Proposition IX.7 H .. a Hilbert space, $T \in \mathcal{L}(H)$

(a) $T^* = T \iff W(T) \subset \mathbb{R}$

$\Big[T^* = T \overset{\text{Prop. 4(c)}}{\iff} \forall x \in H: \langle Tx, x\rangle = \langle T^*x, x\rangle$

$$\langle T^*x,x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

$\iff \forall x \in H: \langle Tx, x\rangle \in \mathbb{R} \iff W(T) \subset \mathbb{R} \Big]$

(b) Suppose $T^* = T$. $a := \inf W(T)$, $b = \sup W(T)$.

Then • $\sigma(T) \subset [a,b]$ (by Prop 4(e), as $[a,b] \supset \overline{W(T)}$. In fact $[a,b] = \overline{W(T)}$ by Prop. 4(d))

• T self-adjoint $\Rightarrow r(T) = \|T\|$. Hence, by Prop. 4(a,f) we get $r(T) = \cancel{w}(T) = \|T\|$. Since $w(T) = \max(|a|, |b|)$ $(= \max(b, -a))$ we get

$$r(T) = \|T\| = \max\{|a|, |b|\}$$

• Since $\sigma(T)$ is compact, $\sigma(T) \subset [a,b]$ and $r(T) = \|T\| = \max\{|a|,|b|\}$, necessarily $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$

• In fact $a, b \in \sigma(T)$:

$$S_1 := T - aI, \quad S_2 = T - bI$$

Then S_1, S_2 are self-adjoint, $\overline{W(S_1)} = [0, b-a]$, $\overline{W(S_2)} = [a-b, 0]$. Thus $b-a \in \sigma(S_1)$, $a-b \in \sigma(S_2)$ $\Rightarrow b \in \sigma(T)$, $a \in \sigma(T)$.

(c) $W(T) \subset [0,\infty) \iff T^* = T$ & $\sigma(T) \subset [0,\infty)$

$\Gamma \Rightarrow$ by (a) and Prop. 4 (e)

$\Leftarrow$ Define a, b as in (b). Then $a, b \in \sigma(T)$. So $a \geq 0$
Thus $W(T) \subset [0,\infty)$. $\lrcorner$