Let (X,τ) be a HTVS with a countable base of nbhds of 0. We will construct an F-norm generating the topology τ.

<u>Step 1</u>: Let $(V_n)_{n=1}^{\infty}$ be a base of nbhds of 0 s.t., moreover, $V_{n+1}+V_{n+1}\subset V_n$ for $n\in\mathbb{N}$.

For $F\subset\mathbb{N}$ nonempty and finite we set

$$q_F=\sum_{n\in F}2^{-n}$$

[$q_F\in(0,1)$, its binary expansion has only finitely many 1's, exactly at the coordinates with numbers in F]

$$V_F=\sum_{n\in F}V_n$$

Set
$$p(x)=\begin{cases}\inf\{q_F;\ x\in V_F\}, & \text{if } x\in\bigcup_F V_F\\ 1, & \text{if } x\in X\setminus\bigcup_F V_F\end{cases}$$

Then p is an F-norm and generates the topology of X. We will prove it in the sequel.

<u>Step 2</u>: Auxilliary properties of $F\mapsto(V_F,q_F)$

(i) $V_{F_1}\subset V_{F_2}\Leftarrow q_{F_1}\leq q_{F_2}$

[$q_{F_1}=q_{F_2}\Rightarrow F_1=F_2$ (by uniqueness of binary expansion)
$\Rightarrow V_{F_1}=V_{F_2}$

$q_{F_1}<q_{F_2}\Rightarrow \exists n\in\mathbb{N}:\ F_1\cap\{1,\dots,n-1\}=F_2\cap\{1,\dots,n-1\}$ and $n\in F_2\setminus F_1$

(by the properties of binary expansions)

Set $H := F_1 \cap \{1, \dots, n-1\}$ $(= F_2 \cap \{1, \dots, n-1\})$

Then • $V_{F_2} \supset V_H + V_n$ (as $F_2 \supset H \cup \{n\}$)

Further, $\forall k \in \mathbb{N}: V_n \supset V_{n+1} + V_{n+2} + \dots + V_{n+k} + V_{n+k}$ (*)

[Induction: $k=1$: $V_n \supset V_{n+1} + V_{n+1}$ by the assumptions

$k \to k+1$

$V_{n+1} + V_{n+2} + \dots + V_{n+k} + \underbrace{V_{n+k+1} + V_{n+k+1}}_{\subset V_{n+k}} \subset$

$\subset V_{n+1} + \dots + V_{n+k} + V_{n+k} \subset V_n$ by the induction hypothesis.]]

So, a fortiori, $V_n \supset V_{n+1} + V_{n+2} + \dots + V_{n+k}$ for $k \in \mathbb{N}$.

F_1 finite $\Rightarrow \exists k \in \mathbb{N}: F_1 \subset H \cup \{n+1, \dots, n+k\}$

So, $V_{F_1} \subset V_H + V_{n+1} + \dots + V_{n+k} \subset V_H + V_n \subset V_{F_2}$

Thus: $q_{F_1} = q_{F_1} \Rightarrow V_{F_1} = V_{F_2}$

$q_{F_1} < q_{F_2} \Rightarrow V_{F_1} \subset V_{F_2}$ $\Bigg\}$ the equivalence follows

$q_{F_1} > q_{F_2} \Rightarrow V_{F_1} \supset V_{F_2}$

(ii) $q_{F_1} + q_{F_2} < 1 \Rightarrow \exists!\, F: q_{F_1} + q_{F_2} = q_F$

(by computation in the binary system $q_{F_1} + q_{F_2}$ has finite binary expansion)

(iii) $q_{F_1} + q_{F_2} = q_F \Rightarrow V_{F_1} + V_{F_2} \subset V_F$

⌈ by induction on the cardinality of F_2

⊙ $F_2 = \{m\}$

⊙⊙ $m \notin F_1 \Rightarrow F = F_1 \cup F_2$, $V_{F_1} + V_{F_2} = V_F$

⊙⊙ $m \in F_1 \Rightarrow \exists\, n < m$ s.t. $n \notin F_1$, $\{n+1, \dots, m\} \subset F_1$

Then $F = \underbrace{(F_1 \cap \{1, \dots, n-1\})}_{H_1} \cup \{n\} \cup \underbrace{(F_1 \cap \{m+1, m+2, \dots\})}_{H_2}$

by the addition rules in binary system

$V_{F_1} + V_m = V_{H_1} + V_{n+1} + \dots + V_m + V_{H_2} + V_m \subset$

$\subset V_{H_1} + V_n + V_{H_2} = V_F$

by (#) from (i)

⊙ Suppose it holds if $|F_2| \leq k$. Let $|F_2| = k+1$.

