Theorem Let A be a Banach algebra. Then $\forall x \in A$: $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$

Proof: (1) A unital $\Rightarrow \sigma(x) = \mathbb{C} \setminus \rho(x) \Rightarrow$ it is closed by Prop. 8(ii)

and by Prop. 8(v) $\sigma(x) \subset \overline{U(0, \|x\|)}$

$\Rightarrow \sigma(x)$ is bdd, hence compact

(2) A not unital $\Rightarrow \sigma(x) = \sigma_{A^+}(x, 0)$ by definition, so it is a compact set by (1)

(3) $\sigma(x) \neq \emptyset$: A not unital $\Rightarrow 0 \in \sigma(x)$

A unital. Suppose $\sigma(x) = \emptyset$, i.e. $\rho(x) = \mathbb{C}$

Then $\forall \varphi \in A^*: \lambda \mapsto \varphi(R(\lambda, a))$ is an entire function (holomorphic on $\mathbb{C}$)

Moreover, for $|\lambda| > \|a\|$ we have (by prop. 8(v))

$$\|R(\lambda, a)\| = \left\| \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{a^n}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \frac{\|a\|^n}{|\lambda|^{n+1}} =$$

$$= \frac{\frac{1}{|\lambda|}}{1 - \frac{\|a\|}{|\lambda|}} = \frac{1}{|\lambda| - \|a\|} \longrightarrow 0 \text{ for } \lambda \to \infty$$

Hence, given $\varphi \in A^*$: $\lim_{\lambda \to \infty} \varphi(R(\lambda, a)) = 0$

Hence, by a consequence of the Liouville theorem $\varphi(R(\lambda, a)) = 0$ for $\lambda \in \mathbb{C}$

Since $\varphi \in A^*$ is arbitrary, a consequence to H-B theorem yields $R(\lambda, a) = 0$ for $\lambda \in \mathbb{C}$.
It is a contradiction, as $(\lambda e - a)^{-1}$ cannot be 0.

Remark: $G \subset \mathbb{C}$ open $\Rightarrow \{a \in A;\ \sigma(a) \subset G\}$ is open in A

- WLOG A unital
- Let $\sigma(a) \subset G$. Then $r := \operatorname{dist}(\sigma(a), \mathbb{C} \setminus G) > 0$

Let $K := \overline{U(0, r + \|a\|) \setminus U(\sigma(a), r)}$.

Then K is a compact subset of $\rho(a)$, $K \neq \emptyset$
$(K \supset \{\lambda;\ |\lambda| = r + \|a\|\}$ as $U(\sigma(a), r) \subset U(0, \|a\| + r))$

$M := \max_{\lambda \in K} \|(\lambda e - a)^{-1}\|$ (cts function on a compact set)

Let now $\|b - a\| < \min\{r, \frac{1}{M}\}$

Then: • $\|b - a\| < r \Rightarrow \|b\| < \|a\| + r \Rightarrow \sigma(b) \subset U(\|a\| + r)$

• $\lambda \in K \Rightarrow \|(\lambda e - b) - (\lambda e - a)\| = \|b - a\| <$

$< \frac{1}{M} \leq \frac{1}{\|(\lambda e - a)^{-1}\|}$. Hence $\lambda e - b$ is invertible (by Lemma 6 (b))

So, $\lambda \in \rho(b)$
It follows that $\sigma(b) \cap K = \emptyset$

So, $\sigma(b) \subset U(0, r + \|a\|) \setminus K = U(\sigma(a), r) \subset G$

Gelfand-Mazur theorem

A unital Banach algebra

A is a field (i.e. $G(A) = A\setminus\{0\}$) $\Leftrightarrow$ $A \approx \mathbb{C}$

Proof: $\Leftarrow$ clear

$\Rightarrow$: $x \in A \Rightarrow \sigma(x) \neq \emptyset$ by Thm 9

Fix $\lambda \in \sigma(x)$. Then $\lambda e - x$ is not invertible,

so by the assumption $\lambda e - x = 0$, hence $x = \lambda e$

Therefore, $T: \mathbb{C} \to A$ defined by $T(\lambda) = \lambda e$

is onto.

It is clear that T is an isometric isomorphism

of Banach algebras.

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Lemma 11 (on spectrum and polynomials)

A unital Banach algebra, $p(\lambda) = \sum_{j=0}^{n} \alpha_j \cdot \lambda^j$ a polynomial

$a \in A$, $p(a) := \sum_{j=0}^{n} \alpha_j a^j$

(a) $p(a) \in G(A) \Leftrightarrow$ the roots of $p \subset \rho(a)$

- p constant $\Rightarrow$
  - $p \neq 0 \Rightarrow p$ has no roots & $p(a) = \alpha_0 e \in G(A)$
  - $p = 0 \Rightarrow p(a) = 0 \notin G(A)$, roots of $p = \mathbb{C} \not\subset \rho(a)$ by Thm 9

- $\deg p \geq 1$ ($\deg p = n$)

$\Rightarrow p(\lambda) = \alpha_n (\lambda - \xi_1) \cdots (\lambda - \xi_n)$, where $\xi_1, \dots, \xi_n$ are roots

Then $p(a) = \alpha_n (a - \xi_1 e) \cdots (a - \xi_n e)$

Since $a - \xi_1 e, \dots, a - \xi_n e$ commute, Prop. 5(c)

yields $p(a) \in G(A) \Leftrightarrow \{a - \xi_1 e, \dots, a - \xi_n e\} \subset G(A)$

$\Updownarrow$

$\{\xi_1, \dots, \xi_n\} \subset \rho(a)$

(b) $\sigma(p(a)) = p(\sigma(a))$

$\ulcorner$. $\mu \in \sigma(p(a)) \Leftrightarrow \mu e - p(a) \notin G(A) \Leftrightarrow$

(Observe that $\mu e - p(a) = (\mu - p)(a)$, so)

$\Leftrightarrow (\mu - p)(a) \notin G(a) \overset{(a)}{\Leftrightarrow} \exists\ \lambda$, a root of $\mu - p$ s.t. $\lambda \notin \varrho(a)$

$\Leftrightarrow \exists\ \lambda \in \sigma(a) : \mu - p(\lambda) = 0 \Leftrightarrow \mu \in p(\sigma(a))$ $\lrcorner$