

Let X be a vector space (over \mathbb{F}) and let \mathcal{U} be a nonempty family of subsets of X with properties:

- (i) Elements of \mathcal{U} are balanced and absorbing
- (ii) $\forall U \in \mathcal{U} \exists V \in \mathcal{U} : V + V \subset U$
- (iii) $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U} : W \subset U \cap V$

Then there is a unique topology \mathcal{T} on X s.t.

(X, \mathcal{T}) is a TVS and \mathcal{U} is a base of nbhds of $0 \in (X, \mathcal{T})$.

Moreover, (X, \mathcal{T}) is Hausdorff iff $\bigcap \mathcal{U} = \{0\}$

Proof:

Step 1: Uniqueness and formula for \mathcal{T} .

Since any linear topology is translation invariant, there is at most one linear topology with a given base of nbhds of 0 . Therefore, uniqueness is clear. Moreover, by the definition of a base of nbhds, the topology \mathcal{T} must be given by

$$\mathcal{T} = \{G \subset X; \forall x \in G \exists U \in \mathcal{U} : x + U \subset G\}$$

So, let us define \mathcal{T} by this formula

Step 2: \mathcal{T} is a topology on X :

- Clearly $\emptyset \in \mathcal{T}, X \in \mathcal{T}$
- $(G_\alpha)_{\alpha \in I}$ any family of elements of \mathcal{T}

$$\Rightarrow \bigcup_{\alpha \in I} G_\alpha \in \mathcal{T} \quad \left[x \in \bigcup_{\alpha \in I} G_\alpha \Rightarrow \exists \alpha : x \in G_\alpha, \text{ so} \right.$$

$$\left. \exists U \in \mathcal{U} \quad x + U \subset G_\alpha \subset \bigcup_{\alpha \in I} G_\alpha \right]$$

- $G_1, G_2 \in \mathcal{T} \Rightarrow G_1 \cap G_2 \in \mathcal{T}$

$$\left[x \in G_1 \cap G_2 \Rightarrow \begin{cases} x \in G_1 \Rightarrow \exists U \in \mathcal{U} : x + U \subset G_1 \\ x \in G_2 \Rightarrow \exists V \in \mathcal{U} : x + V \subset G_2 \end{cases} \right. \left. \begin{array}{l} \text{by (iii) find } W \in \mathcal{U} \\ \text{with } W \subset U \cap V \end{array} \right]$$

Then $x \in W \subset G_1 \cap G_2$]

Step 3: Any $U \in \mathcal{U}$ is a nbhd of 0 in \mathcal{T} ; hence \mathcal{U} is a base of nbhds of 0 in \mathcal{T}

$\Gamma U \in \mathcal{U}$... Let $V := \{x \in U; \exists W \in \mathcal{U} : x+W \subset U\}$
Then clearly

- $V \subset U$
- $0 \in V$ (take $W=U$)

Moreover, $V \in \mathcal{T}$, i.e., V is open:

Γ $x \in V \Rightarrow$ by the definition of $V \exists W \in \mathcal{U}$
with $x+W \subset U$
by (cc) find $\tilde{W} \in \mathcal{U}$ with $\tilde{W}+\tilde{W} \subset W$
Then $x+\tilde{W} \subset V$, as $x+W \subset U$
 $x+W+\tilde{W} \subset x+\tilde{W}+\tilde{W} \subset x+W \subset U \Rightarrow$

So, $0 \in \text{int } V$, hence V is a nbhd of 0 in \mathcal{T} .

Step 4: Addition is continuous in \mathcal{T} :

$\Gamma x, y \in X, G \in \mathcal{T}, x+y \in G$
 $\Rightarrow \exists U \in \mathcal{U} \quad x+y+U \subset G \stackrel{(cc)}{\Rightarrow} \exists V \in \mathcal{U} \quad V+V \subset U$

Then $(x+V)+(y+V) \subset x+y+U \subset G$.

