditions (Theorems 4.1, 4.2 and 4.3 with proofs), examples.

6. Riemann integral of a stochastic process, construction, sufficient condition for existence (Theorem 4.4 with proof), relationship to the mean square continuity (remark with proof).

7. Spectral decomposition of the autocovariance function for a random sequence (Theorem 5.4, proof of the implication that $R(t)$ with the appropriate structure is positive semidefinite). Spectral density and its relationship to the spectral distribution function. Symmetry of the spectral density implies that the autocovariance function is real-valued (proof).

8. Inverse formula for spectral density (Theorem 5.7 with proof, Theorem 5.8 without proof). Spectral density of the white noise sequence.

9. Orthogonal increment process, orthogonal distribution function, Theorem 6.1 with proof. Orthogonal distribution function of the Wiener process.

10. Integral with respect to an orthogonal increment process, construction for simple functions, properties (Theorem 6.2 with proof).

11. Integral with respect to an orthogonal increment process, construction for measurable functions, properties (Theorem 6.3 with proof).

12. Properties of the sequence $X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z(\lambda), t \in \mathbb{Z}$ (Theorem 6.4 with proof). Spectral decomposition of a random sequence (Theorem 6.5 without proof).

13. MA(n) model, definition, properties (Theorem 7.1 with proof).

14. MA(∞) model, definition, properties (Theorem 7.4 with proof).

15. AR(p) model, definition, necessary and sufficient condition for causality (Theorem 7.5 with proof).

16. ARMA(p,q) model, definition, conditions for causality and invertibility (Theorems 7.6 and 7.7 without proofs). Prediction based on infinite history for ARMA(p,q) models, prediction error.

17. Linear filters, definition, properties (Theorem 7.8 without proof), comparison to MA(∞) models. Properties of non-causal AR(1) model.

18. Mean square ergodicity, definition, necessary and sufficient condition for random sequences (Theorem 8.1 with proof), limiting behavior of the variance (Theorem 8.2 without proof). Example of an ergodic sequence, example of a non-ergodic sequence.

19. Projections in a Hilbert space, projection theorem (Theorem 9.2 without proof), properties of the projection mapping (Theorem 9.3 with proof). Prediction based on infinite history for AR(p) models, prediction error.

20. Prediction based on finite history, principles, derivation, prediction error, regularity of Γ_n (Theorem 9.4 without proof). Filtration of signal and noise, comparison with the problem of prediction.

21. Partial autocorrelation function, two definitions and their equivalence, computation (Theorem 10.1 with proof).

22. Estimation of the autocovariance and autocorrelation functions, estimators, properties (Theorem 11.1 without proof). Estimating the parameter of a causal AR(1) model using $\hat{r}_n(1)$.

23. Parameter estimation for AR(p) models, method of moments, derivation of the equations, properties of the estimators (Theorem 12.1 without proof).

24. Parameter estimation for AR(p) models, least squares method and asymptotic properties of the estimators, maximum likelihood method, conditional maximum likelihood method.

25. Parameter estimation for MA(q) models, method of moments, example for MA(1).

26. Parameter estimation for ARMA(p, q) models, method of moments.

27. Periodogram and its properties (including but not limited to Theorem 13.1 with proof).