Let (X,T) be a HLCS whose topology is generated by a sequence $(p_n)_{n\geq 1}^{\infty}$ of seminorms

(1) WLOG $\quad p_1 \leq p_2 \leq p_3 \leq \dots$

$\ulcorner$ $q_n(x) := \max\{p_1(x), \dots, p_n(x)\}$ is also a seminorm and the family (p_n) generates the same topology as (q_n) $\lrcorner$

(2) ~~ϱ~~ $$\varrho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x-y)\}$$

is a translation invariant metric on X

$\ulcorner$ $\varrho(x,x) = 0$ ... clear

$x \neq y \Rightarrow \exists n : p_n(x-y) > 0$ (as X is Hausdorff)

Hence $\varrho(x,y) > 0$

$\varrho(x,y) = \varrho(y,x)$ ... clear, as $p_n(x-y) = p_n(y-x)$

$\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z)$

$\ulcorner$ for each $n \in \mathbb{N}$: $p_n(x-z) \leq p_n(x-y) + p_n(y-z)$

and hence also

$\min\{1, p_n(x-z)\} \leq \min\{1, p_n(x-y)\} + \min\{1, p_n(y-z)\}$ $\lrcorner$

ϱ translation invariant ... clear $\lrcorner$

(3) For each $n \in \mathbb{N}$ and ~~$\varepsilon \in (0,1)$~~ $\varepsilon > 0$ we have

$$\{x ; p_n(x) < \varepsilon\} \subset \{x ; \varrho(x,0) < \varepsilon + 2^{-n}\}$$

$\ulcorner$ $p_n(x) < \varepsilon \Rightarrow \forall k \leq n : p_k(x) \leq p_n(x) < \varepsilon$, so

$$\varrho(x,0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, p_k(x)\} = \sum_{k=1}^{n} \frac{1}{2^k} \min\{1, p_k(x)\} +$$

$$+ \sum_{k=n+1}^{\infty} \frac{1}{2^k} \min\{1, p_k(x)\} < \sum_{k=1}^{n} \frac{1}{2^k} \cdot \varepsilon + \sum_{k=n+1}^{\infty} \frac{1}{2^k} < \varepsilon + \frac{1}{2^n}$$ $\lrcorner$

(4) $\forall \varepsilon \in (0,1)\ \forall n \in \mathbb{N}$: $\{x;\ \rho(t,0) < \frac{\varepsilon}{2^n}\} \subset \{t;\ P_n(t) < \varepsilon\}$

$\lceil\ \rho(t,0) < \frac{\varepsilon}{2^n} \implies \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, P_k(t)\} < \frac{\varepsilon}{2^n}$

$\implies \frac{1}{2^n} \cdot \min\{1, P_n(t)\} < \frac{\varepsilon}{2^n}$

$\implies \min\{1, P_n(t)\} < \varepsilon$

Since $\varepsilon < 1$, it follows $P_n(t) < \varepsilon$ $\rfloor$

(5) ρ generates the topology $\mathcal{T}$

$\lceil$ (3) $\implies \forall r > 0$ $\{t;\ \rho(t,0) < r\}$ is a neighborhood of 0

Γ $r > 0$ ... Fix $\varepsilon > 0$ and $n \in \mathbb{N}$ s.t. $\varepsilon + \frac{1}{2^n} < r$

Then $\{t;\ \rho(t,0) < r\} \supset \{t;\ \rho(t,0) < \varepsilon + \frac{1}{2^n}\} \supset \{t;\ P_n(t) < \varepsilon\}$

(4) $\implies \{x;\ \rho(t,0) < r\}$, $r > 0$, is a base of nbhds of 0.

Hence, the topology generated by ρ has the same nbhds of 0 as $\mathcal{T}$, so it coincides with $\mathcal{T}$. $\rfloor$