Prize problem

I will pay **200 €** to the first person who gives the answer (with a proof) to the following question:

Is there a positive integer $n \geq 2$ and words $u_1, u_2, \ldots, u_n$ such that both equalities

$$\begin{cases} (u_1 u_2 \cdots u_n)^2 = u_1^2 u_2^2 \cdots u_n^2, \\ (u_1 u_2 \cdots u_n)^3 = u_1^3 u_2^3 \cdots u_n^3, \end{cases}$$

hold and the words u_i, $i = 1, \ldots, n$, do not pairwise commute (that is, $u_i u_j \neq u_j u_i$ for at least one pair of indices $i, j \in \{1, 2, \ldots, n\}$)?

Related results

In [Pla03] the negative answer to the question is conjectured.

In [Hol01] it is proved that

$$(u_1 u_2 \cdots u_n)^k = u_1^k u_2^k \cdots u_n^k$$

holds simultaneously for $k = 2, 3, 4$ only for commuting words.

Some other, slightly stronger related results can be found in the paper and in [Hol00].

In [HK97] it is shown that the answer to our question is negative if $n \le 5$.

On the other hand, there are noncommuting words satisfying

$$(u_1 u_2 \cdots u_n)^k = u_1^k u_2^k \cdots u_n^k$$

for each k (see [Hol01] for an example).

Moreover, in [Hol99] noncommuting words $u_1, u_2, \ldots, u_7$ are given satisfying

$$(u_1^2 u_2^2 \cdots u_7^2)^3 = (u_1^3 u_2^3 \cdots u_7^3)^2.$$

Ismo Hakala and Juha Kortelainen.
On the system of word equations $x_1^i x_2^i \cdots x_m^i = y_1^i y_2^i \cdots y_n^i$ $(i = 1, 2, \cdots)$ in a free monoid.
Acta Inform., 34(3):217–230, 1997.

Štěpán Holub.
A solution of the equation $(x_1^2 \cdots x_n^2)^3 = (x_1^3 \cdots x_n^3)^2$.
In *Contributions to general algebra, 11 (Olomouc/Velké Karlovice, 1998)*, pages 105–111. Heyn, Klagenfurt, 1999.

Štěpán Holub.
In search of a word with special combinatorial properties.
In *Computational and geometric aspects of modern algebra (Edinburgh, 1998)*, volume 275 of *London Math. Soc. Lecture Note Ser.*, pages 120–127. Cambridge Univ. Press, Cambridge, 2000.

References 2

Štěpán Holub.
Local and global cyclicity in free semigroups.
Theoret. Comput. Sci., 262(1-2):25–36, 2001.

Wojciech Plandowski.
Test sets for large families of languages.
In *Developments in language theory*, volume 2710 of *Lecture Notes in Comput. Sci.*, pages 75–94. Springer, Berlin, 2003.