

Proposition IX.7  $\Rightarrow$   $H$  is a Hilbert space,  $T \in \mathcal{L}(H)$

(a)  $T^* = T \Leftrightarrow W(T) \subset \mathbb{R}$

$$\begin{aligned} \overline{T^* = T} &\stackrel{\text{Prop. 4(c)}}{\Leftrightarrow} \forall t \in H: \langle Tt, t \rangle = \overline{\langle T^*t, t \rangle} \\ &\stackrel{\text{Prop. 4(c)}}{\Leftrightarrow} \forall t \in H: \langle Tt, t \rangle \in \mathbb{R} \Leftrightarrow W(T) \subset \mathbb{R} \end{aligned}$$

(b) Suppose  $T^* = T$ .  $a := \inf W(T)$ ,  $b = \sup W(T)$ .

Then  $\sigma(T) \subset [a, b]$  (by Prop. 4(e), as  $[a, b] \supset \overline{W(T)}$ .  
In fact  $[a, b] = \overline{W(T)}$  by Prop. 4(d))

•  $T$  self-adjoint  $\Rightarrow r(T) = \|T\|$ . Hence, by Prop. 4(e,f)

$$\text{We get } r(T) = \|T\| = \max\{|a|, |b|\}$$

$$\text{Since } \|T\| = \max\{|a|, |b|\} (= \max\{b, -a\})$$

We get

$$r(T) = \|T\| = \max\{|a|, |b|\}$$

• Since  $\sigma(T)$  is compact,  $\sigma(T) \subset [a, b]$  and

$$r(T) = \|T\| = \max\{|a|, |b|\}, \text{ necessarily } \|T\| \in \sigma(T)$$

$$\text{or } -\|T\| \in \sigma(T)$$

• In fact  $a, b \in \sigma(T)$ :

$$S_1 := T - aI, \quad S_2 = T - bI$$

Then  $S_1, S_2$  are self-adjoint,  $\overline{W(S_1)} = [0, b-a]$

$\overline{W(S_2)} = [a-b, 0]$ . Thus  $b-a \in \sigma(S_1)$ ,  $a-b \in \sigma(S_2)$

$\Rightarrow b \in \sigma(T)$ ,  $a \in \sigma(T)$ .

c)  $W(T) \subset [0, \infty) \Leftrightarrow T^* = T \text{ \& } \sigma(T) \subset [0, \infty)$

$\Gamma \Rightarrow$  by (a) and Prop. 4(e)  $\Rightarrow W(T) \subset [0, \infty) \Leftrightarrow T^* = T$  (A)

$\Leftarrow$  Define  $a, b$  as in (b). Then  $a, b \in \sigma(T)$ . So  $a \geq 0$   
 $\parallel$  Thus  $W(T) \subset [0, \infty)$ .  $\perp$

$\overline{\sigma(T)} = \sigma(T, \infty)$

$\sigma(T, \infty) \subset \mathbb{R} \Leftrightarrow \sigma(T, \infty) \cap i\mathbb{R} = \emptyset$

(b) Suppose  $T^* = T$ .  $a := \inf W(T)$ ,  $b := \sup W(T)$

Then  $\sigma(T) \subset [a, b]$  (by Prop. 4(e))  
 $\sigma(T) \subset [0, \infty)$

Infact  $[a, b] \subset W(T)$  (Prop. 4(e))

Now,  $\|T\| = \max\{|a|, |b|\}$   
 $\|T\| = \max\{|a|, |b|\} = \max\{a, b\}$  (since  $a, b \geq 0$ )

$\|T\| = \max\{a, b\} = \max\{|a|, |b|\}$

Now  $\sigma(T) \subset [0, \infty)$  is compact,  $\sigma(T) \cap i\mathbb{R} = \emptyset$   
 $\|T\| = \max\{|a|, |b|\} = \max\{a, b\}$   
 $\sigma(T) \cap i\mathbb{R} = \emptyset$

$\sigma(T) \cap i\mathbb{R} = \emptyset$

$I - T = 0$ ,  $I - T = 1$

Then  $\sigma(I - T) = \overline{\sigma(T)}$  (Prop. 4(e))  
 $\sigma(I - T) \subset [0, \infty)$  (since  $\sigma(T) \subset [0, \infty)$ )  
 $\sigma(I - T) \cap i\mathbb{R} = \emptyset$