Fix $n \in F_2$. Then

$q_{F_1} + q_{F_2} = q_{F_1} + q_{F_2 \setminus \{n\}} + q_{\{n\}}$ ($F_2 \setminus \{n\}$, $\{n\}$ are disjoint)

Find $\tilde{F}$ s.t. $q_{F_1} + q_{F_2 \setminus \{n\}} = q_{\tilde{F}}$ (by (ii))

Then $V_{F_1} + V_{F_2 \setminus \{n\}} \subset V_{\tilde{F}}$ (by the induction hypothesis)

Moreover, $q_F = q_{\tilde{F}} + q_{\{n\}} \Rightarrow V_{\tilde{F}} + V_n \subset V_F$ by the first induction step

Finally,

$V_{F_1} + V_{F_2} = V_{F_1} + V_{F_2 \setminus \{n\}} + V_n \subset V_{\tilde{F}} + V_n \subset V_F$

$F_2 \setminus \{n\}$, $\{n\}$ disjoint

⌋

Step 3. p has the required properties:

(a) $p(0) = 0$

$\ulcorner$ $\forall n \in \mathbb{N}$ $0 \in V_n$, so $p(0) \leq q_{\{n\}} = 2^{-n}$.
Hence $p(0) = 0$. $\lrcorner$

(b) $\forall x \in X \setminus \{0\}$: $p(x) > 0$

$\ulcorner$ Let $x \in X \setminus \{0\}$. Then, since X is Hausdorff, there is $n \in \mathbb{N}$ such that $x \notin V_n$

If $x \in V_F$, then $V_F \not\subseteq V_n$, hence by step 2(ii),
$q_F > q_{\{n\}} = 2^{-n}$.

So, $p(x) \geq 2^{-n} > 0$. $\lrcorner$

(c) $\forall x \in X$ $\forall \lambda \in \mathbb{F}, |\lambda| \leq 1$: $p(\lambda x) \leq p(x)$

$\ulcorner$ Let $x \in X$ and $\lambda \in \mathbb{F}$, $|\lambda| \leq 1$

Fix any $c > p(x)$.

If $c > 1$, then $p(\lambda x) \leq 1 < c$

If $c \leq 1$, there is F s.t. $x \in V_F$ & $q_F < c$

V_F is balanced (being a sum of a finite number of balanced sets) $\Rightarrow$ $\lambda x \in V_F$

Hence, by the definition,
$p(\lambda x) \leq q_F < c$. $\lrcorner$

(d) $\forall x,y\in X: p(x+y)\le p(x)+p(y)$

$\ulcorner$ Let $x,y\in X$

- If $p(x)+p(y)\ge 1$, then
  $p(x+y)\le 1\le p(x)+p(y)$

- Suppose $p(x)+p(y)<1$
  Fix $\varepsilon>0$ s.t. $p(x)+p(y)+\varepsilon<1$
  an arbitrary

By the definition of p

$\exists F_1: x\in V_{F_1}, q_{F_1}<p(x)+\frac{\varepsilon}{2}$

$\exists F_2: y\in V_{F_2}, q_{F_2}<p(y)+\frac{\varepsilon}{2}$

$q_{F_1}+q_{F_2}<p(x)+p(y)+\varepsilon<1$

$\Rightarrow$ by step 2(ii) there is F s.t. $q_{F_1}+q_{F_2}=q_F$

Moreover, by step 2(iii) $V_{F_1}+V_{F_2}\subset V_F$,
hence $x+y\in V_F$,
so $p(x+y)\le q_F<p(x)+p(y)+\varepsilon$.

Since $\varepsilon>0$ is arbitrary, we get $p(x+y)\le p(x)+p(y)$. $\lrcorner$

(e) $\forall x\in X: \lim\limits_{t\to 0+} p(tx)=0$

$\ulcorner$ Let $x\in X$ be arbitrary. Let $n\in\mathbb{N}$ be arbitrary.
V_n is absorbing $\Rightarrow \exists c_n>0$ s.t. $tx\in V_n$ for $t\in[0,c_n]$

Hence, for $t\in[0,c_n]$ we have $p(tx)\le q_{\{n\}}=2^{-n}$

Since $2^{-n}\to 0$, it follows that $\lim\limits_{t\to 0+} p(tx)=0$. $\lrcorner$

(f) $\{\{x \in X ; p(x) < r\} ; r > 0\}$ is a base of neighborhoods of 0 in X

⌈ Set $U_r := \{x \in X ; p(x) < r\}$ for $r > 0$

We have: $V_{n+1} \subset U_{2^{-n}} \subset V_n$

⌈ $x \in V_{n+1} \Rightarrow p(x) \leq q_{\{n+1\}} = 2^{-n-1} < 2^{-n}$

$x \in U_{2^{-n}} \Rightarrow p(x) < 2^{-n} \Rightarrow \exists F : x \in V_F$ & $q_F < 2^{-n}$
Since $2^{-n} = q_{\{n\}}$, by Step 2(i) we get $V_F \subset V_n$
Hence $x \in V_n$ ⌋

Further $r_1 < r_2 \Rightarrow U_{r_1} \subset U_{r_2}$, so, given $r > 0$, there is $n \in \mathbb{N}$ s.t. $r > 2^{-n}$
Then $U_r \supset U_{2^{-n}} \supset V_{n+1}$.

Hence
- $\forall r > 0$: U_r is a nbhd of 0
- $U_{2^{-n}}$, $n \in \mathbb{N}$ is a base of nbhds of 0. ⌋

Step 4: $\rho(x,y) = p(x-y)$ is a translation invariant metric generating the topology $\mathcal{T}$.

⌈
- ρ translation invariant ... clear by the definition
- $\rho(x,x) = 0$ by (a)
- $\rho(x,y) > 0$ for $x \neq y$ by (b)
- $\rho(x,y) = \rho(y,x)$ by (c) applied to $\lambda = -1$
- $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ by (d)

Hence ρ is a translation invariant metric.
Finally, by (f) nbhds of 0 in $\mathcal{T}$ and in ρ coincide.

Hence, the topology generated by ρ coincides with $\mathcal{T}$ ⌋