Step 5: U is a nbhd of 0 in \mathcal{T} , $\lambda > 0 \Rightarrow \lambda U$ is also a nbhd of 0 in \mathcal{T}

$$\left\{ \begin{array}{l} \lambda \geq 1 : U \text{ nshd of } 0 \Rightarrow \exists V \in \mathcal{U} : V \subset U \\ V \subset \lambda V \subset \lambda U \\ \uparrow V \text{ is balanced by (c)} \end{array} \right\} \Rightarrow \lambda U \text{ is a nshd of } 0$$

$$\lambda = \frac{1}{2} : U \text{ nshd of } 0 \Rightarrow \exists V \in \mathcal{U} \quad V + V \subset U$$

$$\text{Then } V \subset \frac{1}{2}U \quad \left[x \in \frac{1}{2}U \Rightarrow x+x \in V+V \subset U \right]$$

so $\frac{1}{2}U$ is a nshd of 0

By induction: U is a nshd of 0 $\Rightarrow \forall n \in \mathbb{N} \quad \frac{1}{2^n}U$ is a nshd of 0

Hence general $\lambda > 0$: Find $n \in \mathbb{N}$ with $\frac{1}{2^n} < \lambda$. Then

$$\lambda U = (\lambda \cdot 2^n) \cdot \frac{1}{2^n}U \text{ is a nshd of } 0 \text{ whenever } U \text{ is a nshd of } 0$$

Step 6: Continuity of multiplication

$$\Gamma \lambda \in \mathbb{F}, x \in X, G \in \mathcal{T} \text{ s.t. } \lambda x \in G$$

$$\text{Find } U \in \mathcal{U} \text{ s.t. } \lambda x + U \subset G$$

$$\text{By (cc)} \exists V \in \mathcal{U} \text{ with } V + V \subset U$$

$$\text{By (c)} V \text{ is absorbing, so } \exists c > 0 \forall t \in [0, c] : tx \in V$$

Fit such c

$$V \text{ balanced} \Rightarrow \forall t \in \mathbb{F}, |t| \leq c : tx \in V$$

Let $\mu \in \mathbb{F}, y \in X$ be such that $|\mu - \lambda| < c$ & $y \in x + \frac{1}{|\lambda + c|}V$
(this defines a nshd of (λ, x) in $\mathbb{F} \times (X, \mathcal{T})$ by definition and by Step 5). Then

$$\mu y - \lambda x = \underbrace{\mu(y-x)}_{\in \frac{c}{|\lambda+c|}V \subset V} + \underbrace{(\mu-\lambda)x}_{\in V, \text{ as } |\mu-\lambda| < c} \in V + V \subset U$$

$$\in \frac{c}{|\lambda+c|}V \subset V \quad \in V, \text{ as } |\mu-\lambda| < c$$

$$\text{as } \left| \frac{\mu}{\lambda+c} \right| \leq \frac{|\lambda| + |\mu-\lambda|}{|\lambda+c|} < 1 \text{ and } V \text{ is balanced}$$

so, $\mu y \in \lambda x + U \subset G$.

Moreover part: (X, \mathcal{T}) a TVS, \mathcal{U} a base of nbhd of 0

Then X is Hausdorff $\Leftrightarrow \bigcap \mathcal{U} = \{0\}$

\Rightarrow : $x \in X \setminus \{0\} \Rightarrow \exists S_1, S_2 \in \mathcal{T}$ $x \in S_1, 0 \in S_2, S_1 \cap S_2 = \emptyset$

Find $U \in \mathcal{U}$ with $U \subseteq S_2$. Then $x \notin U$
Hence $x \notin \bigcap \mathcal{U}$

\Leftarrow $x, y \in X, x \neq y \Rightarrow x - y \neq 0$.

Find $U \in \mathcal{U}$ with $x - y \notin U$

Find V , a nbhd of 0 s.t. $V + V \subseteq U$

Then $(x + V) \cap (y + V) = \emptyset$

$\forall z \in (x + V) \cap (y + V)$

$\Rightarrow z = x + v_1 = y + v_2, v_1, v_2 \in V$

$\Rightarrow x = z + v_1, y = z - v_2$

$\Rightarrow x - y = v_1 + v_2 \in V + V \subseteq U$

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