

Let A be a unital Banach algebra

(a) $I \subset A$ ideal of codimension 1 $\Rightarrow \exists! h \in \Delta(A)$ s.t. $I = \ker h$

I is closed by Prop. 18

Take A/I and consider $q: A \rightarrow A/I$ the canonical quotient mapping

Then

- A/I is a Banach algebra of dimension 1
- q is a homomorphism.

A/I of dimension 1 $\Rightarrow A/I = \{ \lambda e, \lambda \in \mathbb{C} \} \cong \mathbb{C}$

So, in fact $q \in \Delta(A)$

Uniqueness: $\ker h_1 = \ker h_2 \Rightarrow h_2$ is a multiple of h_1 .

But since $h_1(e) = h_2(e) = 1$, necessarily $h_1 = h_2$

(b) A commutative $\Rightarrow h \mapsto \ker h$ is a bijection of $\Delta(A)$ onto the set of maximal ideals in A .

$h \in \Delta(A) \Rightarrow \ker h$ is an ideal of codimension 1

So, by (a)

$h \mapsto \ker h$ is a bijection of $\Delta(A)$ onto the set of all the ideals of codimension 1.

It remains to prove that any maximal ideal is of codim. 1:

Let I be a closed ideal of codim ≥ 2 . Then

$B := A/I$ is a B -algebra of dimension ≥ 2 . By Gelfand-Mazur there is $x \in B$ not invertible. Let

$J = xB = \{ +y, y \in B \} \Rightarrow J$ is an ideal in $B, J \neq \{0\}$
($+ \in J, e \notin J$)

So, $q^{-1}(J)$ (where $q: A \rightarrow A/I = B$ is the canonical quotient map)

is an ideal in A strictly containing I . So, I is not maximal.